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GEOMETRIC DISTANCE FITTING OF PARABOLAS IN $\mathbb{R}^3$

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Abstract. We are interested in the problem of fitting a parabola to a set of data points in $\mathbb{R}^3$. It can be usually solved by minimizing the geometric distances from the fitted parabola to the given data points. In this paper, a parabola fitting algorithm will be proposed in such a way that the sum of the squares of the geometric distances is minimized in $\mathbb{R}^3$. Our algorithm is mainly based on the steepest descent technique which determines an adequate number λ such that $h(\lambda) = Q(u - \lambda\nabla Q(u)) < Q(u)$. Some numerical examples are given to test our algorithm.

1. Introduction and preliminaries

We are interested in the problem of fitting a curve in $\mathbb{R}^2$ or $\mathbb{R}^3$ (see [2–5, 7, 9–11, 13]). Especially, the problem of fitting a parabola to the given data points will be considered in $\mathbb{R}^3$. This is a basic task that arises in many application areas, e.g., pattern recognition, computer vision and graphics, etc. Many algorithms already have been introduced in $\mathbb{R}^2$ so far, but, those in $\mathbb{R}^3$ have been rarely studied, and become interesting topics recently. In $\mathbb{R}^2$ H. Spath introduced one of the most well-known parametric fitting algorithms which can be found in [12]. Ahn proposed a geometric fitting algorithm as a non-parametric one in [1]. Most algorithms can be usually obtained by minimizing the geometric distances or various approximate algebraic distances from the fitted parabola to the given data points. Geometric distance fitting algorithms can be usually implemented by iterative methods. They sometimes are computationally intensive or divergent. Nevertheless, they are regarded as the most accurate. In contrast, algebraic fitting algorithms are faster, but less precise than geometric fitting algorithms. Thus, we need to propose an efficient algorithm for geometric fitting of parabolas. In connection with it, we consider the problem of fitting a parabola to the given data points $\left\{(\tilde{x}_k, \tilde{y}_k, \tilde{z}_k)\right\}_{k=1}^{n}$ in such a way that the sum of the squares of the geometric distances is minimized.

First, if we let $(b, c, 0)$ be the unknown vertex of a standard parabola C lying on the xy-plane in $\mathbb{R}^3$, and let a be a positive real number such that $p = \frac{1}{4a}$ becomes the distance from the vertex to the focus, then C can be represented

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by the vector valued function $w(a,b,c,t) = \big(x(a,b,c,t), y(a,b,c,t), z(a,b,c,t)\big)^t$ in its parametric form

$$C : x(a,b,c,t) = t + b, \quad y(a,b,c,t) = at^2 + c, \quad z(a,b,c,t) = 0, \tag{1}$$

where $-\infty < t < \infty$. Also, we have $y = a(x-b)^2 + c$ and $z = 0$.

Further, by using the rotation matrix S rotated by the angle γ around the z axis, namely

$$S = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

the general standard parabola $\hat{C}$ which is lying on the xy-plane, can be represented by the vector valued function $\hat{w} = \hat{w}(a,b,c,t,\gamma)$ given by

$$\begin{aligned} \hat{w}(a,b,c,t,\gamma) &= Sw(a,b,c,t) \\ &= \big(\hat{x}(a,b,c,t,\gamma), \hat{y}(a,b,c,t,\gamma), 0\big)^t. \end{aligned} \tag{2}$$

Now, using the rotation matrices R_1 and R_2, which are rotated by the angle α around the x axis and the angle β around the y axis respectively, namely

$$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix}, \quad R_2 = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix},$$

we have the following model parabola $\tilde{C}$ in $\mathbb{R}^3$ which can be expressed by the corresponding vector valued function $\tilde{w} = \tilde{w}(a,b,c,t,\gamma,\alpha,\beta,\delta)$:

$$\begin{aligned} &\tilde{w}(a,b,c,t,\gamma,\alpha,\beta,\delta) \\ &= R\big(\hat{w}(a,b,c,t,\gamma) + \delta(0,0,1)^t\big) \\ &= R\hat{w}(a,b,c,t,\gamma) + \delta\mathbf{n} \\ &= \big(\tilde{x}(a,b,c,t,\gamma,\alpha,\beta,\delta), \tilde{y}(a,b,c,t,\gamma,\alpha,\beta,\delta), \tilde{z}(a,b,c,t,\gamma,\alpha,\beta,\delta)\big)^t, \end{aligned} \tag{3}$$

where

$$R = R_1R_2 = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ -\sin\alpha\sin\beta & \cos\alpha & \sin\alpha\cos\beta \\ -\cos\alpha\sin\beta & -\sin\alpha & \cos\alpha\cos\beta \end{pmatrix}, \tag{4}$$

and

$$\mathbf{n} = (n_1, n_2, n_3)^t = R\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin\beta \\ \sin\alpha\cos\beta \\ \cos\alpha\cos\beta \end{pmatrix} \tag{5}$$

is the unit normal vector of the plane which contains $\tilde{C}$.

Also, the geometric distances $\tilde{d}_k$ of the points $(\tilde{x}_k, \tilde{y}_k, \tilde{z}_k)$ to the model parabola $\tilde{C}$ can be expressed by

$$
\begin{aligned}
\big(\tilde{d}_k\big)^2 =\ & \min_t \Big[\big(\tilde{x}_k - \tilde{x}(a,b,c,t,\gamma,\alpha,\beta,\delta)\big)^2 \\
& + \big(\tilde{y}_k - \tilde{y}(a,b,c,t,\gamma,\alpha,\beta,\delta)\big)^2 + \big(\tilde{z}_k - \tilde{z}(a,b,c,t,\gamma,\alpha,\beta,\delta)\big)^2\Big] \\
=\ & \big(\tilde{x}_k - \tilde{x}(a,b,c,t_k,\gamma,\alpha,\beta,\delta)\big)^2 + \big(\tilde{y}_k - \tilde{y}(a,b,c,t_k,\gamma,\alpha,\beta,\delta)\big)^2 \\
& + \big(\tilde{z}_k - \tilde{z}(a,b,c,t_k,\gamma,\alpha,\beta,\delta)\big)^2,
\end{aligned} \tag{6}
$$

where $\big(\tilde{x}(a,b,c,t_k,\gamma,\alpha,\beta,\delta),\ \tilde{y}(a,b,c,t_k,\gamma,\alpha,\beta,\delta),\ \tilde{z}(a,b,c,t_k,\gamma,\alpha,\beta,\delta)\big)$ are the corresponding nearest points with respect to the points $(\tilde{x}_k, \tilde{y}_k, \tilde{z}_k)$, which are dependent on the unknowns $a, b, c, \gamma, \alpha, \beta, \delta$. Thus, to fit the model parabola $\tilde{C}$ to the given data points we need to determine two parameter vector $p = (a, b, c, \gamma)^t$ and $q = (\alpha, \beta, \delta)^t$ by minimizing $\sum_{k=1}^{n} (\tilde{d}_k)^2 = \min$. The objective function $\tilde{Q}$ to be minimized is given by

$$
\tilde{Q}(\tilde{u}) = \sum_{k=1}^{n} \big(\tilde{d}_k\big)^2, \tag{7}
$$

where the parameter vector $\tilde{u} = (a, b, c, \gamma, \alpha, \beta, \delta; t_1, t_2, \ldots, t_n)^t \in \mathbb{R}^7$ is dependent upon $a, b, c, \gamma, \alpha, \beta, \delta$, and the additional unknowns $\{t_i\}_{i=1}^n$ are also dependent on $a, b, c, \gamma, \alpha, \beta, \delta$. On the other hand, due to the norm invariant property of the transformation R, we have the following general standard parabola $\hat{C}$ with respect to $\tilde{C}$, which lies on the xy-plane in $\mathbb{R}^3$,

$$
\begin{aligned}
\hat{C} : \hat{w} = \hat{w}(a,b,c,t,\gamma) &= R^{-1}\big(\tilde{w}(a,b,c,t,\gamma,\alpha,\beta,\delta) - \delta\mathbf{n}\big) \\
&= Sw(a,b,c,t) \\
&= \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t+b \\ at^2+c \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} (t+b)\cos\gamma - (at^2+c)\sin\gamma \\ (t+b)\sin\gamma + (at^2+c)\cos\gamma \\ 0 \end{pmatrix} \\
&= \big(\hat{x}(a,b,c,t,\gamma), \hat{y}(a,b,c,t,\gamma), \hat{z}(a,b,c,t,\gamma)\big)^t,
\end{aligned} \tag{8}
$$

where $-\infty < t < \infty$.

