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RELATIONSHIP BETWEEN THE STRUCTURE OF A QUOTIENT RING AND THE BEHAVIOR OF CERTAIN ADDITIVE MAPPINGS

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Abstract. The principal aim of this paper is to study the connection between the structure of a quotient ring R/P and the behavior of special additive mappings of R. More precisely, we characterize the commutativity of R/P using derivations (generalized derivations) of R satisfying algebraic identities involving the prime ideal P. Furthermore, we provide examples to show that the various restrictions imposed in the hypothesis of our theorems are not superfluous.

1. Introduction

Throughout this article R will represent an associative ring with center $Z(R)$. The symbols $x \circ y$ and $[x, y]$, where $x, y \in R$, stand for the anti-commutator $xy + yx$ and commutator $xy - yx$, respectively. A proper ideal P of a ring R is said to be prime if for any $a, b \in R$, whenever $aRb \subseteq P$ implies $a \in P$ or $b \in P$. An ideal P of R is minimal if P does not include any proper ideal of R. The ring R is a prime ring if and only if (0) is a prime ideal of R. Let a mapping $d : R \to R$ defined as $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. If d is an additive mapping, then d is said to be a derivation on R. The notion of a generalized derivation was introduced by Brešar in [9]. More precisely, an additive mapping $F : R \to R$ is said to be a generalized derivation if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. For $a, b \in R$, the mapping $F : R \to R$ defined by $F(x) = ax + xb$ for all $x \in R$ is an example of a generalized derivation on R, which is called the inner generalized derivation of R. It is obvious that every derivation is a generalized derivation but the converse is not generally true. Many results in the literature indicate how the global structure of a ring R is often tightly connected to the behaviour of additive mappings defined on R (for example, see [10, 12–14, 16]). A well known result due to Posner [18] states that if d is a derivation of a

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prime ring R such that $[d(x), x] \in Z(R)$ for all $x \in R$, then either $d = 0$ or R is commutative. In [11] Lanski generalized the result of Posner by considering a derivation d such that $[d(x), x] \in Z(R)$ for all x in a nonzero Lie ideal U of R. A number of authors have extended the theorem of Posner in serval ways (for example, see [11] and [17]).

In [4], Ashraf and Rehman proved that if R is a prime ring with a nonzero ideal I of R and d is a derivation of R such that $d(x) \circ d(y) = x \circ y$ for all $x, y \in I$, then R is commutative. Moreover, if d is nonzero and $d(x) \circ d(y) = 0$ for all $x, y \in I$, then R is commutative. Recently, H. E. Bell and Nadeem-Ur Rehman [8] have studied the situation by replacing the derivation d with a generalized derivation F. More precisely, they proved that if R is a prime ring with 1 and $char(R) \neq 2$ such that $F(x) \circ F(y) = 0$ for all $x, y \in R$, then $F = 0$ where, F is a generalized derivation of R associated with a nonzero derivation d. In this line of investigation, Asma Ali et al. [5] have studied the following situations: If R is a 2-torsion free prime ring, U a nonzero Lie ideal of R and $u^2 \in U$, for all $u \in U$ such that d is a derivation of R which acts as an homomorphism or an anti-homomorphism on U, then either $d = 0$ or $U \subseteq Z(R)$. Recently, Rehman [15] studied the above mentioned results of Asma Ali et al. for prime ring with nonzero generalized derivation F. More specifically, he proved that, if R is a 2-torsion free prime ring, $I \neq 0$ an ideal of R and F a nonzero generalized derivation of R with a nonzero derivation d such that either $F(xy) = F(x)F(y)$ or $F(xy) = F(y)F(x)$ for all $x, y \in I$, then R is commutative. Many related generalizations of these results can be found in the literature (see for instance [6] and [1], where further references can be found).

The purpose of the present paper is to continue this line of investigation and study the structure of a quotient ring R/P admitting specific algebraic identities defined in the ring R. Moreover, we consider a more general concept rather than the ring R is prime or semi-prime in the hypothesis of our theorems.

2. Generalized derivations involving prime ideals

The following lemma is very crucial for developing the proof of our main results.

Lemma 2.1 ([2, Theorem 2.2]). *Let R be a ring and P be a prime ideal of R. If d is a derivation of R satisfying $\overline{[d(x), x]} \in Z(R/P)$ for all $x \in R$, then either $d(R) \subseteq P$ or R/P is a commutative integral domain.*

A well known result of Posner [18] states that the existence of a derivation d of a prime ring R such that $[d(x), x] \in Z(R)$ for all $x \in R$, forces that either $d = 0$ or R is commutative.

