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ESTIMATES FOR THE NORM OF A MULTILINEAR FORM ON $\mathbb{R}^n$ WITH THE l_p-NORM

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Abstract. In this paper, we present some estimates for the norm of a multilinear form $T \in \mathcal{L}({}^m l_p^n)$ for $1 \leq p \leq \infty$ and $n, m \geq 2$.

1. Results

Throughout the paper, we let $n, m \in \mathbb{N}$, $n, m \geq 2$. We write B_E and S_E for the closed unit ball and sphere of a real Banach space E. An element $x \in B_E$ is called an *extreme point* of B_E if $y, z \in B_E$ with $x = \frac{1}{2}(y+z)$ implies $x = y = z$. We denote by ext B_E the set of all extreme points of B_E. A mapping $P : E \to \mathbb{R}$ is a continuous n-homogeneous polynomial if there exists a continuous n-linear form T on the product $E \times \cdots \times E$ such that $P(x) = T(x, \ldots, x)$ for every $x \in E$. We denote by $\mathcal{P}({}^n E)$ the Banach space of all continuous n-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$. We denote by $\mathcal{L}({}^n E)$ the Banach space of all continuous n-linear forms on E endowed with the norm $\|T\| = \sup_{\|x_k\|=1} |T(x_1, \ldots, x_n)|$. $\mathcal{L}_s({}^n E)$ denotes the closed subspace of all continuous symmetric n-linear forms on E. The problem of computing the exact value of $\|T\|$ for every $T \in \mathcal{L}({}^n E)$ is important for the classification problem of geometrical structures of the unit ball of a Banach space, i.e., the classification of extreme points, exposed points and smooth points of its unit ball. For more details about the theory of polynomials and multilinear mappings on Banach spaces, we refer to [1].

Let $1 \leq p \leq \infty$ and $l_p^n = \mathbb{R}^n$ with the l_p-norm. In this note we present some estimates for the norm of a multilinear form $T \in \mathcal{L}({}^m l_p^n)$ for $1 \leq p \leq \infty$ and $n, m \geq 2$.

Theorem 1.1. *Let $1 \leq p < \infty$ and $n, m \geq 2$. Let $T \in \mathcal{L}({}^m l_p^n)$ with*

$$T\Big(\Big(x_1^{(1)}, \ldots, x_n^{(1)}\Big), \ldots, \Big(x_1^{(m)}, \ldots, x_n^{(m)}\Big)\Big) = \sum_{1 \leq i_k \leq n,\ 1 \leq k \leq m} a_{i_1 \cdots i_m}\, x_{i_1}^{(1)} \cdots x_{i_m}^{(m)}$$

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for some $a_{i_1\cdots i_m} \in \mathbb{R}$*. Then*

$$\max_{1\le i_k\le n,\ 1\le k\le m} \left|a_{i_1\cdots i_m}\right| \le \|T\| \le n^{m(1-1/p)} \max_{1\le i_k\le n,\ 1\le k\le m} \left|a_{i_1\cdots i_m}\right|.$$

Proof. Let

$$M := \max_{1\le i_k\le n,\ 1\le k\le m} \left|a_{i_1\cdots i_m}\right|.$$

It is obvious that

$$M = \max_{1\le i_k\le n,\ 1\le k\le m} \left|T\Big(e_{i_1},\ldots,e_{i_m}\Big)\right| \le \|T\|.$$

Let $\Big(x_1^{(1)},\ldots,x_n^{(1)}\Big),\ldots,\Big(x_1^{(m)},\ldots,x_n^{(m)}\Big) \in S_{l_p^n}$.

Case 1. $p=1$

$$\begin{aligned}
&\left|T\Big(\Big(x_1^{(1)},\ldots,x_n^{(1)}\Big),\ldots,\Big(x_1^{(m)},\ldots,x_n^{(m)}\Big)\Big)\right| \\
&\le \sum_{1\le i_k\le n,\ 1\le k\le m} \left|a_{i_1\cdots i_m}\right| \left|x_{i_1}^{(1)}\right|\cdots\left|x_{i_m}^{(m)}\right| \\
&\le M \sum_{1\le i_k\le n,\ 1\le k\le m} \left|x_{i_1}^{(1)}\right|\cdots\left|x_{i_m}^{(m)}\right| \\
&= M\Big(\left|x_1^{(1)}\right|+\cdots+\left|x_n^{(1)}\right|\Big)\cdots\Big(\left|x_1^{(m)}\right|+\cdots+\left|x_n^{(m)}\right|\Big) = M,
\end{aligned}$$

which implies that $\|T\| = M$.