In addition, the corresponding transformed data points $\big\{(\hat{x}_k, \hat{y}_k, \hat{z}_k)\big\}_{k=1}^n$ with respect to the given data points $\big\{(\tilde{x}_k, \tilde{y}_k, \tilde{z}_k)\big\}_{k=1}^n$ are given by

$$
\begin{aligned}
& (\hat{x}_k, \hat{y}_k, \hat{z}_k)^t \\
&= R^{-1}(\tilde{x}_k, \tilde{y}_k, \tilde{z}_k)^t - \delta(0,0,1)^t \\
&= R^{-1}\left[(\tilde{x}_k, \tilde{y}_k, \tilde{z}_k)^t - \delta\mathbf{n}\right] \\
&= R^t\left[(\tilde{x}_k, \tilde{y}_k, \tilde{z}_k)^t - \delta\mathbf{n}\right]
\end{aligned} \tag{9}
$$

$$= \begin{pmatrix} \tilde{x}_k \cos\beta - \tilde{y}_k \sin\alpha \sin\beta - \tilde{z}_k \cos\alpha \sin\beta \\ \tilde{y}_k \cos\alpha - \tilde{z}_k \sin\alpha \\ \tilde{x}_k \sin\beta + \tilde{y}_k \sin\alpha \cos\beta + \tilde{z}_k \cos\alpha \cos\beta - \delta \end{pmatrix}.$$

So, the objective function can be replaced by the new function $\hat{Q}(u)$ given by

$$\begin{aligned} \hat{Q}(u) &= \sum_{k=1}^{n} \left(\hat{d}_k\right)^2 \\ &= \sum_{k=1}^{n} \Big[\left\{ \hat{x}_k - (t_k + b)\cos\gamma + (at_k^2 + c)\sin\gamma \right\}^2 \\ &\quad + \left\{ \hat{y}_k - (t_k + b)\sin\gamma - (at_k^2 + c)\cos\gamma \right\}^2 + \left\{ \hat{z}_k \right\}^2 \Big], \end{aligned} \tag{10}$$

where the parameter vector $u = (a, b, c, \gamma, \alpha, \beta, \delta)^t \in \mathbb{R}^7$, and $\hat{d}_k$ denotes the distance of the point $(\hat{x}_k, \hat{y}_k, \hat{z}_k)$ to the general standard parabola $\hat{C}$.

Thus, to determine $u = (a, b, c, \gamma, \alpha, \beta, \delta)^t \in \mathbb{R}^7$, i.e., to obtain two vectors $p = (a, b, c, \gamma)^t$ and $q = (\alpha, \beta, \delta)^t$ simultaneously, we may solve the nonlinear system of seven equations for the parameter vector u induced by the necessary minimum conditions, namely

$$\frac{\partial \hat{Q}}{\partial a} = \frac{\partial \hat{Q}}{\partial b} = \frac{\partial \hat{Q}}{\partial c} = \frac{\partial \hat{Q}}{\partial \gamma} = 0, \tag{11}$$

$$\frac{\partial \hat{Q}}{\partial \alpha} = \frac{\partial \hat{Q}}{\partial \beta} = \frac{\partial \hat{Q}}{\partial \delta} = 0. \tag{12}$$

In this paper, our problem will be divided into two parts. At first, by using the rotation and the translation of data points the orthogonal distance problem in $\mathbb{R}^3$ can be reduced to the problem of parabola fitting in the plane. In Section 2, due to the technique of finding the geometric distance from a point to a parabola we consider a parabola fitting algorithm in the xy-plane. Finally, in Section 3 we propose an iterative algorithm for fitting a parabola in $\mathbb{R}^3$ which is mainly based on the steepest descent method. Our algorithm has the advantage of ensuring the convergence of $\hat{Q}(u)$ to a local minimum. In order to test the proposed algorithm some numerical examples will be given in Section 4.

2. Geometric distance fitting of parabolas in $\mathbb{R}^2$

In this section, we consider the problem of fitting a general standard parabola $\hat{C}$ to a set of data points with coordinates $\left\{(\hat{x}_k, \hat{y}_k)\right\}_{k=1}^{n}$. At first, let us consider the problem of finding the geometric distances from a given set of data points $\left\{(\hat{x}_k, \hat{y}_k)\right\}_{k=1}^{n}$ to a general standard parabola, i.e., a rotated parabola $\hat{C}$ in the xy-plane, which contains a procedure for computing the corresponding values $\left\{t_k\right\}_{k=1}^{n}$ which are dependent upon a, b, c, γ. The following orthogonal distance problem in $\mathbb{R}^2$ can be found in [6].

First, in $\mathbb{R}^2$ the standard parabola

$$C : y = a(x-b)^2 + c \ (a \neq 0) \tag{13}$$

can be expressed by the vector valued function $w = w(a,b,c,t) = \big(x(a,b,c,t), y(a,b,c,t)\big)^t$ in its parametric form

$$\begin{aligned} C : x(a,b,c,t) &= t + b, \\ y(a,b,c,t) &= at^2 + c, \end{aligned} \tag{14}$$

where $-\infty \le t \le \infty$.

Also, we have the general standard parabola $\hat{C}$ rotated with unknown angle γ is given by

$$\hat{C} : \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} = S \begin{pmatrix} x \\ y \end{pmatrix}, \tag{15}$$

where the coordinate transformation matrix $S = \begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix}$.

In other words, the parametric form of $\hat{C}$ with additional parameter γ can be denoted by

$$\begin{aligned} \hat{x} &= (t+b)\cos\gamma - (at^2 + c)\sin\gamma, \\ \hat{y} &= (t+b)\sin\gamma + (at^2 + c)\cos\gamma. \end{aligned} \tag{16}$$

Thus, the orthogonal distance $\hat{d}(\hat{w}_k, \hat{C})$ of $\hat{w}_k = (\hat{x}_k, \hat{y}_k)^t$ from the point $(\hat{x}_k, \hat{y}_k)$ to $\hat{C}$ can be computed by

$$\begin{aligned} \big[\hat{d}(\hat{w}_k, \hat{C})\big]^2 = \min_t \Big[&\big(\hat{x}_k - (t+b)\cos\gamma + (at^2 + c)\sin\gamma\big)^2 \\ &+ \big(\hat{y}_k - (t+b)\sin\gamma - (at^2 + c)\cos\gamma\big)^2 \Big]. \end{aligned} \tag{17}$$

Moreover, due to the norm invariant property of the rotation matrix S we can use the distance $d(w_k, C)$ of $w_k = (x_k, y_k)^t$ from the point (x_k, y_k) to the standard parabola C instead of $\hat{d}(\hat{w}_k, \hat{C})$. Namely, the corresponding rotated data points $\big\{(x_k, y_k)\big\}_{k=1}^n$ with respect to $\big\{(\hat{x}_k, \hat{y}_k)\big\}_{k=1}^n$ are given by

$$\begin{aligned} w_k = \begin{pmatrix} x_k \\ y_k \end{pmatrix} &= S^{-1}\left[\hat{w}_k\right] = R^t \begin{pmatrix} \hat{x}_k \\ \hat{y}_k \end{pmatrix} \\ &= \begin{pmatrix} \hat{x}_k \cos\gamma + \hat{y}_k \sin\gamma \\ -\hat{x}_k \sin\gamma + \hat{y}_k \cos\gamma \end{pmatrix}, \end{aligned} \tag{18}$$

and

$$\begin{aligned}
\big[\hat{d}(\hat{w}_k,\hat{C})\big]^2=\big[d(w_k,C)\big]^2 &= \min_t\Big[\big(x_k-(t+b)\big)^2+\big(y_k-(at^2+c)\big)^2\Big]\\
&= \min_t\Big[\big(\hat{x}_k\cos\gamma+\hat{y}_k\sin\gamma-t-b\big)^2\\
&\qquad +\big(-\hat{x}_k\sin\gamma+\hat{y}_k\cos\gamma-at^2-c\big)^2\Big]\\
&= \big(x_k-x_k^t\big)^2+\big(y_k-y_k^t\big)^2\\
&= \Big[\big(\hat{x}_k\cos\gamma+\hat{y}_k\sin\gamma-t_k-b\big)^2\\
&\qquad +\big(-\hat{x}_k\sin\gamma+\hat{y}_k\cos\gamma-a{t_k}^2-c\big)^2\Big],
\end{aligned} \tag{19}$$

where $\{\phi_k^t\}_{k=1}^n=\{(x_k^t,y_k^t)\}_{k=1}^n=\{(t_k+b,a{t_k}^2+c)\}_{k=1}^n$ denote the corresponding nearest points, lying on C with respect to $\{(x_k,y_k)\}_{k=1}^n$.

Now, according to the perpendicular property between the tangent lines at $\{\phi_k^t\}_{k=1}^n$ and the straight lines of connecting $\{\phi_k^t\}_{k=1}^n$ and $\{(x_k,y_k)\}_{k=1}^n$ it is possible to compute the n values t_k $(k=1,2,\ldots,n)$ easily, such that

$$\begin{aligned}
\big[d(w_k,C)\big]^2 &= \min_t\Big[\big(x_k-(t+b)\big)^2+\big(y_k-(at^2+c)\big)^2\Big]\\
&= \Big[\big(x_k-t_k-b\big)^2+\big(y_k-a{t_k}^2-c\big)^2\Big]\\
&= \big[d(w_k,\phi_k^t)\big]^2.
\end{aligned} \tag{20}$$