Inspired by the above result, our aim in the following theorem is to study the case when the generalized derivations satisfies some conditions involving anti-commutators (instead of commutators). More specifically, we will treat the special identity $F(x) \circ x$ belongs to the center of a quotient ring. Indeed, our results are of more specific interest because we will characterize not only the structure of the ring R/P but we will also prove that the generalized derivation F has its rang in the prime ideal P.

Theorem 2.2. *Let R be a ring and P a prime ideal of R such that R/P is 2-torsion free. If R admits a generalized derivation F associated with a derivation d such that $\overline{F(x)\circ x} \in Z(R/P)$ for all $x \in R$, then $F(R) \subseteq P$ or R/P is a commutative integral domain.*

Proof. We are given that

$$\overline{F(x)\circ x} \in Z(R/P) \quad \text{for all } \ x \in R. \tag{2.1}$$

If $Z(R/P) = \{\overline{0}\}$, then R/P is non-commutative and the relation (2.1) reduces to

$$F(x)\circ x \in P \quad \text{for all } \ x \in R.$$

Linearizing the above expression, we get

$$F(x)\circ y + F(y)\circ x \in P \quad \text{for all } \ x, y \in R. \tag{2.2}$$

Substituting yx for y in (2.2), we find that

$$-y[F(x), x] + y[d(x), x] + (y \circ x)d(x) \in P \quad \text{for all } \ x, y \in R. \tag{2.3}$$

Replacing y by ry in (2.3), we obviously see that

$$[x, r]yd(x) \in P \quad \text{for all } \ r, x, y \in R$$

which implies that

$$[x, r]Rd(x) \subseteq P \quad \text{for all } \ r, x \in R. \tag{2.4}$$

According to the primeness of P, we get $[x, R] \subseteq P$ or $d(x) \in P$ for all $x \in R$. The sets of x for which theses conditions hold are additive subgroups of R with union equal to R; using Brauer's trick we conclude that $d(R) \subseteq P$. Hence (2.3) yields $y[x, F(x)] \in P$. Thus, $[x, F(x)] \in P$ for all $x \in R$ (since a prime ideal is proper). In view of the hypothesis we find that $2F(x)x \in P$. Applying 2-torsion freeness, we get $F(x)x \in P$ for all $x \in R$. A linearization of the preceding relation gives

$$F(x)y + F(y)x \in P \quad \text{for all } \ x, y \in R. \tag{2.5}$$

Replacing y by yr in (2.5) and combining it with the above expression, we may write

$$F(y)[x, r] \in P \quad \text{for all } \ r, x, y \in R \tag{2.6}$$

thereby obtaining

$$F(y)R[x, r] \subseteq P \quad \text{for all } \ r, x, y \in R. \tag{2.7}$$

Once again invoking the primeness of P, we conclude that either $F(R) \subseteq P$ or R/P is an integral domain, contrary to our initial hypothesis; hence $F(R) \subseteq P$. Now if $Z(R/P) \neq \{\overline{0}\}$, then for $\overline{z} \in Z(R/P) \setminus \{\overline{0}\}$, take $x = z$ in (2.1), and by appropriate expansion, obtain $\overline{F(z)} \in Z(R/P)$.

On the other hand, linearizing equation (2.1), we arrive at

$$2\overline{(F(x)z + F(z)x)} \in Z(R/P) \quad \text{for all } x \in R \tag{2.8}$$

in such a way that

$$\overline{[F(x)z + F(z)x, r]} = \overline{0} \quad \text{for all } r, x \in R. \tag{2.9}$$

Writing xr instead of x in (2.9) and using it, we then get

$$\overline{[xd(r)z, r]} = \overline{0} \quad \text{for all } r, x \in R. \tag{2.10}$$

Substituting tx for x in (2.10), we arrive at

$$\overline{[t, r]xd(r)z} = \overline{0} \quad \text{for all } r, t, x \in R \tag{2.11}$$

which implies that $[t, r]Rd(r) \subseteq P$ for all $r, t \in R$. So that R/P is an integral domain or $d(R) \subseteq P$. In the latter case, once again linearizing (2.1), we thereby obtain

$$\overline{F(x) \circ y + F(y) \circ x} \in Z(R/P) \quad \text{for all } x, y \in R. \tag{2.12}$$