Case 2. $1 < p < \infty$

Let $q \in \mathbb{R}$ be such that $1/p + 1/q = 1$. It follows that

$$\begin{aligned}
&\left|T\Big(\Big(x_1^{(1)},\ldots,x_n^{(1)}\Big),\ldots,\Big(x_1^{(m)},\ldots,x_n^{(m)}\Big)\Big)\right| \\
&\le \sum_{1\le i_k\le n,\ 1\le k\le m} \left|a_{i_1\cdots i_m}\right| \left|x_{i_1}^{(1)}\right|\cdots\left|x_{i_m}^{(m)}\right| \\
&\le M \sum_{1\le i_1\le n} \left|x_{i_1}^{(1)}\right| \Big(\sum_{1\le i_k\le n,\ 2\le k\le m} \left|x_{i_2}^{(2)}\right|\cdots\left|x_{i_m}^{(m)}\right|\Big) \\
&\le M\Big(\sum_{1\le i_1\le n} \left|x_{i_1}^{(1)}\right|^p\Big)^{1/p} \Big[\sum_{1\le i_1\le n} \Big(\sum_{1\le i_k\le n,\ 2\le k\le m} \left|x_{i_2}^{(2)}\right|\cdots\left|x_{i_m}^{(m)}\right|\Big)^q\Big]^{1/q} \\
&\quad \text{(by the Hölder inequality)} \\
&= Mn^{1/q} \sum_{1\le i_k\le n,\ 2\le k\le m} \left|x_{i_2}^{(2)}\right|\cdots\left|x_{i_m}^{(m)}\right| \\
&\le Mn^{1/q}\Big(\sum_{1\le i_2\le n} \left|x_{i_2}^{(2)}\right|^p\Big)^{1/p} \Big[\sum_{1\le i_2\le n} \Big(\sum_{1\le i_k\le n,\ 3\le k\le m} \left|x_{i_3}^{(3)}\right|\cdots\left|x_{i_m}^{(m)}\right|\Big)^q\Big]^{1/q} \\
&\quad \text{(by the Hölder inequality)}
\end{aligned}$$

$$= Mn^{2/q} \sum_{1\leq i_k\leq n,\ 3\leq k\leq m} \left|x_{i_3}^{(3)}\right| \cdots \left|x_{i_m}^{(m)}\right|$$
$$\leq$$
$$\vdots$$
$$\leq Mn^{m/q} = Mn^{m(1-1/p)}. \qquad \square$$

Remark 1.2. (a) Notice that the upper estimate $n^{m(1-1/p)}M \to M$ as $p \to 1+$. Hence, $\|T\| \to M$ as $p \to 1+$.

(b) Notice that the upper estimate $n^{m(1-1/p)}M$ of Theorem 1.1 is best possible whenever $p \in \mathbb{N}$. Indeed, let $N \in \mathbb{N}$ and $T \in \mathcal{L}(^m l_N^n)$ with

$$T\Big(\Big(x_1^{(1)}, \ldots, x_n^{(1)}\Big), \ldots, \Big(x_1^{(m)}, \ldots, x_n^{(m)}\Big)\Big) = \sum_{1\leq i_k\leq n,\ 1\leq k\leq m} x_{i_1}^{(1)} \cdots x_{i_m}^{(m)}.$$

We claim that $\|T\| = n^{m(1-\frac{1}{N})}$. It follows that by Theorem 1.1,

$$\begin{aligned} n^{m(1-\frac{1}{N})} \geq \|T\| &\geq \Big|T\Big(\Big(n^{-\frac{1}{N}}, \ldots, n^{-\frac{1}{N}}\Big), \ldots, \Big(n^{-\frac{1}{N}}, \ldots, n^{-\frac{1}{N}}\Big)\Big)\Big| \\ &= \sum_{1\leq i_k\leq n,\ 1\leq k\leq m} n^{-\frac{m}{N}} \\ &= n^m \cdot n^{-\frac{m}{N}} = n^{m(1-\frac{1}{N})}. \end{aligned}$$

