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ON RINGS WHOSE ESSENTIAL MAXIMAL RIGHT IDEALS ARE GP-INJECTIVE

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Abstract. In this paper, we continue to study the von Neumann regularity of rings whose essential maximal right ideals are GP-injective. It is proved that the following statements are equivalent: (1) R is strongly regular; (2) R is a 2-primal ring whose essential maximal right ideals are GP-injective; (3) R is a right (or left) quasi-duo ring whose essential maximal right ideals are GP-injective. Moreover, it is shown that R is strongly regular if and only if R is a strongly right (or left) bounded ring whose essential maximal right ideals are GP-injective. Finally, we prove that a PI-ring whose essential maximal right ideals are GP-injective is strongly π-regular.

Throughout this paper, R denotes an associative ring with identity, and all modules are unitary. A right R-module M is called *right generalized principally injective* (briefly *right GP-injective*) if, for any $0 \neq a \in R$, there exists a positive integer n such that $a^n \neq 0$ and any right R-homomorphism of a^nR into M extends to one of R into M. In particular, M is called *right principally injective* (briefly *right p-injective*) when $n = 1$. Clearly, right p-injective modules are right GP-injective, but the converse is not true by [5]. As a well known fact, a ring R is semisimple Artinian if and only if every right ideal of R is injective if and only if every maximal right ideal of R is injective. While Chen and Ding [4] proved that a ring R is von Neumann regular if and only if every right ideal of R is p-injective if and only if every proper principal right ideal of R is GP-injective if and only if every essential right ideal of R is GP-injective. Von Neumann regularity of rings whose maximal right ideals are GP-injective has studied in [6, 20, 21, etc.]. It is proved that a ring R is strongly regular if and only if R is a ring whose essential maximal right ideals are GP-injective, satisfying any one of the following conditions, respectively: reduced; semicommutative; 2-primal.

Recently, Subedi and Buhphang studied strongly regularity of rings whose essential maximal right ideals are GP-injective. Especially, they [16] proved that R is a strongly regular ring if and only if R is reduced and every essential

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maximal right (left) ideal is GP-injective if and only if R is left quasi-duo and every essential maximal right ideal is GP-injective. We continue in this paper the study of von Neumann regularity of rings whose essential maximal right ideals are GP-injective extending the results in the above. Moreover, this is closely related to the von Neumann regularity of right SF-rings. A ring R is called *right SF* if every simple right R-module is flat. Ramamurthi [12] initiated the study of SF-rings and a question whether a right and left SF-ring is necessarily von Neumann regular. This question has drawn the attention of many authors [9, 13, 22, etc.]. Some positive cases have been found but the question remains open. It is well known that for a maximal right ideal M of R, R/M is flat if and only if for each $a \in M$ there exists $b \in M$ such that $a = ba$. Thus if every maximal right ideal of R is p-injective, then R is a right SF-ring. Moreover, it can be easily checked that if every essential maximal right ideal of R is p-injective, then R is right SF because every maximal right ideal of R is either a direct summand or essential in R. Hence "von Neumann regularity of a ring whose essential maximal right ideals of R are GP-injective" is closely related to von Neumann regularity of right SF-rings.

We denote the prime radical, the Jacobson radical, the left singular ideal, the right socle and the left socle of R by $P(R)$, $J(R)$, $Z_l(R)$, $Soc_r(R)$ and $Soc_l(R)$, respectively. For any nonempty subset S of R, $r_R(S)$ and $\ell_R(S)$ denote the right annihilator and the left annihilator of S in R, respectively.

We begin with the following definition.

Definition 1.