Substituting yx for y in (2.12), we obviously get

$$\overline{[y[F(x), x], x]} = \overline{0} \quad \text{for all } x, y \in R. \tag{2.13}$$

If we write $F(x)y$ instead of y in (2.13), then it follows that $\overline{[F(x), x]} = \overline{0}$. In particular $\overline{[F(x), x]} \in Z(R/P)$ for all $x \in R$. Invoking again the hypothesis, it is obvious to see that $\overline{F(x)x} \in Z(R/P)$. Arguing as above, we are forced to conclude that $\overline{[F(y)[x, r], r]} = \overline{0}$ for all $r, x, y \in R$. Accordingly, putting tx instead of x and using it, we find that

$$\overline{F(y)[x, r][t, r]} = \overline{0} \quad \text{for all } r, t, x, y \in R.$$

It now follows from the above expression that

$$F(y)R[x, r]R[t, r] \subseteq P \quad \text{for all } r, t, x, y \in R.$$

Hence, $F(R) \subseteq P$ or R/P is commutative which completes the proof of our theorem. □

The following corollary is an immediate consequence of the preceding theorem.

Corollary 2.3. *Let R be a 2-torsion free prime ring. If R admits a nonzero generalized derivation F associated with a derivation d, then the following assertions are equivalent:*

(1) $F(x) \circ x \in Z(R)$ *for all* $x \in R$;
(2) $F(x) \circ x + d(x) \circ x \in Z(R)$ *for all* $x \in R$;
(3) $F(x) \circ x - d(x) \circ x \in Z(R)$ *for all* $x \in R$;
(4) R *is a commutative integral domain.*

The next proposition extends Corollary 2.3 to semi-prime rings.

Proposition 2.4. *Let R be a semi-prime ring. If R admits a nonzero generalized derivation F associated with a derivation d satisfying any one of the following conditions*:

(1) $F(x) \circ x \in Z(R)$ *for all* $x \in R$;
(2) $F(x) \circ x + d(x) \circ x \in Z(R)$ *for all* $x \in R$;
(3) $F(x) \circ x - d(x) \circ x \in Z(R)$ *for all* $x \in R$;

then either R is commutative or $\big($*there exists a minimal prime ideal P of R such that* $char(R/P) = 2$ *or* $F(R) \subseteq P\big)$.

Proof. (1) Suppose on the contrary that $char(R/P) \neq 2$ and $F(R) \not\subseteq P$ for any minimal prime ideal P of R such that $F(x) \circ x \in Z(R)$ for all $x \in R$, then

$$[F(x) \circ x, y] = 0$$

for all $x, y \in R$. According to semi-primeness, there exists a family $\mathcal{P}$ of prime ideals P such that $\bigcap_{P \in \mathcal{P}} P = (0)$ and therefore $[F(x) \circ x, y] \in P$ for all $P \in \mathcal{P}$. That is

$$\overline{F(x) \circ x} \in Z(R/P) \quad \text{for all } \ x \in R \ \text{ and } \ \text{for all } \ P \in \mathcal{P}.$$

Accordingly, Theorem 2.2 yields that R/P is commutative. However, for all $x, y \in R$ we get $[x, y] \in P$ (for all $P \in \mathcal{P}$) so that $[x, y] = 0$ proving that R is commutative.

(2) Now if $F(x) \circ x + d(x) \circ x \in Z(R)$ for all $x \in R$ or $F(x) \circ x - d(x) \circ x \in Z(R)$ for all $x \in R$, then using the same techniques as used above with slight modifications we get the required result. □

In 2011 Ashraf and Almas Khan [3] showed that if a 2-torsion free $*$-prime ring R with a $*$-Lie ideal U such that $F[u, v] = [F(u), v]$ or $F(u \circ v) = F(u) \circ v$ for all $u, v \in U$; where F is a generalized derivation associated with a nonzero derivation d, then $U \subseteq Z(R)$.

Motivated by the above results, our aim in the following theorem is to investigate a more general context of differential identities involving a prime ideal by omitting the primeness (semi-primeness) assumption imposed on the ring R and with no further assumption on the characteristic of R. Indeed, we will prove the following result.