For $n, m \geq 2$, let $\mathcal{W}_{n,m} := \Big\{\Big((1, w_2^{(1)}, \ldots, w_m^{(1)}), \ldots, (1, w_2^{(n)}, \ldots, w_m^{(n)})\Big) :$
$$w_j^{(k)} = \pm 1 \text{ for } 1 \leq k \leq n, 1 \leq j \leq m\Big\}.$$

Note that $\mathcal{W}_{n,m}$ has $2^{(m-1)n}$-elements in $S_{l_\infty^m} \times \cdots \times S_{l_\infty^m}$. Let S be a nonempty subset of a real Banach space E. We denote the convex hull of S by $\mathrm{conv}(S)$. Recall that the Krein-Milman theorem [2] says that every nonempty compact convex subset of a Hausdorff locally convex space is the closed convex hull of its set of extreme points. Notice that the unit ball of a finite dimensional space is the convex hull of its extreme points. This finite dimensional version of the Krein-Milman theorem is sometimes referred to Steinitz theorem [3]. Hence, the unit ball of l_∞^m is the convex hull of the set of its extreme points.

The following theorem is a direct consequence of the Krein-Milman theorem.

Theorem 1.3. *Let $n, m \geq 2$. Let $T \in \mathcal{L}(^m l_\infty^n)$ with*

$$T\Big(\Big(x_1^{(1)}, \ldots, x_n^{(1)}\Big), \ldots, \Big(x_1^{(m)}, \ldots, x_n^{(m)}\Big)\Big) = \sum_{1\leq i_k\leq n,\ 1\leq k\leq m} a_{i_1\cdots i_m}\, x_{i_1}^{(1)} \cdots x_{i_m}^{(m)}$$

for some $a_{i_1\cdots i_m} \in \mathbb{R}$. Then

$$\|T\| = \max\Big\{\Big|\sum_{1\leq i_k\leq n,\ 1\leq k\leq m} a_{i_1\cdots i_m} t_{i_1}^{(1)}, \ldots, t_{i_m}^{(m)}\Big| : t_k^{(j)} = \pm 1,\ k = 1, \ldots, n,$$

$$j = 1, \ldots, m \Big\}.$$

Proof. Write

$$\operatorname{ext} B_{l_\infty^m} = \{a_1, \ldots, a_{2^m}\}.$$

By the Krein-Milman theorem,

$$B_{l_\infty^m} = \operatorname{conv}\Big(\{a_1, \ldots, a_{2^m}\}\Big).$$

Let

$$(x_1^{(j)}, \ldots, x_m^{(j)}) \in B_{l_\infty^m} \ (1 \leq j \leq n).$$

Then, there exist $t_1^{(j)}, \ldots, t_{2^m}^{(j)} \in \mathbb{R}$ such that

$|t_1^{(j)}| + \cdots + |t_{2^m}^{(j)}| \leq 1$ and $(x_1^{(j)}, \ldots, x_m^{(j)}) = t_1^{(j)} a_1 + \cdots + t_{2^m}^{(j)} a_{2^m}$ $(1 \leq j \leq n)$.

It follows that

$$\begin{aligned}
&\Big|T\Big(\Big(x_1^{(1)}, \ldots, x_m^{(1)}\Big), \ldots, \Big(x_1^{(n)}, \ldots, x_m^{(n)}\Big)\Big)\Big| \\
&= \Big|T\Big(t_1^{(1)} a_1 + \cdots + t_{2^m}^{(1)} a_{2^m}, \ldots, t_1^{(n)} a_1 + \cdots + t_{2^m}^{(n)} a_{2^m}\Big)\Big| \\
&\leq \sum_{1 \leq j_k \leq 2^m, 1 \leq k \leq n} \Big|t_{j_1}^{(1)}\Big| \cdots \Big|t_{j_n}^{(n)}\Big| \ \Big|T\Big(a_{j_1}, \ldots, a_{j_n}\Big)\Big| \\
&\leq \Big(\sum_{1 \leq j_1 \leq 2^m} \Big|t_{j_1}^{(1)}\Big|\Big) \cdots \Big(\sum_{1 \leq j_n \leq 2^m} \Big|t_{j_n}^{(n)}\Big|\Big) \max_{W \in \mathcal{W}_{n,m}} |T(W)| \\
&\leq \max_{W \in \mathcal{W}_{n,m}} |T(W)|,
\end{aligned}$$

which concludes the proof. □

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