(1) A ring R is called *von Neumann regular* if for any $x \in R$, there exists $y \in R$ such that $x = xyx$.
(2) A ring R is called *strongly regular* if for any $x \in R$, there exists $y \in R$ such that $x = x^2y$.
(3) A ring R is called *semicommutative* [15] if for $a, b \in R$, $ab = 0$ implies $aRb = 0$.
(4) A ring R is called *2-primal* [3] if its prime radical coincides with the set of all nilpotent elements of R.
(5) A ring R is called *right* (resp. *left*) *quasi-duo* [19] if every maximal right (resp. left) ideal of R is a two-sided ideal.
(6) A ring R is called *weakly right* (resp. *left*) *duo* [18] if for any $a \in R$, there exists a positive integer n such that a^nR (resp. Ra^n) is a two-sided ideal of R.
(7) A ring R is called *strongly right bounded* (simply, *SRB*) (resp. *SLB*) [2] if every nonzero right (resp. left) ideal of R contains a nonzero two-sided ideal of R.

Referring [19, Proposition 2.2 and Corollary 2.4], von Neumann regular rings are equivalent to strongly regular rings if any one of the following conditions is satisfied: reduced; semicommutative; 2-primal; right (or left) quasi-duo; weakly right (or left) duo; SRB (or SLB).

Lemma 2. *Let I be a right ideal of R and $0 \neq a \in I$. If I is GP-injective, then there exists a positive integer n such that $a^n \neq 0$ and $a^n = ca^n$ for some $c \in I$.*

Proof. Suppose that a right ideal I of R is GP-injective and $0 \neq a \in I$. Then there exists a positive integer n such that the inclusion map $i : a^nR \to I$ extends to one of R into I, say j. Therefore $a^n = i(a^n) = ca^n$, where $c = j(1) \in I$. □

Proposition 3. *Suppose that every essential maximal right ideal of R is GP-injective. Then*:

(1) *The center $C(R)$ of R is reduced*;
(2) *For a two-sided ideal I of R, if $\bar{R} = R/I$ is a reduced ring, then $\bar{R}$ is strongly regular*;
(3) *If R is right* (*or left*) *quasi-duo, then it is reduced.*

Proof. (1) Suppose that there exists a nonzero $a \in C(R)$ such that $a^2 = 0$. Then $r_R(a) \subseteq M$ for some maximal right ideal M of R, and $r_R(a)$ is right essntial in R. For, assume that $r_R(a) \cap bR = 0$ for some $b \in R$. Then $ba \in r_R(a) \cap bR = 0$ and so $b \in r_R(a) \cap bR = 0$ since $a \in C(R)$ and $aR \subseteq r_R(a)$, this implies that $b = 0$. From which we also note that M is right essential in R. Since M is GP-injective and $a^2 = 0$, there exists $c \in M$ such that $a = ca$ by Lemma 2; whence $1 - c \in r_R(a) \subseteq M$ because $a \in C(R)$, which is a contradiction. Therefore $C(R)$ is reduced.

(2) Let I be a two-sided ideal of R and suppose that $\bar{R} = R/I$ is a reduced ring. Let $\bar{0} \neq \bar{a} \in \bar{R}$. We first claim that $\bar{a}\bar{R} + r_{\bar{R}}(\bar{a}) = \bar{R}$. Suppose not. Then there exists a maximal right ideal $\bar{M} = M/I$ of $\bar{R}$ containing $\bar{a}\bar{R} + r_{\bar{R}}(\bar{a})$. Since $\bar{R}$ is reduced, $\bar{M}$ is right essential in $\bar{R}$. Note that M is also an essential maximal right ideal of R. Thus M is GP-injective. By Lemma 2, there exists $c \in M$ such that $a^n = ca^n$; whence $\bar{1} - \bar{c} \in \ell_{\bar{R}}(\bar{a}^n) = \ell_{\bar{R}}(\bar{a}) = r_{\bar{R}}(\bar{a})$. Hence $\bar{1} \in \bar{M}$, which is a contradiction. Therefore $\bar{R}$ is strongly regular.