Theorem 2.5. *Let R be a ring and P a prime ideal of R. If R admits a generalized derivation F associated with a derivation d satisfying any one of the following conditions*:

(1) $\overline{F[x, y] - [F(x), y]} \in Z(R/P)$ *for all* $x, y \in R$;
(2) $\overline{F(x \circ y) - F(x) \circ y} \in Z(R/P)$ *for all* $x, y \in R$;

then $d(R) \subseteq P$ *or R/P is a commutative integral domain.*

Proof. (1) By given assumption, we have

$$\overline{F[x, y] - [F(x), y]} \in Z(R/P) \quad \text{for all } \ x, y \in R. \tag{2.14}$$

Replacing x by xr in (2.14), we obtain

$$\overline{F\big([x,y]r+x[r,y]\big)-[F(x),y]r-F(x)[r,y]-[xd(r),y]}\in Z(R/P)$$

for all $r,x,y\in R$ by expanding the above equation, we get

$$\big[[x,y]d(r)+xd[r,y]-[xd(r),y],r\big]\in P \quad \text{for all } r,x,y\in R.$$

This can be rewritten as

$$\big[x[r,d(y)],r\big]\in P \quad \text{for all } r,x,y\in R. \tag{2.15}$$

Substituting $d(y)x$ for x in (2.15), we find that

$$[r,d(y)]x[r,d(y)]\in P \quad \text{for all } r,x,y\in R.$$

The primeness of P, leads to that $[r,d(y)]\in P$ for all $r,y\in R$. As a special case of the last equation, we may write that $\overline{[r,d(r)]}\in Z(R/P)$ for all $r\in R$. By view of Lemma 2.1, we conclude that $d(R)\subseteq P$ or R/P is a commutative integral domain.

(2) Now if we consider

$$\overline{F(x\circ y)-F(x)\circ y}\in Z(R/P) \quad \text{for all } x,y\in R$$

then proceeding as in (1) with necessary variations, we arrive at $d(R)\subseteq P$ or R/P is a commutative integral domain. □

As an application of Theorem 2.5, we get the following result.

Corollary 2.6. *Let R be a prime ring. If R admits a generalized derivation F associated with a nonzero derivation d, then the following assertions are equivalent*:

(1) $F[x,y]-[F(x),y]\in Z(R)$ *for all* $x,y\in R$;
(2) $F(x\circ y)-F(x)\circ y\in Z(R)$ *for all* $x,y\in R$;
(3) *R is a commutative integral domain.*

Proposition 2.7. *Let R be a semi-prime ring. If R admits a generalized derivation F associated with a nonzero derivation d satisfying any one of the following conditions*:

(1) $F[x,y]-[F(x),y]\in Z(R)$ *for all* $x,y\in R$;
(2) $F(x\circ y)-F(x)\circ y\in Z(R)$ *for all* $x,y\in R$;

then either R is commutative or $d(R)\subseteq P$ for some minimal prime ideal P.

3. Some special derivations

In [6], Bell and Kappe studied derivations acting as an homomorphism and an anti-homomorphism on a nonempty subset of a ring R. They proved that if R is a prime ring, U a nonzero right ideal of R and d a derivation of R which acts as an homomorphism or (an anti-homomorphism) on U, i.e., $d(xy)-d(x)d(y)=0$ for all $x,y\in U$ $\big($resp. $d(xy)-d(y)d(x)=0$ for all $x,y\in U\big)$, then $d=0$ on R. Further Asma Ali et al. [5, Theorem 3.1] extended this results to a Lie ideal.

Motivated by the above results, our next aim is to suggest a more general situation by considering differential identities involving two derivations d and g satisfying $d(x)d(y) \pm g(yx)$ belongs to the center of a quotient ring.

Theorem 3.1. *Let R be a ring and P a prime ideal of R. If R admits two derivations d and g satisfying any one of the following conditions*:
(1) $\overline{d(x)d(y) - g(yx)} \in Z(R/P)$ *for all* $x, y \in R$;
(2) $\overline{d(x)d(y) + g(yx)} \in Z(R/P)$ *for all* $x, y \in R$;
then, $\big(d(R) \subseteq P$ *and* $g(R) \subseteq P\big)$ *or* R/P *is a commutative integral domain.*

Proof. (1) Suppose that

$$\overline{d(x)d(y) - g(yx)} \in Z(R/P) \quad \text{for all } x, y \in R. \tag{3.1}$$

