

A NOTE ON STATIC MANIFOLDS AND ALMOST RICCI SOLITONS

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ABSTRACT. In this short paper, we investigate the existence of non-trivial almost Ricci solitons on static manifolds. As a result we show any compact nontrivial static manifold is isometric to a Euclidean sphere.

1. Introduction

In [2], Corvino studied localized scalar curvature deformation of a Riemannian metric and introduced the following definition:

Definition 1.1. A Riemannian metric g is called static on a manifold M if the linearized scalar curvature map at g has a nontrivial cokernel, i.e., if there exists a nontrivial function f on M such that

$$(1.1) \quad -\Delta(f)g + \nabla^2 f - fRic = 0.$$

Here ∇^2 , Δ and Ric denote the Hessian, the Laplacian and the Ricci curvature of g , respectively.

A nontrivial solution f to (1.1) has been called a static potential if it exists. It is proved that a static metric (as defined above) must have constant scalar curvature [2]. When this constant is zero (which is always the case for an asymptotically flat, static metric), (1.1) becomes

$$(1.2) \quad \nabla^2 f = fRic \quad \text{and} \quad \Delta f = 0.$$

Some investigations around static manifolds can be found in [3, 5, 6].

On the other hand, the concept of almost Ricci soliton was introduced in a recent paper due to Pigola et al. [4], where essentially they modified the definition of Ricci solitons by adding the condition on the parameter λ to be a variable function. More precisely:

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Definition 1.2. A Riemannian manifold (M, g) is an almost Ricci soliton if there exist a complete vector field X and a smooth soliton function $\lambda : M \rightarrow \mathbb{R}$ satisfying:

$$(1.3) \quad Ric + \frac{1}{2}L_X g = \lambda g,$$

where L_X denotes the Lie derivative in the direction of X .

Almost Ricci soliton will be called expanding, steady or shrinking, respectively if $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$. When the vector field X is a gradient of a smooth function $f : M \rightarrow \mathbb{R}$, the manifold will be called a gradient almost Ricci soliton. In this case the soliton equation (1.3) turns out to be:

$$(1.4) \quad Ric + \nabla^2 f = \lambda g.$$

Moreover, when X is a Killing vector field, the almost Ricci soliton will be called trivial, otherwise it will be a nontrivial almost Ricci soliton [4].

Barros et al. proved that:

Theorem 1.3 ([1]). *Every compact nontrivial almost Ricci soliton with constant scalar curvature is gradient.*

2. Main results

In this section, we investigate the existence of gradient almost Ricci solitons on static manifolds. As a result we prove any compact nontrivial static manifold is isometric to a Euclidean sphere. At first, we show:

Theorem 2.1. *Almost Ricci solitons on static manifold (M, g) are gradient.*

Proof. Since static manifold (M, g) has constant scalar curvature, the result is obtained from Theorem 1.3. \square

Finally, we obtain the following theorem:

Theorem 2.2. *Every compact static manifold (M, g) with static potential f has a gradient almost Ricci soliton.*

Proof. Let (M, g) be a compact static manifold with static potential f . Taking the trace of (1.1), we have

$$(2.1) \quad \Delta f - n\Delta f - fR = 0,$$

where R is the scalar curvature of g . Thus we have

$$(2.2) \quad (1 - n)\Delta f - fR = 0.$$

Consequently

$$(2.3) \quad \Delta f = \frac{R}{1 - n}f.$$

On the other hand, computing the trace of equation (1.4) yields that

$$(2.4) \quad R + \Delta f = n\lambda.$$

Hence,

$$(2.5) \quad \Delta f = n\lambda - R.$$

Then, from equations (2.5) and (2.3) we arrive at

$$(2.6) \quad n\lambda - R + \frac{R}{n-1}f = 0.$$

Therefore we acquire the smooth function λ as follows:

$$(2.7) \quad \lambda = \left(\frac{1}{n} - \frac{f}{n(n-1)} \right) R.$$

So we proved on the compact static manifold (M, g) there exist a gradient almost Ricci soliton with potential function f and a soliton function λ as given by equation (2.7). \square

Corollary 2.3. *Any compact nontrivial static manifold is isometric to a Euclidean sphere.*

Proof. Since we have shown every compact static manifold (M, g) with static potential f has a gradient almost Ricci solitons, applying Corollary 1 in [1] we conclude that the every compact nontrivial static manifold isometric to a Euclidean sphere. \square

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