(3) Assume that R is right quasi-duo. Then by [19, Corollary 2.4], $R/J(R)$ is reduced. By (2), $R/J(R)$ is strongly regular and hence R is left quasi-duo. We claim that R is reduced. Suppose that $a^2 = 0$ with $a \neq 0$. Then $\ell_R(a) \subseteq M$ for some maximal left ideal M of R. Since R is left quasi-duo, M is itself maximal right ideal of R. Note that M is right essential in R. Since M is GP-injective and $a^2 = 0$, there exists $c \in M$ such that $a = ca$ by Lemma 2, whence $1 - c \in \ell_R(a) \subseteq M$ and so $1 \in M$, which is a contradiction. Thus R is reduced. □

By help of Lemma 2 and Proposition 3, we investigate the strong regularity of a ring whose essential maximal right ideals are GP-injective.

Proposition 4. *Let R be a semiprime ring whose essential maximal right ideals are GP-injective. Then the center $C(R)$ of R is strongly regular.*

Proof. We first show that $aR + r_R(a) = R$ for any nonzero $a \in C(R)$. Suppose not. Then there exists a maximal right ideal M of R containing $aR + r_R(a)$.

Note that $aR + r_R(a)$ is right essential in R by same method as in the proof of Proposition 3(1). Thus M is right essential in R and hence it is GP-injective. By Lemma 2, there exists $c \in M$ such that $a^n = ca^n$; whence $((1-c)aR)^n = 0$. This implies that $(1-c)a = 0$ since R is semiprime. So $1 - c \in r_R(a) \subseteq M$, which is a contradiction. Thus $aR + r_R(a) = R$ and so $a = aba$ for some $b \in R$. By [8, Proof of Theorem 1.14], we have $C(R)$ is von Neumann regular. □

Note that R is semicommutative if and only if the right (left) annihilator of R is a two-sided ideal [15, Lemma 1.2]. Shin also proved that semicommutative rings are 2-primal in [15, Theorem 1.5]. Obviously, reduced rings are 2-primal.

We need the following lemma to prove our main results.

Lemma 5 ([4, Corollary 2.4])**.** *A ring R is von Neumann regular if and only if every right R-module is GP-injective.*

Using Lemma 5 and Proposition 3, we obtain the following result which extends known results [20, Theorem 5.1 and Proposition 7] and [16, Theorem 2.5].

Theorem 6. *For a ring R, the following statements are equivalent*:

(1) *R is a strongly regular ring*;
(2) *R is a reduced ring whose essential maximal right ideals are GP-injective*;
(3) *R is a semicommutative ring whose essential maximal right ideals are GP-injective*;
(4) *R is a 2-primal ring whose essential maximal right ideals are GP-injective.*

Proof. By Lemma 5, (1) implies (2). (2) ⇒ (3) and (3) ⇒ (4) are obvious. (4) ⇒ (1): Assume that R is 2-primal. Then $\bar{R} = R/P(R)$ is a reduced ring. By Proposition 3, $\bar{R}$ is strongly regular and so R is left and right quasi-duo. By Proposition 3, R is reduced and so $P(R) = 0$; whence R is a strongly regular ring. □

Yu [19, Proposition 2.2] proved that every weakly right (left) duo ring is right (left) quasi-duo. The following theorem also extends results [21, Proposition 7] and [16, Theorem 2.15].

Theorem 7. *For a ring R, the following statements are equivalent*:

(1) *R is a strongly regular ring*;
(2) *R is a weakly right* (*or left*) *duo ring whose essential maximal right ideals are GP-injective*;
(3) *R is a right* (*or left*) *quasi-duo ring whose essential maximal right ideals are GP-injective.*

Proof. By Lemma 5 and [19, Proposition 2.2], (1) ⇒ (2) ⇒ (3). (3) ⇒ (1): By Proposition 3, R is reduced. Thus R is strongly regular by Theorem 6. □

We recall that a ring R is *abelian* if every idempotent element of R is central. A ring R is called π-*regular* if for every $x \in R$, there exists a positive integer n, depending on x, such that $x^n = x^n y x^n$ for some $y \in R$. Ohori [10] proved that the set of all nilpotent elements of an abelian π-regular ring is a two-sided ideal. Using this result, it can be easily checked that an abelian π-regular ring is left and right quasi-duo. Hence we have the following.

