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COEFFICIENT ESTIMATES FOR FUNCTIONS ASSOCIATED WITH VERTICAL STRIP DOMAIN

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Abstract. In this paper, we consider a convex univalent function $f_{\alpha,\beta}$ which maps the open unit disc $\mathbb{U}$ onto the vertical strip domain

$$\Omega_{\alpha,\beta} = \{w \in \mathbb{C} : \alpha < \Re(w) < \beta\}$$

and introduce new subclasses of both close-to-convex and bi-close-to-convex functions with respect to an odd starlike function associated with $\Omega_{\alpha,\beta}$. Also, we investigate the Fekete-Szegö type coefficient bounds for functions belonging to these classes.

1. Introduction

Assume that $\mathcal{H}$ is the class of analytic functions in the open unit disc

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\},$$

and let the class $\mathcal{P}$ be defined by

$$\mathcal{P} = \{\, p \in \mathcal{H} : p(0) = 1 \quad \text{and} \quad \Re(p(z)) > 0 \ (z \in \mathbb{U})\}.$$

For two functions $f, g \in \mathcal{H}$, we say that the function f is subordinate to g in $\mathbb{U}$, and write

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}),$$

if there exists a Schwarz function

$$\omega \in \Lambda := \{\omega \in \mathcal{H} : \omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \ (z \in \mathbb{U})\},$$

such that

$$f(z) = g(\omega(z)) \quad (z \in \mathbb{U}).$$

Indeed, it is known that

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Rightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

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Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let $\mathcal{A}$ denote the subclass of $\mathcal{H}$ consisting of functions f normalized by

$$f(0) = f'(0) - 1 = 0.$$

Each function $f \in \mathcal{A}$ can be expressed as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathbb{U}). \tag{1}$$

We also denote by $\mathcal{S}$ the class of all functions in the normalized analytic function class $\mathcal{A}$ which are univalent in $\mathbb{U}$.

A function $f \in \mathcal{A}$ is said to be *starlike of order* $\alpha\,(0 \leq \alpha < 1)$, if it satisfies the inequality

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathbb{U}).$$

We denote the class which consists of all functions $f \in \mathcal{A}$ that are starlike of order α by $\mathcal{S}^*(\alpha)$. It is well-known that $\mathcal{S}^*(\alpha) \subset \mathcal{S}^*(0) = \mathcal{S}^* \subset \mathcal{S}$.

Furthermore a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{M}(\beta)\,(\beta > 1)$ if it satisfies the inequality

$$\Re\left(\frac{zf'(z)}{f(z)}\right) < \beta \qquad (z \in \mathbb{U}).$$

This class was introduced by Uralegaddi *et al.* [12].

Motivated by the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{M}(\beta)$, Kuroki and Owa [7] introduced the subclass $\mathcal{S}(\alpha, \beta)$ of analytic functions $f \in \mathcal{A}$ which is given by Definition 1 below.

Definition 1 ([7])**.** Let $\mathcal{S}(\alpha, \beta)$ be a class of functions $f \in \mathcal{A}$ which satisfy the inequality

$$\alpha < \Re\left(\frac{zf'(z)}{f(z)}\right) < \beta \qquad (z \in \mathbb{U})$$

for some real number α $(\alpha < 1)$ and some real number β $(\beta > 1)$.

The class $\mathcal{S}(\alpha, \beta)$ is non-empty. For example, the function $f \in \mathcal{A}$ given by

$$f(z) = z \exp\left\{\frac{\beta - \alpha}{\pi} i \int_0^z \frac{1}{t} \log\left(\frac{1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} t}{1 - t}\right) dt\right\}$$

is in the class $\mathcal{S}(\alpha, \beta)$.

Also for $f \in \mathcal{S}(\alpha, \beta)$, if $\alpha \geq 0$ then $f \in \mathcal{S}^*(\alpha)$ in $\mathbb{U}$, which implies that $f \in \mathcal{S}$.