If $Z(R/P) = \{\overline{0}\}$, then the hypothesis reduces to

$$d(x)d(y) - g(yx) \in P \quad \text{for all } x, y \in R. \tag{3.2}$$

Replacing y by yr in (3.2), and using (3.2), we find that

$$d(x)yd(r) + g(y)[x, r] + y[g(x), r] - yg(r)x \in P \quad \text{for all } r, x, y \in R. \tag{3.3}$$

Substituting rx for r in (3.3), we obviously get

$$\begin{aligned} &d(x)yd(r)x + d(x)yrd(x) + g(y)[x, r]x + y[g(x), r]x \\ &+ yr[g(x), x] - yg(r)x^2 - yrg(x)x \in P. \end{aligned} \tag{3.4}$$

Using (3.3) together with (3.4), we arrive at

$$d(x)yrd(x) - yrxg(x) \in P \quad \text{for all } r, x, y \in R. \tag{3.5}$$

Left multiplying the above expression by t and subtracting from (3.5), it follows that

$$[d(x), t]yrd(x) \in P \quad \text{for all } r, t, x, y \in R. \tag{3.6}$$

Writing $yd(x)$ instead of y in (3.6), we obtain

$$[d(x), t]yd(x)rd(x) \in P \quad \text{for all } r, t, x, y \in R.$$

In particular,

$$[d(x), t]R[d(x), t]R[d(x), t] \subseteq P \quad \text{for all } t, x \in R. \tag{3.7}$$

The primeness of P assures that, $[d(x), t] \in P$ for all $t, x \in R$. Hence Lemma 2.1 proving that $d(R) \subseteq P$. In this case the relation (3.2) reduces to $g(yx) \in P$ for all $x, y \in R$. If we write xr for x, then we get $yxg(r) \in P$ for all $r, x, y \in R$. Letting $y = g(r)$, we have $g(r)Rg(r) \subseteq P$ for all $r \in R$. Hence, it follows that $g(R) \subseteq P$.

Now suppose that $Z(R/P) \neq \{\overline{0}\}$, then there exists $\overline{z}(\neq \overline{0}) \in Z(R/P)$; substituting yz for y in (3.1), one can easily verify that

$$\overline{d(x)yd(z) - y[g(z), x] - yxg(z)} \in Z(R/P) \quad \text{for all } x, y \in R. \tag{3.8}$$

Putting ry instead of y in (3.8), we get

$$\overline{\big[[d(x),r]yd(z),r\big]} = \overline{0} \quad \text{for all } r,x,y \in R \tag{3.9}$$

which may be restated as

$$\overline{[d(x),r]y[d(z),r]} + \overline{\big[[d(x),r]y,r\big]d(z)} = \overline{0} \quad \text{for all } r,x,y \in R. \tag{3.10}$$

Replacing y by $yd(z)$ in (3.10), it is obvious to see that

$$\overline{[d(x),r]yd(z)[d(z),r]} = \overline{0} \quad \text{for all } r,x,y \in R.$$

In light of the primeness of P, we get for each $r \in R$ either $[d(x),r] \in P$ or $d(z)[d(z),r] \in P$. Let us set $H = \{r \in R \,/\, [d(x),r] \in P \text{ for all } x \in R\}$ and $K = \{r \in R \,/\, d(z)[d(z),r] \in P\}$. Then it can be seen that H and K are two additives subgroups of R whose union is R. Using Brauer's trick we have either $R = H$ or $R = K$.

Assume that $R = K$, then $d(z)[d(z),r] \in P$. Accordingly $[d(z),r]R[d(z),r] \subseteq P$, by virtue of the primeness, we can see that $\overline{d(z)} \in Z(R/P)$. In this case, (3.9) becomes $\overline{\big[[d(x),r]y,r\big]d(z)} = \overline{0}$ for all $r,x,y \in R$. Therefore, $\overline{d(z)} = \overline{0}$ or $[d(x),r] \in P$. By the first case, the hypothesis leads to that $\overline{g(z)x + zg(x)} \in Z(R/P)$ for all $x \in R$, now if we put xr instead of x, then we find that $\overline{[x,r]xg(r)z} = \overline{0}$. So that $[R,R] \subseteq P$ or $g(R) \subseteq P$. On the other hand, if $g(R) \subseteq P$, then expanding the expression (3.1), we are forced to get R/P is commutative or $d(R) \subseteq P$.