In other words, for $k=1,2,\ldots,n$

$$\left[\frac{dy}{dx}\right]_{x=x_k^t}\cdot\frac{(y_k-y_k^t)}{(x_k-x_k^t)}=2a\big(x_k^t-b\big)\cdot\frac{(y_k-y_k^t)}{(x_k-x_k^t)}=2at_k\cdot\frac{(y_k-a{t_k}^2-c)}{(x_k-t_k-b)}=-1$$

allows us to have the following n cubic equations for $\{t_k\}_{k=1}^n$:

$$ {t_k}^3+p_kt_k+q_k=0 \quad (k=1,2,\ldots,n) \tag{21}$$

with

$$p_k=\frac{1-2a(y_k-c)}{2a^2},$$
$$q_k=\frac{(b-x_k)}{2a^2}.$$

If we let D_k be the discriminant such that

$$D_k=\left(\frac{p_k}{3}\right)^3+\left(\frac{q_k}{2}\right)^2 \quad (k=1,2,\ldots,n),$$

then, in the case of $D_k>0$ each equation (21) has the unique one real root:

$$t_k=A_k-\frac{p_k}{3A_k} \quad (k=1,2,\ldots,n),$$

where

$$A_k = \left(-\frac{q_k}{2} + \sqrt{D_k}\right)^{\frac{1}{3}}.$$

When $D_k \leq 0$, there are two or three real roots for each t_k:

$$s_1 = B_k - \frac{p_k}{3B_k},$$
$$s_2 = wB_k - w^2\frac{p_k}{3B_k},$$
$$s_3 = w^2B_k - w\frac{p_k}{3B_k},$$

where

$$w = \exp\left(\frac{2\pi}{3}i\right),$$
$$B_k = \sqrt{\frac{|p_k|}{3}}\exp\left(\frac{\theta_k}{3}i\right)$$

with

$$\cos\theta_k = -\frac{q_k}{2\sqrt{\left(\frac{|p_k|}{3}\right)^3}}.$$

Thus, in this case the value $t_k = s_m$ can be chosen such that

$$\begin{aligned}&\big(x_k - (s_m + b)\big)^2 + \big(y_k - (a{s_m}^2 + c)\big)^2 \\ = &\min_{l=1,2,3}\left[\big(x_k - (s_l + b)\big)^2 + \big(y_k - (a{s_l}^2 + c)\big)^2\right],\end{aligned}$$

and we obtain the corresponding nearest contact-points $\{\phi_k^t\}_{k=1}^n = \{(x_k^t, y_k^t)\}_{k=1}^n = \{(t_k + b, a{t_k}^2 + c)\}_{k=1}^n$, lying on C with respect to $\{(x_k, y_k)\}_{k=1}^n$. Finally, the distances $\hat{d}(\hat{w}_k, \hat{C})$ of $\hat{w}_k$ from the given data points $\{(\hat{x}_k, \hat{y}_k)\}_{k=1}^n$ to $\hat{C}$ are given by

$$\hat{d}(\hat{w}_k, \hat{C}) = d(w_k, C) = \sqrt{\big(x_k - t_k - b\big)^2 + \big(y_k - a{t_k}^2 - c\big)^2}, \tag{22}$$

where

$$w_k = \begin{pmatrix} x_k \\ y_k \end{pmatrix} = S^{-1}[\hat{w}_k] = S^t\begin{pmatrix} \hat{x}_k \\ \hat{y}_k \end{pmatrix} = \begin{pmatrix} \hat{x}_k\cos\gamma + \hat{y}_k\sin\gamma \\ -\hat{x}_k\sin\gamma + \hat{y}_k\cos\gamma \end{pmatrix}.$$

The following procedure will be useful for finding the corresponding values $\{t_k(a, b, c, \gamma)\}_{k=1}^n$ with respect to the given points $\{(\hat{x}_k, \hat{y}_k)\}_{k=1}^n$. These values can be given by $t_k = t_k(a, b, c, \gamma) = find\_pt(a, b, c, \gamma, \hat{x}_k, \hat{y}_k)$, $k = 1, 2, \ldots, n$.

Procedure A : Subroutine $find\_pt\big(a, b, c, \gamma, \hat{x}, \hat{y}\big)$

$$u = \big(a, b, c, \gamma\big)^t, \hat{w} = (\hat{x}, \hat{y})^t.$$

STEP 1 : Compute

$$x = \hat{x}\cos\gamma + \hat{y}\sin\gamma,$$
$$y = -\hat{x}\sin\gamma + \hat{y}\cos\gamma.$$

STEP 2 : Solve the cubic equation: $t^3 + pt + q = 0$ with

$$p = \frac{1 - 2a(y - c)}{2a^2},$$
$$q = \frac{b - x}{2a^2}.$$

Let $D = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2$. If $D > 0$, then

$$t = A - \frac{p}{3A},$$

where

$$A = \left(-\frac{q}{2} + \sqrt{D}\right)^{\frac{1}{3}}.$$

In the case of $D \leq 0$, compute

$$s_1 = B - \frac{p}{3B},$$
$$s_2 = wB - w^2\frac{p}{3B},$$
$$s_3 = w^2B - w\frac{p}{3B},$$

where

$$w = \exp\left(\frac{2\pi}{3}i\right),$$
$$B = \sqrt{\frac{|p|}{3}}\,\exp\left(\frac{\theta}{3}i\right)$$

with

$$\cos\theta = -\frac{q}{2\sqrt{\left(\frac{|p|}{3}\right)^3}}.$$

Take the value $t = s_m$ such that

$$\begin{aligned}&\big(x - (s_m + b)\big)^2 + \big(y - (a{s_m}^2 + c)\big)^2 \\ &= \min_{l=1,2,3}\left[\big(x - (s_l + b)\big)^2 + \big(y - (a{s_l}^2 + c)\big)^2\right].\end{aligned}$$

STEP 3 : Return $find\_pt(a, b, c, \gamma, \hat{x}, \hat{y}) := t$.

Finally, let us now consider the problem of fitting a general standard parabola $\hat{C}$ to a set of data points with coordinates $\left\{(\hat{x}_k, \hat{y}_k)\right\}_{k=1}^{n}$.

First, we have the representation of a general standard parabola $\hat{C}$ rotated with unknown angle γ :

$$\hat{C} : \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} = S \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\gamma - y\sin\gamma \\ x\sin\gamma + y\cos\gamma \end{pmatrix}, \tag{23}$$

where $S = \begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix}$ is the rotation matrix, and the corresponding standard parabola $C : y = a(x-b)^2 + c$ can be expressed by its parametric form:

$$C : \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t + b \\ at^2 + c \end{pmatrix}. \tag{24}$$

Thus, our problem is to determine the general standard parabolas (23) such that the following quadratic function $\hat{Q}(u)$ is minimized:

$$\begin{aligned} \hat{Q}(u) &= \sum_{k=1}^{n} \left(\min_t \left[\big(\hat{x}_k - \hat{x}(a,b,c,t,\gamma)\big)^2 + \big(\hat{y}_k - \hat{y}(a,b,c,t,\gamma)\big)^2 \right] \right) \\ &= \sum_{k=1}^{n} \left[\big(\hat{x}_k - \hat{x}(a,b,c,t_k,\gamma)\big)^2 + \big(\hat{y}_k - \hat{y}(a,b,c,t_k,\gamma)\big)^2 \right], \end{aligned} \tag{25}$$

where $\big(\hat{x}(a,b,c,t_k,\gamma), \hat{y}(a,b,c,t_k,\gamma)\big)$, $k = 1,2,\ldots,n$ are the corresponding nearest contact-points with respect to the points $(\hat{x}_k, \hat{y}_k)$, $k = 1,2,\ldots,n$, which are dependent on the unknowns a,b,c,γ.

Due to the transformation invariant property we have

$$\begin{aligned} \hat{Q}(u) = Q(u) &= \sum_{k=1}^{n} \big(d_k\big)^2 \\ &= \sum_{k=1}^{n} \left[\big(x_k - \{t_k + b\}\big)^2 + \big(y_k - \{a{t_k}^2 + c\}\big)^2 \right], \end{aligned} \tag{26}$$

where the parameter vector $u = (a,b,c,\gamma)^t \in \mathbb{R}^4$, and the corresponding rotated data points $(x_k, y_k) = \big(x_k(\gamma), y_k(\gamma)\big)$ are given by the following vector valued functions ρ_k of a variable γ:

$$\begin{aligned} \rho_k(\gamma) &= \big(x_k(\gamma), y_k(\gamma)\big)^t \\ &= S^{-1}\left[(\hat{x}_k, \hat{y}_k)^t\right] \\ &= \begin{pmatrix} \hat{x}_k\cos\gamma + \hat{y}_k\sin\gamma \\ -\hat{x}_k\sin\gamma + \hat{y}_k\cos\gamma \end{pmatrix}, \end{aligned} \tag{27}$$

and the corresponding values $\big\{t_k\big\}_{k=1}^n = \big\{t_k(a,b,c,\gamma)\big\}_{k=1}^n$ are dependent upon the unknowns a,b,c,γ and become the roots of the cubic equations:

$$\begin{aligned} &{t_k}^3 + p_k t_k + q_k = 0 \ \ (k = 1,2,\ldots,n) \text{ with} \\ &p_k = \frac{1 - 2a(y_k - c)}{2a^2} \text{ and } q_k = \frac{b - x_k}{2a^2}. \end{aligned} \tag{28}$$