Lemma 1.1 ([7])**.** *Let $f \in \mathcal{A}$ and $\alpha < 1 < \beta$. Then $f \in \mathcal{S}(\alpha, \beta)$ if and only if*

$$\frac{zf'(z)}{f(z)} \prec 1 + \frac{\beta-\alpha}{\pi} i \log\left(\frac{1-e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} z}{1-z}\right) \qquad (z \in \mathbb{U}).$$

Lemma 1.1 means that the function $f_{\alpha,\beta} : \mathbb{U} \to \mathbb{C}$ defined by

$$f_{\alpha,\beta}(z) = 1 + \frac{\beta-\alpha}{\pi} i \log\left(\frac{1-e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} z}{1-z}\right) \tag{2}$$

is analytic in $\mathbb{U}$ with $f_{\alpha,\beta}(0) = 1$ and maps the open unit disk $\mathbb{U}$ onto the vertical strip domain

$$\Omega_{\alpha,\beta} = \{w \in \mathbb{C} : \alpha < \Re(w) < \beta\} \tag{3}$$

conformally.

We note that the function $f_{\alpha,\beta}$ defined by (2) is a convex univalent function in $\mathbb{U}$ and has the form

$$f_{\alpha,\beta}(z) = 1 + \sum_{n=1}^{\infty} \varphi_n z^n, \tag{4}$$

where

$$\varphi_n = \frac{\beta-\alpha}{n\pi} i \left(1 - e^{2n\pi i \frac{1-\alpha}{\beta-\alpha}}\right) \qquad (n \in \mathbb{N}). \tag{5}$$

Kowalczyk and Leś-Bomba [6] introduced the subclass $\mathcal{K}_s(\alpha)$ of close-to-convex analytic functions as follows:

Definition 2 ([6])**.** Let the function f be analytic in $\mathbb{U}$ defined by (1). We say that $f \in \mathcal{K}_s(\alpha)$ $(0 \leq \alpha < 1)$, if there exists a function $g \in \mathcal{S}^*(1/2)$ such that

$$\Re\left(\frac{z^2 f'(z)}{-g(z)g(-z)}\right) > \alpha \qquad (z \in \mathbb{U}).$$

In particular, we have the class $\mathcal{K}_s(0) = \mathcal{K}_s$ introduced and studied by Gao and Zhou [2].

Lemma 1.2 ([2])**.** *If $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^*(1/2)$, then*

$$\psi(z) = \frac{-g(z)g(-z)}{z} = z + \sum_{n=2}^{\infty} B_{2n-1} z^{2n-1} \in \mathcal{S}^* \subset \mathcal{S}, \tag{6}$$

where the coefficients of the odd-starlike function ψ satisfy the condition

$$|B_{2n-1}| = \left|2b_{2n-1} - 2b_2 b_{2n-2} + \cdots + 2(-1)^n b_{n-1} b_{n+1} + (-1)^{n+1} b_n^2\right| \leq 1 \tag{7}$$

for $n \geq 2$.

The aforecited work of Kowalczyk and Leś-Bomba [6] was followed by such works as those by Goyal and Goswami [3], Goyal and Singh [4], Wang and Chen [13], Wang *et al.* [14] and Xu *et al.* [15].

Here, in our present sequel to the aforecited works of Kuroki and Owa [7] and Kowalczyk and Leś-Bomba [6], we first introduce the following subclasses of analytic functions.

Definition 3. Let α and β be real such that $0 \leq \alpha < 1 < \beta$. We denote by $\mathcal{K}_s(\alpha, \beta)$ the class of functions $f \in \mathcal{A}$ satisfying

$$\alpha < \Re\left(\frac{z^2 f'(z)}{-g(z)g(-z)}\right) < \beta \qquad (z \in \mathbb{U}),$$

where $g \in \mathcal{S}^*(1/2)$.

Remark 1.3. (i) If we let $\beta \to \infty$ in Definition 3, then the class $\mathcal{K}_s(\alpha, \beta)$ reduces to the class $\mathcal{K}_s(\alpha)$.

(ii) If we let $\alpha = 0$ and $\beta \to \infty$ in Definition 3, then the class $\mathcal{K}_s(\alpha, \beta)$ reduces to the class $\mathcal{K}_s$.

Using (3) and by the principle of subordination, we can immediately obtain Lemma 1.4.

Lemma 1.4. *Let α and β be real numbers such that $0 \leq \alpha < 1 < \beta$ and let the function $f \in \mathcal{A}$ be defined by* (1)*. Then $f \in \mathcal{K}_s(\alpha, \beta)$ if and only if*

$$\frac{z^2 f'(z)}{-g(z)g(-z)} \prec f_{\alpha,\beta}(z),$$

where $f_{\alpha,\beta}(z)$ is defined by (2).