Corollary 8. *For a ring R, the following statements are equivalent*:

(1) *R is a strongly regular ring*;
(2) *R is an abelian π-regular ring whose essential maximal right ideals are GP-injective.*

Following Theorem 7, it is natural to ask whether MERT (or MELT) rings whose essential maximal right ideal of R is GP-injective are von Neumann regular, where MERT (resp. MELT) rings mean every essential maximal right (resp. left) ideal is two-sided. We have the following result.

Theorem 9. *Let R be a MERT ring. Then the following statements are equivalent*:

(1) *R is von Neumann regular*;
(2) *Every essential maximal right ideal of R is GP-injective and $R/Soc_r(R)$ is semiprimitive.*

Proof. Obviously, (1) implies (2). (2) $\Rightarrow$ (1): Since R is MERT, $\bar{R} = R/Soc_r(R)$ is right quasi-duo. Then by [19, Corollary 2.4], $\bar{R}/J(\bar{R})$ is reduced and hence $\bar{R}$ is reduced. By Proposition 3, $\bar{R}$ is strongly regular. By hypothesis, $J(R) \subseteq Soc_r(R)$ which implies $J(R)^2 \subseteq Soc_r(R)J(R) = 0$. We next claim that $J(R) = 0$. Suppose that $J(R) \neq 0$. Then there exists a nonzero element $x \in J(R)$. Now $Soc_r(R) \subseteq \ell_R(J(R)) \subseteq \ell_R(x)$. Since $\bar{R}$ is strongly regular, $\ell_R(x)/Soc_r(R)$ is a two-sided ideal of $\bar{R}$ and so $\ell_R(x)$ is a two-sided ideal of R. Then there exists a maximal right ideal K of R containing $\ell_R(x)$ and also K is right essential in R. Since K is GP-injective and $x^2 = 0$, $x = yx$ for some $y \in K$ by Lemma 2. Thus $1 - y \in \ell_R(x) \subseteq K$; whence $1 \in K$, which is a contradiction. Hence $J(R) = 0$ and so $Soc_r(R)$ is von Neumann regular by [17, Remark 3.2]. Therefore R is von Neumann regular. □

In Theorem 9, the condition (1) cannot be replaced with "R is strongly regular". For example, the 2-by-2 full matrix ring over a field is MERT (MELT) von Neumann regular, but it is not strongly regular.

The following result is one of very useful properties of rings in which every essential maximal right ideal is GP-injective, which also gives good clues to our next investigations.

Lemma 10. *Suppose that every essential maximal right ideal of R is GP-injective. Then $Z_l(R) \subseteq J(R)$. Furthermore, $Z_l(R) = J(R)$ if $J(R)^2 = 0$.*

Proof. For any $x \in Z_l(R)$, then $\ell_R(x)$ is an essential left ideal of R. Note that $\ell_R(x) \cap \ell_R((1-x)^m) = 0$ for every positive integer m. Thus $\ell_R((1-x)^m) = 0$. We claim that $(1-x)R = R$. Suppose not. Then there exists a maximal right ideal M of R containing $(1-x)R$. Moreover, M is right essential in R. For if M were not essential, then we can write $M = r_R(e)$ where $0 \neq e = e^2 \in R$. Since $1 - x \in M = r_R(e)$, $e(1-x) = 0$ and so $e \in \ell_R(1-x) = 0$, which is a contradiction. Hence M is an essential maximal right ideal of R. By Lemma 2, there exists $c \in M$ such that $(1-x)^n = c(1-x)^n$ and so $(1-c)(1-x)^n = 0$. Thus $1 - c \in \ell_R((1-x)^n) = 0$ and so $1 \in M$, which is also a contradiction. Therefore $(1-x)R = R$, and so $Z_l(R) \subseteq J(R)$.