Further, we have the following:

$$\frac{\partial Q}{\partial a} = -2\sum_{k=1}^{n}\left[(x_k - t_k - b)\left(\frac{\partial t_k}{\partial a}\right) + (y_k - a{t_k}^2 - c)\left({t_k}^2 + 2at_k\frac{\partial t_k}{\partial a}\right)\right], \tag{29}$$

$$\frac{\partial Q}{\partial b} = -2\sum_{k=1}^{n}\left[(x_k - t_k - b)\left(\frac{\partial t_k}{\partial b} + 1\right) + (y_k - a{t_k}^2 - c)\left(2at_k\frac{\partial t_k}{\partial b}\right)\right], \tag{30}$$

$$\frac{\partial Q}{\partial c} = -2\sum_{k=1}^{n}\left[(x_k - t_k - b)\left(\frac{\partial t_k}{\partial c}\right) + (y_k - a{t_k}^2 - c)\left(2at_k\frac{\partial t_k}{\partial c} + 1\right)\right], \tag{31}$$

$$\begin{aligned}\frac{\partial Q}{\partial \gamma}&\\ = -2\sum_{k=1}^{n}&\left[(x_k - t_k - b)\left(\frac{\partial t_k}{\partial \gamma} - \frac{\partial x_k}{\partial \gamma}\right) + (y_k - a{t_k}^2 - c)\left(2at_k\frac{\partial t_k}{\partial \gamma} - \frac{\partial y_k}{\partial \gamma}\right)\right].\end{aligned} \tag{32}$$

Also, due to (27) and (28) the following partial derivatives can be obtained:

$$\frac{\partial x_k}{\partial \gamma} = -\hat{x}_k \sin\gamma + \hat{y}_k \cos\gamma, \tag{33}$$

$$\frac{\partial y_k}{\partial \gamma} = -\hat{x}_k \cos\gamma - \hat{y}_k \sin\gamma, \tag{34}$$

$$\frac{\partial t_k}{\partial a} = -\frac{\frac{\partial p_k}{\partial a}t_k + \frac{\partial q_k}{\partial a}}{3{t_k}^2 + p_k} = -\frac{at_k(y_k - c) - t_k + x_k - b}{a^3\left(3{t_k}^2 + p_k\right)}, \tag{35}$$

$$\frac{\partial t_k}{\partial b} = -\frac{\frac{\partial q_k}{\partial b}}{3{t_k}^2 + p_k} = -\frac{1}{2a^2\left(3{t_k}^2 + p_k\right)}, \tag{36}$$

$$\frac{\partial t_k}{\partial c} = -\frac{\frac{\partial p_k}{\partial c}t_k}{3{t_k}^2 + p_k} = -\frac{t_k}{a\left(3{t_k}^2 + p_k\right)}, \tag{37}$$

$$\begin{aligned}\frac{\partial t_k}{\partial \gamma} &= -\frac{\frac{\partial p_k}{\partial \gamma}t_k + \frac{\partial q_k}{\partial \gamma}}{3{t_k}^2 + p_k}\\ &= -\frac{\hat{x}_k(2at_k\cos\gamma + \sin\gamma) + \hat{y}_k(2at_k\sin\gamma - \cos\gamma)}{2a^2\left(3{t_k}^2 + p_k\right)}.\end{aligned} \tag{38}$$

On the other hand, the necessary conditions for a minimum of $Q(u)$ give the four non-linear equations:

$$\frac{\partial Q}{\partial a} = \frac{\partial Q}{\partial b} = \frac{\partial Q}{\partial c} = \frac{\partial Q}{\partial \gamma} = 0. \tag{39}$$

Thus, at a glance, this problem looks not so difficult in order to be solved and one may try to apply minimization algorithms to (39) or solve the nonlinear system of the above four equations for $u = (a, b, c, \gamma)^t$. But, as seen in (28)-(32) this system of non-linear equations is far more complicated and not easy to be solved directly by using some well known algorithms. So, we need to establish an efficient iterative algorithm which consists of a root-finding procedure and a steepest descent algorithm. Now, we describe an algorithm for fitting a general standard parabola $\hat{C}$ to a set of data points with coordinates $\big\{(\hat{x}_k, \hat{y}_k)\big\}_{k=1}^{n}$ as follows:

Algorithm B :

STEP 1 : Give $a^{(0)}, b^{(0)}, c^{(0)}, \gamma^{(0)}$ as four initial values for unknowns a, b, c and γ. That is, determine $u^{(0)} = \big(a^{(0)}, b^{(0)}, c^{(0)}, \gamma^{(0)}\big)^t$.

Compute the corresponding rotated data points $\big\{(x_k, y_k)\big\}_{k=1}^{n}$ with respect to $\big\{(\hat{x}_k, \hat{y}_k)\big\}_{k=1}^{n}$:

$$\begin{aligned} x_k &= \hat{x}_k \cos\gamma^{(0)} + \hat{y}_k \sin\gamma^{(0)}, \\ y_k &= -\hat{x}_k \sin\gamma^{(0)} + \hat{y}_k \cos\gamma^{(0)}. \end{aligned}$$

Solve the n cubic equations:

$$t_k{}^3 + p_k t_k + q_k = 0 \;\; (k = 1, 2, \ldots, n) \text{ with}$$

$$\begin{aligned} p_k &= \frac{1 - 2a^{(0)}(y_k - c^{(0)})}{2\big\{a^{(0)}\big\}^2}, \\ q_k &= \frac{b^{(0)} - x_k}{2\big\{a^{(0)}\big\}^2}. \end{aligned}$$

For $k = 1, 2, \ldots, n$, apply **Procedure A** with $u^{(0)} = \big(a^{(0)}, b^{(0)}, c^{(0)}, \gamma^{(0)}\big)^t$. That is, find $t_k = find\_pt(a^{(0)}, b^{(0)}, c^{(0)}, \gamma^{(0)}, \hat{x}_k, \hat{y}_k)$ $(k = 1, 2, \ldots, n)$.

Finally, compute

$$Q(u^{(0)}) = \sum_{k=1}^{n} \Big[\big(x_k - \{t_k + b^{(0)}\}\big)^2 + \big(y_k - \{a^{(0)} t_k{}^2 + c^{(0)}\}\big)^2 \Big].$$

Set $j := 0$.

STEP 2 : Apply the steepest descent method to the problem of minimizing the objective function $Q\big(u\big)$ with $u = u^{(j)}$.

First, due to (28)-(31) and (32)-(38) the gradient vector $\nabla Q\big(u^{(j)}\big) = \big(\hat{a}^{(j)}, \hat{b}^{(j)}, \hat{c}^{(j)}, \hat{\gamma}^{(j)}\big)$ can be computed at $u = u^{(j)}$ as follows:

We can find a new approximation $u^{(j+1)} = u^{(j)} - \lambda \nabla Q\big(u^{(j)}\big)$ such that the value of λ is determined so as to minimize the following single variable function:

$$\begin{aligned} h(\lambda) = \sum_{k=1}^{n} \Bigg[& \Big\{ x_k(\gamma^{(j)} - \lambda\hat{\gamma}^{(j)}) - t_k^{(j)}(\lambda) - (b^{(j)} - \lambda\hat{b}^{(j)}) \Big\}^2 \\ & + \Big\{ y_k(\gamma^{(j)} - \lambda\hat{\gamma}^{(j)}) - (a^{(j)} - \lambda\hat{a}^{(j)})\big(t_k^{(j)}(\lambda)\big)^2 - (c^{(j)} - \lambda\hat{c}^{(j)}) \Big\}^2 \Bigg], \end{aligned}$$

where

$$x_k(\gamma^{(j)} - \lambda\hat{\gamma}^{(j)}) = \hat{x}_k \cos(\gamma^{(j)} - \lambda\hat{\gamma}^{(j)}) + \hat{y}_k \sin(\gamma^{(j)} - \lambda\hat{\gamma}^{(j)}),$$
$$y_k(\gamma^{(j)} - \lambda\hat{\gamma}^{(j)}) = -\hat{x}_k \sin(\gamma^{(j)} - \lambda\hat{\gamma}^{(j)}) + \hat{y}_k \cos(\gamma^{(j)} - \lambda\hat{\gamma}^{(j)}),$$

and $t_k^{(j)}(\lambda) = t_k^{(j)}\left(u^{(j)} - \lambda\nabla Q(u^{(j)})\right)$ are computed by $t_k^{(j)}(\lambda) = find\_pt\big(a^{(j)} - \lambda\hat{a}^{(j)}, b^{(j)} - \lambda\hat{b}^{(j)}, c^{(j)} - \lambda\hat{c}^{(j)}, \gamma^{(j)} - \lambda\hat{\gamma}^{(j)}\big)$ $(k = 1, 2, \ldots, n)$.

Finally, compute

$$Q(u^{(j+1)}) = \sum_{k=1}^{n}\left[\left(x_k - \{t_k + b^{(j+1)}\}\right)^2 + \left(y_k - \{a^{(j+1)}{t_k}^2 + c^{(j+1)}\}\right)^2\right].$$

In other words, compute the corresponding rotated data points $\{(x_k, y_k)\}_{k=1}^{n}$ with respect to $\{(\hat{x}_k, \hat{y}_k)\}_{k=1}^{n}$:

$$x_k = \hat{x}_k \cos\gamma^{(j+1)} + \hat{y}_k \sin\gamma^{(j+1)},$$
$$y_k = -\hat{x}_k \sin\gamma^{(j+1)} + \hat{y}_k \cos\gamma^{(j+1)}.$$

Solve the n cubic equations:

$${t_k}^3 + p_k t_k + q_k = 0 \ \ (k = 1, 2, \ldots, n) \text{ with}$$

$$p_k = \frac{1 - 2a^{(j+1)}(y_k - c^{(j+1)})}{2\{a^{(j+1)}\}^2},$$
$$q_k = \frac{b^{(j+1)} - x_k}{2\{a^{(j+1)}\}^2}.$$

For $k = 1, 2, \ldots, n$, apply **Procedure A** with $u^{(j+1)} = \left(a^{(j+1)}, b^{(j+1)}, c^{(j+1)}, \gamma^{(j+1)}\right)^t$. That is, find $t_k = find\_pt(a^{(j+1)}, b^{(j+1)}, c^{(j+1)}, \gamma^{(j+1)}, \hat{x}_k, \hat{y}_k)$ $(k = 1, 2, \ldots, n)$.