(2) Suppose that $\overline{d(x)d(y) + g(yx)} \in Z(R/P)$ for all $x,y \in R$, since $(-g)$ is also a derivation of R. Then, we have by assertion (1) either $\big(d(R) \subseteq P$ and $g(R) \subseteq P\big)$ or R/P is an integral domain. □

As an application of Theorem 3.1, we obtain the following corollary which constitutes an improved version of [5, Theorem 1.2].

Corollary 3.2. *Let R be a prime ring. If R admits two derivations d and g such that either d or g is nonzero, then the following assertions are equivalent*:

(1) $d(x)d(y) - g(yx) \in Z(R)$ *for all* $x,y \in R$;
(2) $d(x)d(y) + g(yx) \in Z(R)$ *for all* $x,y \in R$;
(3) R *is a commutative integral domain.*

The following proposition gives a generalization of Bell and Kappe's result.

Proposition 3.3. *Let d and g be derivations of a semi-prime ring R. If $d(x)d(y) \pm g(yx) = 0$ for all $x,y \in R$; then either $d = g = 0$ or R contains a nonzero central ideal.*

Proof. Assume that $d(x)d(y) \pm g(yx) = 0$ for all $x,y \in R$. By view of the semi-primeness of the ring R, there exists a family Γ of prime ideals such that $\bigcap_{P\in\Gamma} P = (0)$, thereby obtaining $d(x)d(y) \pm g(yx) \in P$ for all $P \in \Gamma$. Invoking the proof of Theorem 3.1, which in view of (3.7), reduces to

$$[d(x),x]R[d(x),x]R[d(x),x] \subseteq P \quad \text{for all } x \in R \text{ and for all } P \in \Gamma.$$

Therefore, one can see that

$$[d(x), x] \in \bigcap_{P \in \Gamma} P = (0) \quad \text{for all } \ x \in R.$$

Applying [7, Theorem 3], it follows that $d = 0$ or R contains a nonzero central ideal. In the first case, our hypothesis reduces to $g(yx) = 0$ for all $x, y \in R$. Replacing x by xr, we deduce that $yxg(r) = 0$ so that $g = 0$. □

In [4] Ashraf and Rehman proved that, if R is a 2-torsion free prime ring , I is a nonzero ideal of R and d is a derivation of R such that $d(x) \circ d(y) = x \circ y$ for all $x, y \in I$; then R is commutative. In fact, this result is false because in the particular case when $d = 0$, we get $R = \{0\}$, which is a contradiction.

The fundamental aim of the next theorem is to establish a generalization of the above result by investigating the behavior of the more general expressions. More precisely, we will treat the following special identities:

(i) $\overline{d(x) \circ d(y) - g(x) \circ y} \in Z(R/P)$ for all $x, y \in R$;

(ii) $\overline{d(x) \circ d(y) + g(x) \circ y} \in Z(R/P)$ for all $x, y \in R$.

Theorem 3.4. *Let R be a ring and P a prime ideal of R such that R/P is a 2-torsion free. If R admits two derivations d and g satisfying any one of the following conditions*:

(1) $\overline{d(x) \circ d(y) - g(x) \circ y} \in Z(R/P)$ *for all* $x, y \in R$;

(2) $\overline{d(x) \circ d(y) + g(x) \circ y} \in Z(R/P)$ *for all* $x, y \in R$;

then, $\big(d(R) \subseteq P$ *and* $g(R) \subseteq P\big)$ *or R/P is a commutative integral domain.*

Proof. We have only to prove assertion (1), while the assertion (2) can be proved similarly.

(1) Assuming that

$$\overline{d(x) \circ d(y) - g(x) \circ y} \in Z(R/P) \quad \text{for all } \ x, y \in R. \tag{3.11}$$

If $Z(R/P) = \{\overline{0}\}$, then the expression (3.11) becomes

$$d(x) \circ d(y) - g(x) \circ y \in P \quad \text{for all } \ x, y \in R. \tag{3.12}$$

Replacing y by yr in (3.12) and applying it, we obtain

$$-d(y)[d(x), r] + [d(x), y]d(r) + y\big(d(x) \circ d(r)\big) + y[g(x), r] \in P \quad \text{for all } \ r, x, y \in R. \tag{3.13}$$

Putting ty instead of y in (3.13), and subtracting it from (3.13), we find that

$$-d(t)y[d(x), r] + [d(x), t]yd(r) \in P \quad \text{for all } \ r, t, x, y \in R. \tag{3.14}$$