Conversely, suppose that $a \in J(R)$ but $a \notin Z_l(R)$. Then there exists $0 \neq b \in R$ such that $\ell_R(a) \cap Rb = 0$. Since $J(R)^2 = 0$, $b \notin J(R)$. If $b \in J(R)$, then $ba = 0$ and so $Rb \subseteq \ell_R(a)$, which is a contradiction. Thus there exists a maximal right ideal M of R such that $b \notin M$. We claim that M is right essential in R. If not, then there exists a nonzero right ideal I of R such that $M \cap I = 0$ and so $M \oplus I = R$. This implies $x + y = 1$ for some $x \in M$ and $y \in I$. Thus $xba + yba = ba$ and so $ba - xba = yba \in M \cap I = 0$ because $0 \neq ba \in J(R) \subseteq M$. Hence $(b - xb)a = 0$ and so $b - xb \in \ell_R(a) \cap Rb = 0$. Therefore $b = xb \in M$, which is a contradiction. This implies that M is an essential maximal right ideal of R. By Lemma 2, there exists $c \in M$ such that $ba = cba$ and so $(1-c)ba = 0$. Then $(1-c)b \in \ell_R(a) \cap Rb = 0$ and so $b = cb \in M$, which is also a contradiction. Hence $Z_l(R) = J(R)$. □

Using Lemma 10, we can observe the von Neumann regularity of MELT rings whose essential maximal right ideals are GP-injective.

Proposition 11. *Let R be a MELT ring. Then the following statements are equivalent*:

(1) *R is von Neumann regular*;
(2) *Every essential maximal right ideal of R is GP-injective and $R/Soc_l(R)$ is semiprimitive.*

Proof. It is enough to show that (2) ⇒ (1): Since R is MELT, $\bar{R} = R/Soc_l(R)$ is left quasi-duo. Then by [19, Corollary 2.4], $\bar{R}/J(\bar{R})$ is reduced and hence $\bar{R}$ is reduced. By Proposition 3, $\bar{R}$ is strongly regular. By hypothesis, $J(R) \subseteq Soc_l(R)$ which implies $J(R)^2 \subseteq J(R)Soc_l(R) = 0$. We next claim that $J(R) = 0$. Suppose that $J(R) \neq 0$ and $x \in J(R)$. Then by Lemma 10, $x \in J(R) = Z_l(R)$ and so $\ell_R(x)$ is an essential left ideal of R. Then there exists an essential maximal left ideal L of R such that $\ell_R(x) \subseteq L$. Since R is MELT, L is two-sided and so $L \subseteq Q$ for some maximal right ideal Q of R. Note that Q is right essential in R. By Lemma 2, there exists $y \in Q$ such that $x = yx$ and so $1 - y \in \ell_R(x) \subseteq Q$, which is a contradiction. Hence $Soc_l(R)$ is von Neumann regular by [17, Remark 3.2] and therefore R is von Neumann regular. □

By help of Lemma 10, we also have the following result.

Theorem 12. *For a ring R, the following statements are equivalent*:

(1) *R is a strongly regular ring*;
(2) *R is an SRB ring whose essential maximal right ideals are GP-injective*;
(3) *R is an SLB ring whose essential maximal right ideals are GP-injective.*

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are obvious. (2) $\Rightarrow$ (1): We claim that R is reduced. Suppose that $a^2 = 0$ with $a \neq 0$. Then $r_R(\ell_R(a))$ is nozero right ideal of R and so there exists a nonzero two-sided ideal I of R such that $I \subseteq r_R(\ell_R(a))$ and so $\ell_R(a) \subseteq \ell_R(I)$. So $\ell_R(I) \subseteq M$ for some maximal right ideal M of R because $\ell_R(I)$ is two-sided. Note that M is right essential in R. Since M is GP-injective and $a^2 = 0$, there exists $c \in M$ such that $a = ca$ by Lemma 2, whence $1 - c \in \ell_R(a) \subseteq \ell_R(I) \subseteq M$, which is a contradiction. Thus R is reduced. Therefore R is strongly regular by Theorem 6.