On the other hand, since univalent functions are one-to-one, they are invertible and the inverse functions need not be defined on the entire unit disk $\mathbb{U}$. In fact, the Koebe one-quarter theorem [1] ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{S}$ contains a disk of radius 1/4. Thus every function $f \in \mathcal{A}$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U})$$

and

$$f\left(f^{-1}(w)\right) = w \qquad \left(|w| < r_0(f);\ r_0(f) \geq \frac{1}{4}\right).$$

In fact, the inverse function $F = f^{-1}$ is given by

$$F(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{8}$$

If both the function f and its inverse function f^{-1} are univalent in $\mathbb{U}$, then the function f is called *bi-univalent*. We will denote the class which consists of functions f that are bi-univalent by Σ, [8].

Now, we introduce a new subclass of bi-univalent functions as follows:

Definition 4. Let α and β be real such that $0 \leq \alpha < 1 < \beta$. A function $f \in \Sigma$ given by (1) is said to be in the class $\mathcal{K}_{\Sigma_s}(\alpha, \beta)$ if there exist the functions

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^*(1/2), \qquad G(w) = w + \sum_{n=2}^{\infty} d_n w^n \in \mathcal{S}^*(1/2)$$

and the following conditions are satisfied:

$$\alpha < \Re\left(\frac{z^2 f'(z)}{-g(z)g(-z)}\right) < \beta \qquad (z \in \mathbb{U}) \tag{9}$$

and

$$\alpha < \Re\left(\frac{w^2 F'(w)}{-G(w)G(-w)}\right) < \beta \qquad (w \in \mathbb{U}), \tag{10}$$

where the function $F = f^{-1}$ is defined by (8).

Remark 1.5. (i) Letting $\beta \to \infty$, we have the class $\mathcal{K}_{\Sigma}^{s}(\alpha)$ of bi-close-to-convex functions of order α satisfying the conditions

$$\Re\left(\frac{z^2 f'(z)}{-g(z)g(-z)}\right) > \alpha \qquad (z \in \mathbb{U})$$

and

$$\Re\left(\frac{w^2 F'(w)}{-G(w)G(-w)}\right) > \alpha \qquad (w \in \mathbb{U}).$$

This class introduced and studied by Şeker and Sümer Eker [11].

(ii) Letting $\alpha = 0$ and $\beta \to \infty$, we have the class $\mathcal{K}_{\Sigma}^{s}$ of bi-close-to-convex functions satisfying the conditions

$$\Re\left(\frac{z^2 f'(z)}{-g(z)g(-z)}\right) > 0 \qquad (z \in \mathbb{U})$$

and

$$\Re\left(\frac{w^2 F'(w)}{-G(w)G(-w)}\right) > 0 \qquad (w \in \mathbb{U}).$$

2. Preliminary lemmas

Lemma 2.1 ([10])**.** *Let the function* $\mathfrak{g}$ *given by*

$$\mathfrak{g}(z) = \sum_{k=1}^{\infty} \mathfrak{b}_k z^k \qquad (z \in \mathbb{U})$$

be convex in $\mathbb{U}$. *Also let the function* $\mathfrak{f}$ *given by*

$$\mathfrak{f}(z) = \sum_{k=1}^{\infty} \mathfrak{a}_k z^k \qquad (z \in \mathbb{U})$$

be holomorphic in $\mathbb{U}$. *If*

$$\mathfrak{f}(z) \prec \mathfrak{g}(z) \qquad (z \in \mathbb{U}),$$

then

$$|\mathfrak{a}_k| \leq |\mathfrak{b}_1| \qquad (k \in \mathbb{N}).$$

Lemma 2.2 ([9])**.** *Let $p \in \mathcal{P}$ with $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$. Then for any complex number ν,*

$$\left|c_2 - \nu c_1^2\right| \leq 2 \max\{1, |2\nu - 1|\}.$$

Lemma 2.3 ([5])**.** *For $0 \leq \beta < 1$, let $f \in \mathcal{A}$ given by (1) belong to the function class $\mathcal{S}^*(\beta)$. Then for any real number μ,*

$$\left|a_3 - \mu a_2^2\right| \leq (1-\beta)\max\{1, |3 - 2\beta - 4\mu(1-\beta)|\}.$$

Lemma 2.4 ([16])**.** *Let $k, l \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{C}$. If $|z_1| < R$ and $|z_2| < R$, then*

$$|(k+l) z_1 + (k-l) z_2| \leq \begin{cases} 2R|k| & , \quad |k| \geq |l| \\ \\ 2R|l| & , \quad |k| \leq |l| \end{cases}.$$

3. Coefficient estimates for functions in $\mathcal{K}_s(\alpha, \beta)$

In this section, we find the upper bound for general coefficient of functions belonging to the class $\mathcal{K}_s(\alpha, \beta)$ and also solve Fekete-Szegö problem.