Lastly, we obtain

$$Q(u^{(j+1)}) = \sum_{k=1}^{n}\left[\left(x_k - \{t_k + b^{(j+1)}\}\right)^2 + \left(y_k - \{a^{(j+1)}{t_k}^2 + c^{(j+1)}\}\right)^2\right].$$

STEP3 : If $\left|Q\left(u^{(j+1)}\right) - Q\left(u^{(j)}\right)\right| > TOL$, then set $j := j + 1$ and go back to **STEP2**.

3. Geometric parabola fitting in $\mathbb{R}^3$

In this section we consider the problem of fitting a parabola to the given data points $\{(\tilde{x}_k, \tilde{y}_k, \tilde{z}_k)\}_{k=1}^{n}$ in $\mathbb{R}^3$.

First, from (8) we can define the following general standard parabola $\hat{C}$ lying on the xy-plane:

$$\begin{aligned} \hat{C} : \hat{w} &= \hat{w}(a,b,c,t,\gamma) = \big(\hat{x}(a,b,c,t,\gamma), \hat{y}(a,b,c,t,\gamma), \hat{z}(a,b,c,t,\gamma)\big)^t \\ &= \begin{pmatrix} (t+b)\cos\gamma - (at^2+c)\sin\gamma \\ (t+b)\sin\gamma + (at^2+c)\cos\gamma \\ 0 \end{pmatrix}, \end{aligned} \tag{40}$$

where $-\infty < t < \infty$.

The model parabola $\tilde{C}$ is expressed by the following vector valued function $\tilde{w} = \tilde{w}(a,b,c,t,\gamma,\alpha,\beta,\delta)$:

$$\begin{aligned} \tilde{C} :\ \tilde{w} &= \tilde{w}(a,b,c,t,\gamma,\alpha,\beta,\delta) \\ &= \big(\tilde{x}(a,b,c,t,\gamma), \tilde{y}(a,b,c,t,\gamma), \tilde{z}(a,b,c,t,\gamma)\big)^t \\ &= R\hat{w}(a,b,c,t,\gamma) + \delta\mathbf{n} \\ &= \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ -\sin\alpha\sin\beta & \cos\alpha & \sin\alpha\cos\beta \\ -\cos\alpha\sin\beta & -\sin\alpha & \cos\alpha\cos\beta \end{pmatrix} \\ &\quad \begin{pmatrix} (t+b)\cos\gamma - (at^2+c)\sin\gamma \\ (t+b)\sin\gamma + (at^2+c)\cos\gamma \\ 0 \end{pmatrix} \\ &\quad + \delta\big(\ \sin\beta, \sin\alpha\cos\beta, \cos\alpha\cos\beta\ \big)^t = (\eta_1, \eta_2, \eta_3)^t, \end{aligned} \tag{41}$$

where

$$\begin{aligned} \eta_1 &= (t+b)\cos\gamma\cos\beta - (at^2+c)\sin\gamma\sin\beta + \delta\sin\beta, \\ \eta_2 &= (t+b)(\sin\gamma\cos\alpha - \cos\gamma\sin\alpha\sin\beta) \\ &\quad + (at^2+c)(\cos\gamma\cos\alpha + \sin\gamma\sin\alpha\sin\beta) + \delta\sin\alpha\cos\beta, \\ \eta_3 &= -(t+b)(\sin\gamma\sin\alpha + \cos\gamma\cos\alpha\sin\beta) \\ &\quad - (at^2+c)(\cos\gamma\sin\alpha - \sin\gamma\cos\alpha\sin\beta) + \delta\cos\alpha\cos\beta. \end{aligned}$$

The objective function $\tilde{Q}(u)$ can be given by

$$\begin{aligned} \tilde{Q}(u) &= \sum_{k=1}^{n} \big(\tilde{d}_k\big)^2 \\ &= \sum_{k=1}^{n} \Big(\min_t \Big[\big(\tilde{x}_k - \tilde{x}(\phi)\big)^2 + \big(\tilde{y}_k - \tilde{y}(\phi)\big)^2 + \big(\tilde{z}_k - \tilde{z}(\phi)\big)^2\Big]\Big) \\ &= \sum_{k=1}^{n} \Big(\big(\tilde{x}_k - \tilde{x}(\phi_k)\big)^2 + \big(\tilde{y}_k - \tilde{y}(\phi_k)\big)^2 + \big(\tilde{z}_k - \tilde{z}(\phi_k)\big)^2\Big), \end{aligned} \tag{42}$$

where $\phi = (a,b,c,t,\gamma,\alpha,\beta,\delta)$ is a parameter vector and $\big(\tilde{x}(\phi_k), \tilde{y}(\phi_k), \tilde{z}(\phi_k)\big) = \big(\tilde{x}(a,b,c,t_k,\gamma,\alpha,\beta,\delta), \tilde{y}(a,b,c,t_k,\gamma,\alpha,\beta,\delta), \tilde{z}(a,b,c,t_k,\gamma,\alpha,\beta,\delta)\big)$ is the corresponding nearest point with respect to the point $(\tilde{x}_k, \tilde{y}_k, \tilde{z}_k)$.

Further, it follows from (8) that the transformed data points $\{(\hat{x}_k, \hat{y}_k, \hat{z}_k)\}_{k=1}^n$ with respect to $\{(\tilde{x}_k, \tilde{y}_k, \tilde{z}_k)\}_{k=1}^n$ are given by

$$\begin{aligned}(\hat{x}_k, \hat{y}_k, \hat{z}_k)^t &= R^{-1}(\tilde{x}_k, \tilde{y}_k, \tilde{z}_k)^t - \delta(0,0,1)^t \\ &= \begin{pmatrix} \tilde{x}_k \cos\beta - \tilde{y}_k \sin\alpha \sin\beta - \tilde{z}_k \cos\alpha \sin\beta \\ \tilde{y}_k \cos\alpha - \tilde{z}_k \sin\alpha \\ \tilde{x}_k \sin\beta + \tilde{y}_k \sin\alpha \cos\beta + \tilde{z}_k \cos\alpha \cos\beta - \delta \end{pmatrix}.\end{aligned}$$

Thus, the objective function $\tilde{Q}(u)$ can be replaced by the following new function $\hat{Q}(u)$:

$$\begin{aligned}\hat{Q}(u) &= \sum_{k=1}^{n} \left(\hat{d}_k\right)^2 \\ &= \sum_{k=1}^{n} \Big[\left\{\hat{x}_k - (t_k + b)\cos(\gamma) + (a{t_k}^2 + c)\sin(\gamma)\right\}^2 \\ &\quad + \left\{\hat{y}_k - (t_k + b)\sin(\gamma) - (a{t_k}^2 + c)\cos(\gamma)\right\}^2 + \left\{\hat{z}_k\right\}^2 \Big],\end{aligned} \tag{43}$$

where $\hat{d}_k$ denotes the distance of the point $(\hat{x}_k, \hat{y}_k, \hat{z}_k)$ to the general standard parabola $\hat{C}$ and $\big(x(a,b,c,t_k,\gamma,\alpha,\beta,\delta), y(a,b,c,t_k,\gamma,\alpha,\beta,\delta), z(a,b,c,t_k,\gamma,\alpha,\beta,\delta)\big)$ are the corresponding nearest points with respect to the points (x_k, y_k, z_k) and $u = (a,b,c,\gamma,\alpha,\beta,\delta)^t \in \mathbb{R}^7$.

Here, if we let $\hat{Q}(u) = \bar{Q}(u) + \sum_{k=1}^{n}\{\hat{z}_k\}^2$, i.e., if we define a new objective function

$$\begin{aligned}\bar{Q}(u) = \sum_{k=1}^{n} \Big[&\left\{\hat{x}_k - (t_k + b)\cos(\gamma) + (a{t_k}^2 + c)\sin(\gamma)\right\}^2 \\ &+ \left\{\hat{y}_k - (t_k + b)\sin(\gamma) - (a{t_k}^2 + c)\cos(\gamma)\right\}^2 \Big],\end{aligned} \tag{44}$$

then this can be exactly reduced to the same problem of fitting a general standard parabola $\hat{C}$ to a set of data points with coordinates $\{(\hat{x}_k, \hat{y}_k)\}_{k=1}^n$ equipped with the objective function $\hat{Q}(u) = \bar{Q}(u)$ in $\mathbb{R}^2$.

Further, we have

$$\frac{\partial \hat{Q}}{\partial a} = \frac{\partial \bar{Q}}{\partial a}, \ \frac{\partial \hat{Q}}{\partial b} = \frac{\partial \bar{Q}}{\partial b}, \ \frac{\partial \hat{Q}}{\partial c} = \frac{\partial \bar{Q}}{\partial c}, \ \frac{\partial \hat{Q}}{\partial \gamma} = \frac{\partial \bar{Q}}{\partial \gamma}.$$

The necessary minimum conditions for $\hat{Q}(u)$ yield the following system of 7 equations:

$$\frac{\partial \hat{Q}}{\partial a} = \frac{\partial \hat{Q}}{\partial b} = \frac{\partial \hat{Q}}{\partial c} = \frac{\partial \hat{Q}}{\partial \gamma} = \frac{\partial \hat{Q}}{\partial \alpha} = \frac{\partial \hat{Q}}{\partial \beta} = \frac{\partial \hat{Q}}{\partial \delta} = 0. \tag{45}$$

We now propose an iterative algorithm for fitting a parabola to the set of data points $\{(\tilde{x}_k, \tilde{y}_k, \tilde{z}_k)\}_{k=1}^n$ in $\mathbb{R}^3$.