Taking $r = d(x)$ in (3.14), we obviously get

$$[d(x), t]yd^2(x) \in P \quad \text{for all } \ t, x, y \in R.$$

Using the primeness of P, we deduce that either $[d(x), t] \in P$ or $d^2(x) \in P$ for all $x \in R$. Clearly $R = R_1 \cup R_2$ with $R_1 = \{x \in R \ / \ [d(x), t] \in P$ for all $t \in R\}$ and $R_2 = \{x \in R \ / \ d^2(x) \in P\}$. Since a group cannot be union of its subgroups then $R = R_1$ in which, one can see from Lemma 2.1 that $d(R) \subseteq P$. Now if

$R = R_2$, i.e., $d^2(x) \in P$ for all $x \in R$, then replacing x by xy, one can verify that $2d(x)d(y) \in P$ for all $x, y \in R$. Substituting xr for x in the last expression, we get $2d(x)rd(y) \in P$ for all $r, x, y \in R$. In view of 2-torsion freeness, we may conclude that $d(R) \subseteq P$. On the other hand, the equation (3.12) forces that $g(x) \circ y \in P$ for all $x, y \in R$. Writing xy instead of x in this relation, we get $[y, x]g(y) \in P$ and thus by putting rx instead of x, we obtain $[y, r]Rg(y) \subseteq P$ for all $r, y \in R$. So that, we have necessarily $g(R) \subseteq P$. However, $d(R) \subseteq P$ and $g(R) \subseteq P$.

Analogously, if $Z(R/P) \neq \{\overline{0}\}$, then writing yz instead of y in (3.11), where $\overline{z} \in Z(R/P) \setminus \{\overline{0}\}$, we find that

$$\overline{\big(d(x) \circ d(y) - g(x) \circ y\big)z + y(d(x) \circ d(z)) + [d(x), y]d(z)} \in Z(R/P)$$

this may be restated as

$$\overline{y(d(x) \circ d(z)) + [d(x), y]d(z)} \in Z(R/P). \tag{3.15}$$

Now replacing y by ry in (3.15), one can verify that

$$\overline{\big[[d(x), r]yd(z), r\big]} = \overline{0} \quad \text{for all} \quad r, x, y \in R. \tag{3.16}$$

Since (3.16) is the same as (3.9), reasoning in the same manner as in the proof of Theorem 3.1, we arrive at R/P is commutative or $d(R) \subseteq P$ or $\overline{d(z)} = \overline{0}$. By the last case taking $y = z$ in our hypothesis; and expanding it, one can verify that $\overline{g(x)} \in Z(R/P)$ for all $x \in R$. In light of Lemma 2.1, we conclude that $g(R) \subseteq P$ or R/P is an integral domain. On the other hand, if $d(R) \subseteq P$ then by developing equation (3.11), we get $g(R) \subseteq P$ or R/P is commutative. Consequently, it follows that either $\big(d(R) \subseteq P$ and $g(R) \subseteq P\big)$ or R/P is a commutative integral domain. □

Applying Theorem 3.4, we get an improved result of Ashraf and Rehman as follows:

Corollary 3.5. *Let R be a 2-torsion free prime ring. If R admits two derivations d and g such that either d or g is nonzero, then the following assertions are equivalent*:

(1) $d(x) \circ d(y) - g(x) \circ y \in Z(R)$ *for all* $x, y \in R$;
(2) $d(x) \circ d(y) + g(x) \circ y \in Z(R)$ *for all* $x, y \in R$;
(3) *R is a commutative integral domain.*

The following example proves that the condition "R/P is 2-torsion free" in Theorems 2.2 and 3.4 is necessary.

Example. (1) Let us consider $R = M_2(\mathbb{Z}/2\mathbb{Z})$ and $P = \{0\}$, it is straightforward to check that R/P is a prime ring with $char(R/P) = 2$. Moreover, if we take the generalized derivation defined by $F = id_R$, where id_R denote the identity map defined on R by $id_R(r) = r$ for all $r \in R$. Then, we have $\overline{F(X) \circ X} \in Z(R/P)$ for all $X \in R$. Hence F satisfies the condition of Theorem 2.2, but R/P is not commutative.

(2) Let us consider R and P as in the preceding example. Furthermore, we define the derivations d and g by

$$d\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix} \text{ and } g\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0_R$$

then it is obvious to verify that d and g satisfying the condition of Theorem 3.4. However, R/P is a non commutative ring.

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