Theorem 3.1. *Let α and β be real numbers such that $0 \leq \alpha < 1 < \beta$ and let the function $f \in \mathcal{A}$ be defined by (1). If $f \in \mathcal{K}_s(\alpha, \beta)$, then*

$$|a_{2n}| \leq \frac{(\beta - \alpha)}{\pi} \sin \frac{\pi(1-\alpha)}{\beta - \alpha} \qquad (n \in \mathbb{N})$$

and

$$|a_{2n+1}| \leq \frac{1 + \frac{2(\beta-\alpha)n}{\pi} \sin \frac{\pi(1-\alpha)}{\beta-\alpha}}{(2n+1)} \qquad (n \in \mathbb{N}).$$

Proof. Let the function $f \in \mathcal{K}_s(\alpha, \beta)$ be of the form (1). Therefore, there exists a function $g \in \mathcal{S}^*(1/2)$ so that

$$\alpha < \Re\left(\frac{z^2 f'(z)}{-g(z)g(-z)}\right) < \beta.$$

Let us set

$$\frac{z^2 f'(z)}{-g(z)g(-z)} = \frac{z f'(z)}{\frac{-g(z)g(-z)}{z}} = \frac{z f'(z)}{\psi(z)} \qquad (z \in \mathbb{U}), \tag{11}$$

where the function ψ is defined by (6). Furthermore, by Lemma 1.2, we have following equations:

$$\psi(z) = \frac{-g(z)g(-z)}{z} = z + \sum_{n=2}^{\infty} B_{2n-1} z^{2n-1} \in \mathcal{S}^* \quad \text{and} \quad |B_{2n-1}| \leq 1. \tag{12}$$

Let us define the function p by

$$p(z) = \frac{z f'(z)}{\psi(z)} \qquad (z \in \mathbb{U}). \tag{13}$$

Then according to the assertion of Lemma 1.4, we get

$$p(z) \prec f_{\alpha,\beta}(z) \qquad (z \in \mathbb{U}), \tag{14}$$

where $f_{\alpha,\beta}(z)$ is defined by (2). Hence, using Lemma 2.1, we obtain

$$\left|\frac{p^{(m)}(0)}{m!}\right| = |c_m| \le |\varphi_1| \qquad (m \in \mathbb{N}), \tag{15}$$

where

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots \qquad (z \in \mathbb{U}) \tag{16}$$

and by (5)

$$|\varphi_1| = \left|\frac{\beta-\alpha}{\pi} i \left(1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}}\right)\right| = \frac{2(\beta-\alpha)}{\pi} \sin \frac{\pi(1-\alpha)}{\beta-\alpha}. \tag{17}$$

Also from (13), we find

$$zf'(z) = p(z)\psi(z). \tag{18}$$

Since G is an odd starlike function with $B_1 = 1$, in view of (18), we obtain

$$2n\, a_{2n} = c_{2n-1} + c_{2n-3}B_3 + \cdots + c_1 B_{2n-1} \qquad (n \in \mathbb{N}) \tag{19}$$

and

$$(2n+1)\, a_{2n+1} = c_{2n} + c_{2n-2}B_3 + \cdots + c_2 B_{2n-1} + B_{2n+1} \qquad (n \in \mathbb{N}). \tag{20}$$

Using (15), we get from the equalities (19) and (20)

$$2n\, |a_{2n}| = n\, |\varphi_1| \qquad (n \in \mathbb{N}) \tag{21}$$

and

$$(2n+1)\, |a_{2n+1}| = 1 + n\, |\varphi_1| \qquad (n \in \mathbb{N}), \tag{22}$$

respectively. The desired result is obtain from the equalities (21) and (22) by considering (17). □

Letting $\beta \to \infty$ in Theorem 3.1, we have the coefficient bounds for functions belong to the class $\mathcal{K}_s(\alpha)$.