Algorithm :

STEP1 : Find three initial approximations $\alpha^{(0)}, \beta^{(0)}, \delta^{(0)}$. Here, by using a plane-fitting algorithm in $\mathbb{R}^3$ we can take good starting values $\left(\alpha^{(0)}, \beta^{(0)}, \delta^{(0)}\right)^t$ (see Algorithm in [8]).

Compute the transformed data points $\left\{\left(\hat{x}_k^{(0)}, \hat{y}_k^{(0)}, \hat{z}_k^{(0)}\right)\right\}_{k=1}^n$, i.e., find

$$\begin{aligned}
\hat{x}_k^{(0)} &= \tilde{x}_k \cos\left(\beta^{(0)}\right) - \tilde{y}_k \sin\left(\alpha^{(0)}\right) \sin\left(\beta^{(0)}\right) - \tilde{z}_k \cos\left(\alpha^{(0)}\right) \sin\left(\beta^{(0)}\right), \\
\hat{y}_k^{(0)} &= \tilde{y}_k \cos\left(\alpha^{(0)}\right) - \tilde{z}_k \sin\left(\alpha^{(0)}\right), \\
\hat{z}_k^{(0)} &= \tilde{x}_k \sin\left(\beta^{(0)}\right) + \tilde{y}_k \sin\left(\alpha^{(0)}\right) \cos\left(\beta^{(0)}\right) + \tilde{z}_k \cos\left(\alpha^{(0)}\right) \cos\left(\beta^{(0)}\right) - \delta^{(0)}.
\end{aligned}$$

From (26), given any fixed values of $\alpha^{(0)}, \beta^{(0)}$, and $\delta^{(0)}$ the objective function may be connected with a new function $Q(a, b, c, \gamma)$ for the data points $\left\{\left(\hat{x}_k^{(0)}, \hat{y}_k^{(0)}\right)\right\}_{k=1}^n$,

$$Q(a, b, c, \gamma) = \sum_{k=1}^n \left[\left(x_k - \{t_k + b\}\right)^2 + \left(y_k - \{a{t_k}^2 + c\}\right)^2\right], \tag{46}$$

where the corresponding rotated data points $(x_k, y_k) = \left(x_k(\gamma), y_k(\gamma)\right)$ are given by the following vector valued functions ρ_k of a variable γ:

$$\begin{aligned}
\rho_k(\gamma) &= (x_k(\gamma), y_k(\gamma))^t \\
&= S^{-1}\left[(\hat{x}_k^{(0)}, \hat{y}_k^{(0)})^t\right] \\
&= \begin{pmatrix} \hat{x}_k^{(0)} \cos\gamma + \hat{y}_k^{(0)} \sin\gamma \\ -\hat{x}_k^{(0)} \sin\gamma + \hat{y}_k^{(0)} \cos\gamma \end{pmatrix},
\end{aligned} \tag{47}$$

The minimization problem $Q(a, b, c, \gamma) = min$ is exactly the same thing as that in $\mathbb{R}^2$. So, by using **Algorithm B**, find $a^{(0)}, b^{(0)}, c^{(0)}, \gamma^{(0)}$.

Finally, set a starting data approximation $u^{(0)} = \left(a^{(0)}, b^{(0)}, c^{(0)}, \gamma^{(0)}, \alpha^{(0)}, \beta^{(0)}, \delta^{(0)}\right)^t$, and compute the following value of the objective function $\hat{Q}(u)$ at $u^{(0)}$:

$$\begin{aligned}
&\hat{Q}(u^{(0)}) \\
&= Q(u^{(0)}) + \sum_{k=1}^n \left\{\hat{z}_k^{(0)}\right\}^2 \\
&= \sum_{k=1}^n \left[\left(x_k - \{t_k + b\}\right)^2 + \left(y_k - \{a{t_k}^2 + c\}\right)^2\right] + \sum_{k=1}^n \left\{\hat{z}_k^{(0)}\right\}^2 \\
&= \sum_{k=1}^n \Big[\left\{\hat{x}_k^{(0)} - (t_k + b^{(0)}) \cos(\gamma^{(0)}) + (a^{(0)}{t_k}^2 + c^{(0)}) \sin(\gamma^{(0)})\right\}^2 \\
&\qquad + \left\{\hat{y}_k^{(0)} - (t_k + b^{(0)}) \sin(\gamma^{(0)}) - (a^{(0)}{t_k}^2 + c^{(0)}) \cos(\gamma^{(0)})\right\}^2 + \left\{\hat{z}_k^{(0)}\right\}^2\Big].
\end{aligned} \tag{48}$$

Set $j := 0$.

STEP2 : Apply the steepest descent method to the problem of minimizing the objective function $\hat{Q}(u)$, and get a new approximation

$$u^{(j+1)} = \big(a^{(j+1)}, b^{(j+1)}, c^{(j+1)}, \gamma^{(j+1)}, \alpha^{(j+1)}, \beta^{(j+1)}, \delta^{(j+1)}\big)^t.$$

From (43) we have the following objective function:

$$\begin{aligned}
(49)\quad & \hat{Q}(u^{(j)}) \\
&= Q(u^{(j)}) + \sum_{k=1}^{n} \{\hat{z}_k^{(j)}\}^2 \\
&= \sum_{k=1}^{n} \Big[\big(x_k - \{t_k + b\}\big)^2 + \big(y_k - \{a{t_k}^2 + c\}\big)^2\Big] + \sum_{k=1}^{n} \{\hat{z}_k^{(0)}\}^2 \\
&= \sum_{k=1}^{n} \Big[\big\{\hat{x}_k^{(j)} - (t_k + b^{(j)})\cos(\gamma^{(j)}) + (a^{(j)}{t_k}^2 + c^{(j)})\sin(\gamma^{(j)})\big\}^2 \\
&\quad + \big\{\hat{y}_k^{(j)} - (t_k + b^{(j)})\sin(\gamma^{(j)}) - (a^{(j)}{t_k}^2 + c^{(j)})\cos(\gamma^{(j)})\big\}^2 + \big\{\hat{z}_k^{(j)}\big\}^2\Big],
\end{aligned}$$

where $u^{(j)} = \big(a^{(j)}, b^{(j)}, c^{(j)}, \gamma^{(j)}, \alpha^{(j)}, \beta^{(j)}, \delta^{(j)}\big)^t$.

Let $\nabla\hat{Q} = \big(\hat{a}^{(j)}, \hat{b}^{(j)}, \hat{c}^{(j)}, \hat{\gamma}^{(j)}, \hat{\alpha}^{(j)}, \hat{\beta}^{(j)}, \hat{\delta}^{(j)}\big)^t$ be the gradient of $\hat{Q}(u)$ at $u = u^{(j)}$. Then, it follows from (49) that

$$\begin{aligned}
\hat{a}^{(j)} &= \sum_{k=1}^{n} 2{t_k}^2 \Big[\hat{x}_k^{(j)}\sin(\gamma^{(j)}) - \hat{y}_k^{(j)}\cos(\gamma^{(j)}) + (a^{(j)}{t_k}^2 + c^{(j)})\Big], \\
\hat{b}^{(j)} &= -\sum_{k=1}^{n} 2\Big[\hat{x}_k^{(j)}\cos(\gamma^{(j)}) + \hat{y}_k^{(j)}\sin(\gamma^{(j)}) - (t_k + b^{(j)})\Big], \\
\hat{c}^{(j)} &= \sum_{k=1}^{n} 2\Big[\hat{x}_k^{(j)}\sin(\gamma^{(j)}) - \hat{y}_k^{(j)}\cos(\gamma^{(j)}) + (a^{(j)}{t_k}^2 + c^{(j)})\Big], \\
\hat{r}^{(j)} &= \sum_{k=1}^{n} 2\Big[\hat{x}_k^{(j)}\big\{(t_k + b^{(j)})\sin(\gamma^{(j)}) + (a^{(j)}{t_k}^2 + c^{(j)})\cos(\gamma^{(j)})\big\} \\
&\qquad - \hat{y}_k^{(j)}\big\{(t_k + b^{(j)})\cos(\gamma^{(j)}) - (a^{(j)}{t_k}^2 + c^{(j)})\sin(\gamma^{(j)})\big\}\Big].
\end{aligned}$$