(3) $\Rightarrow$ (1): We first claim that $R/Z_l(R)$ is reduced. Assume that there exists $a \notin Z_l(R)$ such that $a^2 \in Z_l(R)$. Then there exists a nonzero left ideal I of R such that $\ell_R(a) \oplus I \subseteq \ell_R(a^2)$. Since R is SLB, there exists a nonzero two-sided ideal L such that $L \subseteq I$. Then $La^2 = 0$ and so $La \subseteq \ell_R(a) \cap L \subseteq \ell_R(a) \cap I = 0$. Thus $La = 0$ and hence $L \subseteq \ell_R(a) \cap I = 0$, which is a contradiction. Therefore $R/Z_l(R)$ is reduced. By Proposition 3, $R/Z_l(R)$ is strongly regular. Thus $J(R) \subseteq Z_l(R)$ and hence $J(R) = Z_l(R)$ by Lemma 10. This implies that $R/J(R)$ is strongly regular and so R is right quasi-duo. Therefore by Theorem 7, R is strongly regular. □

Finally, we turn our attention to a PI-ring whose essential maximal right ideals are GP-injective. Recall that a ring R is *right* (*left*) *weakly regular* [11] if $I^2 = I$ for each right (left) ideal I of R; equivalently, $a \in aRaR$ ($a \in RaRa$) for every $a \in R$. R is *weakly regular* if it is both right and left weakly regular. A ring R is called *strongly π-regular* if for every $x \in R$, there exists a positive integer n, depending on x, such that $x^n = x^{n+1}y$ for some $y \in R$.

Theorem 13. *Suppose that R is a PI-ring whose essential maximal right ideals are GP-injective. Then R is strongly π-regular.*

Proof. We first show that every prime factor ring of R is right weakly regular. If not, then there exists a prime ideal P such that $\bar{R} = R/P$ is not right weakly regular. This implies that $\bar{R}\bar{a}\bar{R} \neq \bar{R}$ for some $\bar{0} \neq \bar{a} \in \bar{R}$. Then there exists maximal right ideal $\bar{M} = M/P$ of $\bar{R}$ such that $\bar{R}\bar{a}\bar{R} \subseteq \bar{M}$. Since $\bar{R}$ is a prime PI-ring, by adapting a result of [14, Theorem 6.1.28] there exists $\bar{0} \neq \bar{x} \in \bar{R}\bar{a}\bar{R} \cap C(\bar{R})$. Observe that M must be an essential right ideal of R. For if M were not essential, then we can write $M = r_R(e)$ where $0 \neq e = e^2 \in R$. Since $RaR \subseteq M$, $eRaR = \{0\} \subseteq P$. But $a \notin P$ and so $eR \subseteq P \subseteq M = r_R(e)$; whence $e = 0$. It is a contradiction. Thus M is GP-injective, there exists $y \in M$ such that $x^n = yx^n$. Thus $\bar{x}^n = \bar{y}\bar{x}^n$ and so $(\bar{1} - \bar{y})\bar{x}^n = \bar{0}$. Since $\bar{x} \in C(\bar{R})$, $(\bar{1} - \bar{y})\bar{R}\bar{x}^n = \bar{0}$. If $\bar{x}^n = \bar{0}$, then $(\bar{x}\bar{R})^n = \bar{x}^n\bar{R} = \bar{0}$. Since $\bar{R}$ is prime, $\bar{x}\bar{R} = \bar{0}$ and so $\bar{x} = \bar{0}$, which is a contradiction. Hence $\bar{1} - \bar{y} = \bar{0}$ and so $\bar{1} \in \bar{M}$. It is also a contradiction. Therefore every prime factor ring of R is right weakly regular and so is von Neumann regular by [1, Theorem 1]. Hence by [7, Theorem 2.3], R is strongly π-regular. □

In Theorem 13, we cannot replace "R is strongly π-regular" with "R is strongly regular". By Lemma 5, von Neumann regular PI-rings satisfy the hypothesis of Theorem 13. But, there exists a von Neumann regular PI-ring which is not strongly regular, for example, the 2-by-2 full matrix ring over a field.

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