Corollary 3.2. *Let α be a real number such that $0 \le \alpha < 1$ and let the function $f \in \mathcal{A}$ be defined by* (1). *If $f \in \mathcal{K}_s(\alpha)$, then*

$$|a_{2n}| \le 1 - \alpha \qquad (n \in \mathbb{N})$$

and

$$|a_{2n+1}| \le \frac{1 + 2(1-\alpha)n}{2n+1} \qquad (n \in \mathbb{N}).$$

Letting $\alpha = 0$ and $\beta \to \infty$ in Theorem 3.1, we have the coefficient bounds for functions belong to the class $\mathcal{K}_s$.

Corollary 3.3 ([2, Theorem 2])**.** *Let the function $f \in \mathcal{A}$ be defined by* (1). *If $f \in \mathcal{K}_s$, then*

$$|a_n| \le 1 \qquad (n = 2, 3, \ldots).$$

Theorem 3.4. *Let α and β be real numbers such that $0 \leq \alpha < 1 < \beta$ and let the function $f \in \mathcal{A}$ be defined by (1). If $f \in \mathcal{K}_s(\alpha,\beta)$, then for any real number μ,*

$$\left|a_3-\mu a_2^2\right| \leq \frac{1}{3}+\frac{2(\beta-\alpha)}{3\pi}\sin\frac{\pi(1-\alpha)}{\beta-\alpha} \times \max\left\{1,\left|\cos\frac{\pi(1-\alpha)}{\beta-\alpha}-\mu\frac{3(\beta-\alpha)}{2\pi}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}\right|\right\}. \tag{23}$$

Proof. Let $f \in \mathcal{K}_s(\alpha,\beta)$ be given by (1) and by means of the function $f_{\alpha,\beta}$ given by (4), let us define the function $u(z)$ by

$$u(z)=\frac{1+f_{\alpha,\beta}^{-1}(p(z))}{1-f_{\alpha,\beta}^{-1}(p(z))}=1+u_1z+u_2z^2+\cdots\in\mathcal{P} \qquad (z\in\mathbb{U}),$$

where the function p is given by (16) and it satisfies (13). So we have

$$p(z)=f_{\alpha,\beta}\left(\frac{u(z)-1}{u(z)+1}\right) \qquad (z\in\mathbb{U}).$$

From (4) and (13) we obtain

$$f_{\alpha,\beta}\left(\frac{u(z)-1}{u(z)+1}\right)=1+\frac{1}{2}\varphi_1u_1z+\left[\frac{1}{2}\varphi_1\left(u_2-\frac{1}{2}u_1^2\right)+\frac{1}{4}\varphi_2u_1^2\right]z^2+\cdots$$

and

$$p(z)=1+2a_2z+(3a_3-B_3)z^2+\cdots,$$

respectively, which implies that

$$2a_2=\frac{1}{2}\varphi_1u_1$$

and

$$3a_3=B_3+\frac{1}{2}\varphi_1\left(u_2-\frac{1}{2}u_1^2\right)+\frac{1}{4}\varphi_2u_1^2.$$

So we get

$$a_3-\mu a_2^2=\frac{B_3}{3}+\frac{1}{6}\varphi_1\left(u_2-\frac{1}{2}u_1^2\right)+\frac{1}{12}\varphi_2u_1^2-\frac{\mu}{16}\varphi_1^2u_1^2. \tag{24}$$

If we choose $n=2$ in (7), then we find that

$$B_3=2b_3-b_2^2. \tag{25}$$

Thus we have

$$a_3-\mu a_2^2=\frac{2}{3}\left(b_3-\frac{1}{2}b_2^2\right)+\frac{1}{6}\varphi_1\left(u_2-\nu u_1^2\right),$$

where

$$\nu=\frac{1}{2}\left(1-\frac{\varphi_2}{\varphi_1}+\frac{3\mu}{4}\varphi_1\right).$$

Hence, from Lemma 2.2 and Lemma 2.3, we obtain the inequality (23). □

Letting $\beta \to \infty$ in Theorem 3.4, we have the coefficient bounds for functions belong to the class $\mathcal{K}_s(\alpha)$.