Further, $\hat{\alpha}, \hat{\beta}, \hat{\delta}$ can be given by the following:

$$\begin{aligned}
\hat{\alpha}^{(j)} &= \sum_{k=1}^{n} \left[\frac{\partial\hat{Q}}{\partial\hat{x}_k}\frac{\partial\hat{x}_k}{\partial\alpha} + \frac{\partial\hat{Q}}{\partial\hat{y}_k}\frac{\partial\hat{y}_k}{\partial\alpha} + \frac{\partial\hat{Q}}{\partial\hat{z}_k}\frac{\partial\hat{z}_k}{\partial\alpha}\right], \\
\hat{\beta}^{(j)} &= \sum_{k=1}^{n} \left[\frac{\partial\hat{Q}}{\partial\hat{x}_k}\frac{\partial\hat{x}_k}{\partial\beta} + \frac{\partial\hat{Q}}{\partial\hat{y}_k}\frac{\partial\hat{y}_k}{\partial\beta} + \frac{\partial\hat{Q}}{\partial\hat{z}_k}\frac{\partial\hat{z}_k}{\partial\beta}\right],
\end{aligned}$$

$$\hat{\delta}^{(j)} = \sum_{k=1}^{n} \left[\frac{\partial \hat{Q}}{\partial \hat{x}_k} \frac{\partial \hat{x}_k}{\partial \delta} + \frac{\partial \hat{Q}}{\partial \hat{y}_k} \frac{\partial \hat{y}_k}{\partial \delta} + \frac{\partial \hat{Q}}{\partial \hat{z}_k} \frac{\partial \hat{z}_k}{\partial \delta} \right] = \sum_{k=1}^{n} \frac{\partial \hat{Q}}{\partial \hat{z}_k} \frac{\partial \hat{z}_k}{\partial \delta}.$$

It follows from (43) that

$$\begin{aligned}
\frac{\partial \hat{Q}}{\partial \hat{x}_k} &= 2 \left[\hat{x}_k^{(j)} - (t_k + b^{(j)}) \cos(\gamma^{(j)}) + (a^{(j)} {t_k}^2 + c^{(j)}) \sin(\gamma^{(j)}) \right], \\
\frac{\partial \hat{Q}}{\partial \hat{y}_k} &= 2 \left[\hat{y}_k^{(j)} - (t_k + b^{(j)}) \sin(\gamma^{(j)}) - (a^{(j)} {t_k}^2 + c^{(j)}) \cos(\gamma^{(j)}) \right], \\
\frac{\partial \hat{Q}}{\partial \hat{z}_k} &= 2 \hat{z}_k^{(j)}.
\end{aligned}$$

Since the coordinates $\{(\hat{x}_k, \hat{y}_k, \hat{z}_k)\}_{k=1}^{n}$ are given by

$$\begin{aligned}
\hat{x}_k &= \tilde{x}_k \cos(\beta) - \tilde{y}_k \sin(\alpha) \sin(\beta) - \tilde{z}_k \cos(\alpha) \sin(\beta), \\
\hat{y}_k &= \tilde{y}_k \cos(\alpha) - \tilde{z}_k \sin(\alpha), \\
\hat{z}_k &= \tilde{x}_k \sin(\beta) + \tilde{y}_k \sin(\alpha) \cos(\beta) + \tilde{z}_k \cos(\alpha) \cos(\beta) - \delta,
\end{aligned}$$

we have the following:

$$\begin{aligned}
\frac{\partial \hat{x}_k}{\partial \alpha} &= -\tilde{y}_k \cos(\alpha) \sin(\beta) + \tilde{z}_k \sin(\alpha) \sin(\beta), \\
\frac{\partial \hat{x}_k}{\partial \beta} &= -\tilde{x}_k \sin(\beta) - \tilde{y}_k \sin(\alpha) \cos(\beta) - \tilde{z}_k \cos(\alpha) \cos(\beta), \\
\frac{\partial \hat{y}_k}{\partial \alpha} &= -\tilde{y}_k \sin(\alpha) - \tilde{z}_k \cos(\alpha), \\
\frac{\partial \hat{y}_k}{\partial \beta} &= 0, \\
\frac{\partial \hat{z}_k}{\partial \alpha} &= \tilde{y}_k \cos(\alpha) \cos(\beta) - \tilde{z}_k \sin(\alpha) \cos(\beta), \\
\frac{\partial \hat{z}_k}{\partial \beta} &= \tilde{x}_k \cos(\beta) - \tilde{y}_k \sin(\alpha) \sin(\beta) - \tilde{z}_k \cos(\alpha) \sin(\beta), \\
\frac{\partial \hat{z}_k}{\partial \delta} &= -1.
\end{aligned}$$

Finally, using the steepest descent technique we can find a new approximation

$$\begin{aligned}
u^{(j+1)} &= \left(a^{(j+1)}, b^{(j+1)}, c^{(j+1)}, \gamma^{(j+1)}, \alpha^{(j+1)}, \beta^{(j+1)}, \delta^{(j+1)}\right)^t \\
&= u^{(j)} - \lambda \nabla Q\big(\hat{u}^{(j)}\big),
\end{aligned}$$

where λ is an adequate number which can be computed in such a way that the corresponding single variable function $h(\lambda) = \hat{Q}(u^{(j)} - \lambda \nabla Q\big(\hat{u}^{(j)}\big))$ is less than $\hat{Q}(u^{(j)})$.

STEP3 : If $\left|Q\big(u^{(j+1)}\big) - Q\big(u^{(j)}\big)\right| > TOL$, then set $j := j + 1$ and go back to **STEP2**.

4. Numerical examples

In this section, to test our algorithm four examples are given. In each case, we find the best fitted parabola to the given data points in $\mathbb{R}^3$ (or in $\mathbb{R}^2$), and observe the convergence of the corresponding objective function $Q(u)$.

The following example shows that our algorithm is also useful and efficient for solving the problem of fitting a parabola to the given data points in $\mathbb{R}^2$.

Example 1. Let us consider the following data points $\{(\hat{x}_k, \hat{y}_k)\}_{k=1}^{6}$ which are lying on the xy-plane, taken from [1].

$$\{(-7,9),(-3,5),(0,4),(1,3),(1,5),(0,8)\}.$$

By using the Ahn algorithm a parabola which is fitted to the above data points was obtained in [1]. This good fitted parabola was represented by rotating the standard parabola $C : (y - y_c)^2 = 2p(x - x_c)$ with angle α at the center (x_c, y_c). The values $p = 0.526$, $x_c = 0.668$, $y_c = 3.228$ and $\alpha = 2.170$ were obtained.

If the standard parabola $C : y = a(x - b)^2 + c$ $(a \neq 0)$ is expressed in its parametric form

$$\begin{aligned} C : x(a,b,c,t) &= t + b, \\ y(a,b,c,t) &= at^2 + c, \end{aligned}$$

where $-\infty \le t \le \infty$, the general standard parabola $\hat{C}$ can be obtained by rotating with angle γ which is given by

$$\hat{C} : \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} = S \begin{pmatrix} x \\ y \end{pmatrix},$$

where the coordinate transformation matrix $S = \begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix}$.

That is, the parametric form of $\hat{C}$ with γ can be denoted by

$$\begin{aligned} \hat{x} &= (t + b)\cos\gamma - (at^2 + c)\sin\gamma, \\ \hat{y} &= (t + b)\sin\gamma + (at^2 + c)\cos\gamma. \end{aligned}$$

Furthermore, the fitted parabola that was obtained in [1] is exactly the same as the general standard parabola $\hat{C}$ with the values $a = 1/(2p) = 0.9506, \gamma = \alpha - \pi/2 = 0.5992$ and

$$\begin{pmatrix} b \\ c \end{pmatrix} = S^{-1} \begin{pmatrix} x_c \\ y_c \end{pmatrix} = \begin{pmatrix} x_c\cos\gamma + y_c\sin\gamma \\ -\sin\gamma x_c + \cos\gamma y_c \end{pmatrix} = \begin{pmatrix} 2.3722 \\ 2.2889 \end{pmatrix}.$$

The value of $Q(u)$ is $Q(u) = 0.8014$.

On the other hand, by using our algorithm with starting values $(a,b,c,\gamma)^t = (0.5000, 2.1577, 2.3117, 0.4292)^t$ and $Q(u) = 5.6284$ we obtain $(a,b,c,\gamma)^t =$

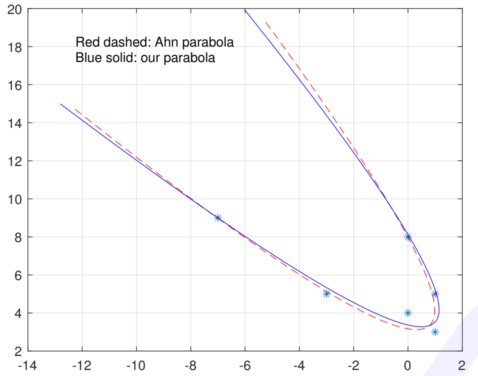

FIGURE 1. The fitted parabola is shown for Example 1

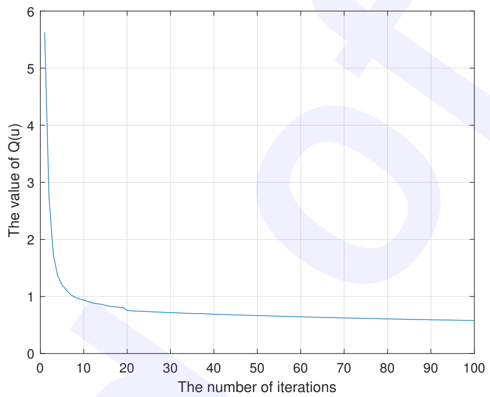

FIGURE 2. The values of $Q(u)$ are shown for Example 1

$(0.9883, 2.6948, 2.2240, 0.6305)^t$ and the value $Q(u) = 0.5785$ after 100 iterations. Thus, we have a good fitted parabola, which is given by

$$\hat{x} = (t+b)\cos\gamma - (at^2+c)\sin\gamma,$$
$$\hat{y} = (t+b)\sin\gamma + (at^2+c)\cos\gamma$$

with $(a, b, c, \gamma)^t = (0.9883, 2.6948, 2.2240, 0.6305)^t$.