Corollary 3.5. *Let α be a real number such that $0 \le \alpha < 1$ and let the function $f \in \mathcal{A}$ be defined by (1). If $f \in \mathcal{K}_s(\alpha)$, then for any real number μ,*

$$\left|a_3 - \mu a_2^2\right| \le \frac{1}{3} + \frac{2(1-\alpha)}{3} \max\left\{1, \left|1 - \frac{3(1-\alpha)}{2}\mu\right|\right\}.$$

Letting $\alpha = 0$ and $\beta \to \infty$ in Theorem 3.4, we have the coefficient bounds for functions belong to the class $\mathcal{K}_s$.

Corollary 3.6. *Let the function $f \in \mathcal{A}$ be defined by (1). If $f \in \mathcal{K}_s$, then for any real number μ,*

$$\left|a_3 - \mu a_2^2\right| \le \frac{1}{3} + \frac{2}{3} \max\left\{1, \left|1 - \frac{3}{2}\mu\right|\right\}.$$

4. Coefficient estimates for functions in $\mathcal{K}_{\Sigma_s}(\alpha, \beta)$

In this section, we find the upper bounds for initial coefficients of functions belonging to the class $\mathcal{K}_{\Sigma_s}(\alpha, \beta)$ and also solve Fekete-Szegö problem.

Theorem 4.1. *Let α and β be real such that $0 \le \alpha < 1 < \beta$. If a function f given by (1) is in $\mathcal{K}_{\Sigma_s}(\alpha, \beta)$, then*

$$|a_2| \le \frac{(\beta-\alpha)}{\pi} \sin \frac{\pi(1-\alpha)}{\beta-\alpha} \tag{26}$$

and

$$|a_3| \le \frac{1 + \frac{2(\beta-\alpha)}{\pi} \sin \frac{\pi(1-\alpha)}{\beta-\alpha}}{3}. \tag{27}$$

Proof. Let $f \in \mathcal{K}_{\Sigma_s}(\alpha, \beta)$ be given by (1). Then by Definition 4, there exist the functions

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^*(1/2), \qquad G(w) = w + \sum_{n=2}^{\infty} d_n w^n \in \mathcal{S}^*(1/2)$$

satisfying (9) and (10). Firstly, we will re-arrange the relations in (9) and (10) as follows:

$$p(z) = \frac{z^2 f'(z)}{-g(z)g(-z)} = \frac{zf'(z)}{\frac{-g(z)g(-z)}{z}} = \frac{zf'(z)}{\psi(z)} \prec f_{\alpha,\beta}(z) \qquad (z \in \mathbb{U}) \tag{28}$$

and

$$q(w) = \frac{w^2 F'(w)}{-G(w)G(-w)} = \frac{wF'(w)}{\frac{-G(w)G(-w)}{w}} = \frac{wF'(w)}{\Omega(w)} \prec f_{\alpha,\beta}(w) \quad (w \in \mathbb{U}), \tag{29}$$

respectively, where

$$\psi(z) := \frac{-g(z)g(-z)}{z} \qquad \text{and} \qquad \Omega(w) := \frac{-G(w)G(-w)}{w}.$$

Let p and q be two functions with positive real part defined by

$$p(z) := 1 + c_1 z + c_2 z^2 + \cdots$$

and

$$q(w) := 1 + q_1 w + q_2 w^2 + \cdots ,$$

respectively. The relations (28) and (29) imply by Lemma 2.1 that for all $m \in \mathbb{N}$,

$$|c_m| \leq |\varphi_1| \tag{30}$$

and

$$|q_m| \leq |\varphi_1| . \tag{31}$$

Furthermore, by Lemma 1.2, we have the following equations:

$$\psi(z) = \frac{-g(z)g(-z)}{z} := z + \sum_{n=2}^{\infty} B_{2n-1} z^{2n-1} \in \mathcal{S}^* \text{ and } |B_{2n-1}| \leq 1, \tag{32}$$

$$\Omega(w) = \frac{-G(w)G(-w)}{w} := w + \sum_{n=2}^{\infty} D_{2n-1} w^{2n-1} \in \mathcal{S}^* \text{ and } |D_{2n-1}| \leq 1. \tag{33}$$

Now, upon equating the coefficients in (28) and (29), we obtain

$$2a_2 = c_1, \tag{34}$$

$$3a_3 - B_3 = c_2, \tag{35}$$

$$-2a_2 = q_1, \tag{36}$$

$$3(2a_2^2 - a_3) - D_3 = q_2. \tag{37}$$

From (34) and (36), we get

$$c_1 = -q_1$$

and

$$8a_2^2 = c_1^2 + q_1^2. \tag{38}$$

We thus find (by (30), (31), (32) and (33)) that

$$|a_2| \leq \frac{|\varphi_1|}{2}. \tag{39}$$

Further, from the equalities (35) and (37), we find

$$6a_2^2 - B_3 - D_3 = c_2 + q_2. \tag{40}$$

Consequently (by (30), (31), (32) and (33)), we have

$$|a_2| \leq \sqrt{\frac{1 + |\varphi_1|}{3}}. \tag{41}$$

Hence we get the desired result on the coefficient a_2 as asserted in (26) from the inequalities (39) and (41).

Now, in order to obtain the bound on the coefficient a_3, we subtract (37) from (35). We thus get

$$6(a_3 - a_2^2) - B_3 + D_3 = c_2 - q_2$$

or

$$a_3 = a_2^2 + \frac{c_2 - q_2 + B_3 - D_3}{6}. \tag{42}$$

Upon substituting the value of a_2^2 from (38) into (42), it follows that

$$a_3 = \frac{c_1^2 + q_1^2}{8} + \frac{c_2 - q_2 + B_3 - D_3}{6}.$$

We thus find (by (30), (31), (32) and (33)) that

$$|a_3| \leq \frac{|\varphi_1|^2}{4} + \frac{1 + |\varphi_1|}{3}. \tag{43}$$

On the other hand, upon substituting the value of a_2^2 from (40) into (42), it follows that

$$a_3 = \frac{c_2 + q_2 + B_3 + D_3}{6} + \frac{c_2 - q_2 + B_3 - D_3}{6} = \frac{c_2 + B_3}{3}.$$

Consequently (by (30), (31), (32) and (33)), we have

$$|a_3| \leq \frac{1 + |\varphi_1|}{3}. \tag{44}$$

Combining (43) and (44), we get the desired result on the coefficient a_3 as asserted in (27). □

Letting $\beta \to \infty$ in Theorem 4.1, we have the coefficient bounds for functions belonging to the class $\mathcal{K}_{\Sigma_s}(\alpha)$.

Corollary 4.2. *Let α be real such that $0 \leq \alpha < 1$. If a function f given by* (1) *is in $\mathcal{K}_{\Sigma_s}(\alpha)$, then*

$$|a_2| \leq 1 - \alpha$$

and

$$|a_3| \leq \frac{3 - 2\alpha}{3}.$$

Remark 4.3. We note that Corollary 4.2 is an improvement of the estimates obtained by Şeker and Sümer Eker [11, Theorem 3.2].

Letting $\alpha = 0$ and $\beta \to \infty$ in Theorem 4.1, we have the coefficient bounds for functions belonging to the class $\mathcal{K}_{\Sigma_s}$.

Corollary 4.4. *If a function f given by* (1) *is in $\mathcal{K}_{\Sigma_s}$, then*

$$|a_2| \leq 1 \qquad \text{and} \qquad |a_3| \leq 1.$$

Theorem 4.5. *Let α and β be real such that $0 \leq \alpha < 1 < \beta$. If a function f given by* (1) *is in $\mathcal{K}_{\Sigma_s}(\alpha, \beta)$, then for any real number δ,*

$$\left|a_3 - \delta a_2^2\right| \leq \frac{1 + \frac{2(\beta-\alpha)}{\pi} \sin \frac{\pi(1-\alpha)}{\beta-\alpha}}{3} \begin{cases} |1-\delta|, & \delta \in (-\infty, 0] \cup [2, \infty), \\ 1, & \delta \in [0, 2]. \end{cases}$$

Proof. By using the equality (42) in the proof of Theorem 4.1, we obtain

$$a_3 - \delta a_2^2 = (1-\delta) a_2^2 + \frac{c_2 - q_2 + B_3 - D_3}{6}.$$

Upon substituting the value of a_2^2 from (40) into the above equality, it follows that

$$\begin{aligned} a_3 - \delta a_2^2 &= (1-\delta) \frac{c_2 + q_2 + B_3 + D_3}{6} + \frac{c_2 - q_2 + B_3 - D_3}{6} \\ &= \frac{1}{6} \left[(2-\delta)(c_2 + B_3) - \delta(q_2 + D_3)\right]. \end{aligned}$$

Thus by Lemma 2.4, we get desired estimate. $\square$

Letting $\beta \to \infty$ in Theorem 4.5, we have the coefficient bounds for functions belonging to the class $\mathcal{K}_{\Sigma_s}(\alpha)$.