We see our good fitted parabola and Ahn's one in Figure 1. The convergence of our algorithm to a local minimum is shown in Figure 2.

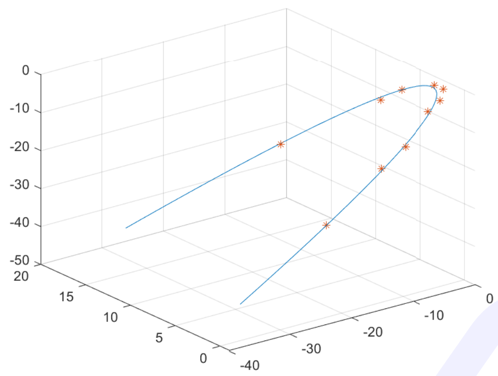

FIGURE 3. The fitted parabola is shown for Example 2

Example 2. When a standard parabola is given by the equation $C : y = \frac{1}{2}(x-1)^2 + 2, z = 0$, lying on the xy-plane, using the rotation matrix S with $\gamma = \pi/3$:

$$S = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we have the following general standard parabola $\hat{C}$ which is represented by

$$\begin{pmatrix} \hat{x}(t) \\ \hat{y}(t) \\ \hat{z}(t) \end{pmatrix} = S \begin{pmatrix} t+1 \\ \frac{1}{2}t^2 + 2 \\ 0 \end{pmatrix} \qquad -\infty < t < \infty,$$

By rotating this general standard parabola $\hat{C}$ with $\alpha = \pi/6$ and $\beta = -\pi/4$, and translating it with $\delta = 2$ in the direction of $\mathbf{n} = (\sin\beta, \sin\alpha\cos\beta, \cos\alpha\cos\beta)^t$ The corresponding model parabola $\tilde{C}$ can be obtained in $\mathbb{R}^3$. Finally, by choosing 10 data points at $t = -8, -6, -5, -3, -2, 0, 1, 3, 4, 7$, lying exactly on the parabola $\tilde{C}$, and truncating all their fractional parts, the following coordinates $\{(\tilde{x}_k, \tilde{y}_k, \tilde{z}_k)\}_{k=1}^{10}$ of 10 data points can be given:

$$\{(-24,-1,-24), (-15,-1,-13), (-11,-1,-9), (-6,0,-3), (-4,0,-1), \\ (-2,1,0), (-2,2,0), (-3,5,-4), (-5,6,-7), (-14,11,-20))\}$$

The objective function $Q(u)$ to be minimized is given by

$$Q(u) = \hat{Q}(u) = \sum_{k=1}^{n} \left(\hat{d}_k\right)^2$$

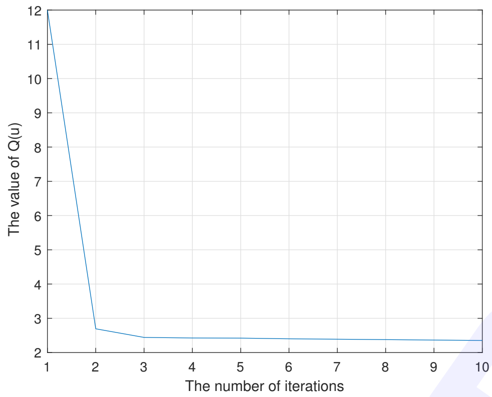

FIGURE 4. The values of $Q(u)$ are shown for Example 2

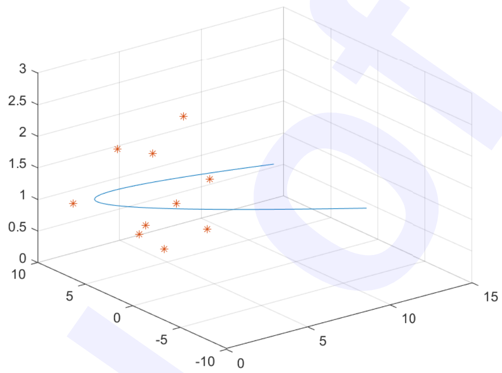

FIGURE 5. The fitted parabola is shown for Example 3

$$= \sum_{k=1}^{n} \Big[\big\{ \hat{x}_k - (t_k + b)\cos\gamma + (a t_k^2 + c)\sin\gamma \big\}^2 + \big\{ \hat{y}_k - (t_k + b)\sin\gamma - (a t_k^2 + c)\cos\gamma \big\}^2 + \big\{ \hat{z}_k \big\}^2 \Big],$$

where

$$\begin{aligned} x_k &= \tilde{x}_k \cos\beta - \tilde{y}_k \sin\alpha \sin\beta - \tilde{z}_k \cos\alpha \sin\beta, \\ y_k &= \tilde{y}_k \cos\alpha - \tilde{z}_k \sin\alpha, \\ z_k &= \tilde{x}_k \sin\beta + \tilde{y}_k \sin\alpha \cos\beta + \tilde{z}_k \cos\alpha \cos\beta - \delta. \end{aligned}$$

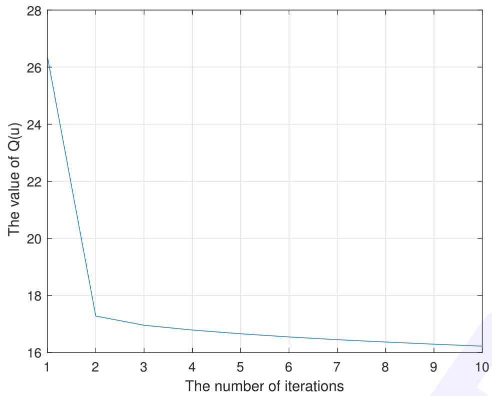

FIGURE 6. The values of $Q(u)$ are shown for Example 3

Using our algorithm to fit a parabola to the above data points we obtain $(a,b,c,\gamma,\alpha,\beta,\delta)^t = (0.5154, 0.9960, 1.9894, 1.0021, 0.5822, -0.7679, 1.9975)^t$ and the corresponding value $Q(u) = 2.3531$ which is very small. These good results are visualized in Figure 3. Further, the convergence of $Q(u)$ to a local minimum can be shown in Figure 4.

Example 3. The following 10 data points are given:

$$\{(3,-3,2),(2,-2,3),(4,-1,1),(1,1,1),(3,3,2),(1,8,1),(6,7,0),(5,4,1),(2,5,2),(2,2,1)\}.$$

By using our algorithm the values $(a,b,c,\gamma,\alpha,\beta,\delta)^t = (-0.5422, 5.3762, 5.2634, 0.5392, 0.1198, 0.1205, 2.1167)^t$ and the value $Q(u) = 16.2246$ can be obtained.

We can see the fitted parabola in Figure 5. The convergence of $Q(u)$ to a local minimum is shown in Figure 6.

Example 4. In order to reduce the number of data points we choose specially 5 data points from the data points given in example 3. Let us consider the following data points:

$$\{(3,-3,2),(4,-1,1),(3,3,2),(6,7,0),(2,5,2)\}.$$

Using our algorithm we have $(a,b,c,\gamma,\alpha,\beta,\delta)^t = (-0.3808, 5.5113, 5.1811, 0.5484, 0.0336, 0.5070, 3.0099)^t$ and the value $Q(u) = 2.2096$. The best fitted parabola can be shown in Figure 7. The convergence of our algorithm is shown in Figure 8.

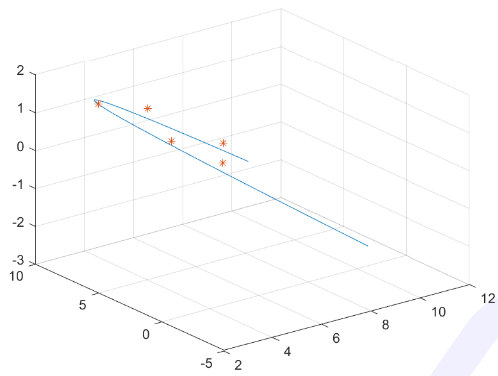

FIGURE 7. The fitted parabola is shown for Example 4

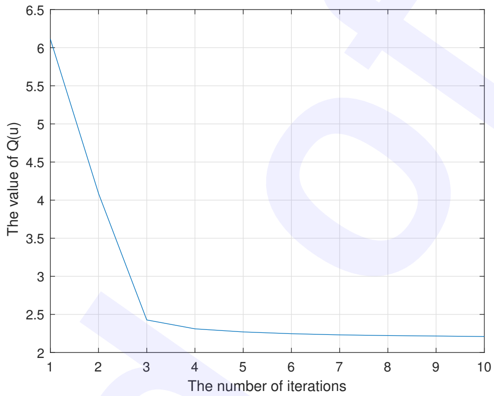

FIGURE 8. The values of $Q(u)$ are shown for Example 4

5. Conclusion

In this paper, we are interested in the problem of fitting a parabola to a set of the given data points in $\mathbb{R}^3$. So, a new iterative algorithm has been suggested for finding a fitted parabola to the given data points. Further, our algorithm is one of the geometric fitting algorithms, which contains a steepest descent technique. Thus, it confirms that the convergence of $Q(u)$ to a local minimum can be ensured. Numerical examples show that the proposed algorithm is a good approach for the fitting of parabolas in $\mathbb{R}^3$.

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