Corollary 4.6. *Let α be real such that $0 \leq \alpha < 1$. If a function f given by* (1) *is in $\mathcal{K}_{\Sigma_s}(\alpha)$, then for any real number δ,*

$$\left|a_3 - \delta a_2^2\right| \leq \frac{3 - 2\alpha}{3} \begin{cases} |1-\delta|, & \delta \in (-\infty, 0] \cup [2, \infty), \\ 1, & \delta \in [0, 2]. \end{cases}$$

Letting $\alpha = 0$ and $\beta \to \infty$ in Theorem 4.5, we have the coefficient bounds for functions belong to the class $\mathcal{K}_{\Sigma_s}$.

Corollary 4.7. *If a function f given by* (1) *is in $\mathcal{K}_{\Sigma_s}$, then for any real number δ,*

$$\left|a_3 - \delta a_2^2\right| \leq \begin{cases} |1-\delta|, & \delta \in (-\infty, 0] \cup [2, \infty), \\ 1, & \delta \in [0, 2]. \end{cases}$$

References

[1] P. L. Duren, *Univalent functions*, Grundlehren der Mathematischen Wissenschaften, 259, Springer-Verlag, New York, 1983.

[2] C. Gao and S. Zhou, *On a class of analytic functions related to the starlike functions*, Kyungpook Math. J. **45** (2005), no. 1, 123–130.

[3] S. P. Goyal and P. Goswami, *On certain properties for a subclass of close-to-convex functions*, J. Class. Anal. **1** (2012), no. 2, 103–112. https://doi.org/10.7153/jca-01-11

[4] S. P. Goyal and O. Singh, *Certain subclasses of close-to-convex functions*, Vietnam J. Math. **42** (2014), no. 1, 53–62. https://doi.org/10.1007/s10013-013-0032-4

[5] F. R. Keogh and E. P. Merkes, *A coefficient inequality for certain classes of analytic functions*, Proc. Amer. Math. Soc. **20** (1969), 8–12. https://doi.org/10.2307/2035949

[6] J. Kowalczyk and E. Leś-Bomba, *On a subclass of close-to-convex functions*, Appl. Math. Lett. **23** (2010), no. 10, 1147–1151. https://doi.org/10.1016/j.aml.2010.03.004
[7] K. Kuroki and S. Owa, *Notes on new class for certain analytic functions*, RIMS Kokyuroku **1772** (2011), 21–25.
[8] M. Lewin, *On a coefficient problem for bi-univalent functions*, Proc. Amer. Math. Soc. **18** (1967), 63–68. https://doi.org/10.2307/2035225
[9] V. Ravichandran, A. Gangadharan, and M. Darus, *Fekete-Szegő inequality for certain class of Bazilevic functions*, Far East J. Math. Sci. (FJMS) **15** (2004), no. 2, 171–180.
[10] W. Rogosinski, *On the coefficients of subordinate functions*, Proc. London Math. Soc. (2) **48** (1943), 48–82. https://doi.org/10.1112/plms/s2-48.1.48
[11] B. Şeker and S. S. Eker, *On subclasses of bi-close-to-convex functions related to the odd-starlike functions*, Palest. J. Math. **6** (2017), Special Issue II, 215–221.
[12] B. A. Uralegaddi, M. D. Ganigi, and S. M. Sarangi, *Univalent functions with positive coefficients*, Tamkang J. Math. **25** (1994), no. 3, 225–230.
[13] Z.-G. Wang and D.-Z. Chen, *On a subclass of close-to-convex functions*, Hacet. J. Math. Stat. **38** (2009), no. 2, 95–101.
[14] Z. Wang, C. Gao, and S. Yuan, *On certain subclass of close-to-convex functions*, Acta Math. Acad. Paedagog. Nyházi. (N.S.) **22** (2006), no. 2, 171–177.
[15] Q.-H. Xu, H. M. Srivastava, and Z. Li, *A certain subclass of analytic and close-to-convex functions*, Appl. Math. Lett. **24** (2011), no. 3, 396–401. https://doi.org/10.1016/j.aml.2010.10.037
[16] P. Zaprawa, *Estimates of initial coefficients for bi-univalent functions*, Abstr. Appl. Anal. **2014** (2014), Art. ID 357480, 6 pp. https://doi.org/10.1155/2014/357480

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