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ON THE CONFORMAL TRIHARMONIC MAPS

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Abstract. In this paper, we give the necessary and sufficient condition for the conformal mapping $\phi : (\mathbb{R}^n, g_0) \to (N^n, h)$ $(n \geq 3)$ to be triharmonic where we prove that the gradient of its dilation is a solution of a fourth-order elliptic partial differential equation. We construct some examples of triharmonic maps which are not biharmonic and we calculate the trace of the stress-energy tensor associated with the triharmonic maps.

1. Introduction

Let (M^m, g) and (N^n, h) be smooth Riemannian manifolds with M compact. A smooth map $\phi : (M^m, g) \longrightarrow (N^n, h)$ is called harmonic if it is a critical point of the energy functional:

$$E(\phi) = \frac{1}{2} \int_M |d\phi|^2 \, dv_g,$$

where $|d\phi|$ is the Hilbert-Schmidt norm of ϕ. Equivalently, ϕ is harmonic if it satisfies the Euler-Lagrange equations:

$$\tau(\phi) = Tr_g \nabla d\phi = 0,$$

where $\tau(\phi)$ is the tension field of ϕ. A map $\phi : (M^m, g) \longrightarrow (N^n, h)$ is called biharmonic if it is a critical point of the bi-energy functional:

$$E_2(\phi) = \frac{1}{2} \int_M |\tau(\phi)|^2 \, dv_g.$$

The corresponding Euler-Lagrange equations are the fourth-order elliptic system:

$$\tau_2(\phi) = -Tr_g \left(\nabla^\phi\right)^2 \tau(\phi) - Tr_g R^N(\tau(\phi), d\phi)d\phi = 0,$$

where R^N is the Riemannian curvature on N and ∇^ϕ is the connection in the pull-back bundle $\phi^{-1}(TN)$ and, if $(e_i)_{1 \leq i \leq m}$ is a local orthonormal frame field

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on M, then

$$Tr_g \left(\nabla^{\phi}\right)^2 \tau(\phi) = \left(\nabla^{\phi}_{e_i} \nabla^{\phi}_{e_i} - \nabla^{\phi}_{\nabla^M_{e_i} e_i}\right) \tau(\phi),$$

where we sum over repeated indices. We will call the operator $\tau_2(\phi)$, the bi-tension field of the map ϕ. A generalization of harmonic and biharmonic maps are polyharmonic maps. Such maps are critical points of the following functional

$$E_{2s}(\phi) = \int_M \left|\Delta^{s-1}\tau(\phi)\right|^2 dv_g, \tag{1.1}$$

and

$$E_{2s+1}(\phi) = \int_M \left|\nabla\Delta^{s-1}\tau(\phi)\right|^2 dv_g, \tag{1.2}$$

where $s = 1, 2, \ldots$ is a natural number. The first variation of E_k is developed in [9]. The associated Euler-Lagrange equation is given by

(1) If $k = 2s, s = 1, 2, \ldots$, the critical points of (1.1) are given by

$$\begin{aligned} \tau_{2s}(\phi) &= \Delta^{2s-1}\tau(\phi) - R^N\left(\Delta^{2s-2}\tau(\phi), d\phi(e_i)\right) d\phi(e_i) \\ &+ \sum_{l=1}^{s-1} R^N\left(\Delta^{s-l-1}\tau(\phi), \nabla_{e_i}\Delta^{s+l-2}\tau(\phi)\right) d\phi(e_i) \\ &- \sum_{l=1}^{s-1} R^N\left(\nabla_{e_i}\Delta^{s-l-1}\tau(\phi), \Delta^{s+l-2}\tau(\phi)\right) d\phi(e_i) = 0. \end{aligned} \tag{1.3}$$

(2) If $k = 2s+1, s = 0, 1, 2, \ldots$, the critical points of (1.2) are given by

$$\begin{aligned} \tau_{2s+1}(\phi) &= \Delta^{2s}\tau(\phi) - R^N\left(\Delta^{2s-1}\tau(\phi), d\phi(e_i)\right) d\phi(e_i) \\ &+ \sum_{l=1}^{s-1} R^N\left(\nabla_{e_i}\Delta^{s+l-1}\tau(\phi), \Delta^{s-l-1}\tau(\phi)\right) d\phi(e_i) \\ &- \sum_{l=1}^{s-1} R^N\left(\Delta^{s+l-1}\tau(\phi), \nabla_{e_i}\Delta^{s-l-1}\tau(\phi)\right) d\phi(e_i) \\ &- R^N\left(\nabla_{e_i}\Delta^{s-1}\tau(\phi), \Delta^{s-1}\tau(\phi)\right) d\phi(e_i) = 0. \end{aligned} \tag{1.4}$$

Where $(e_i)_{1\leq i\leq m}$ is an orthonormal frame on M,

$$\Delta\tau(\phi) = -Tr_g\left(\nabla^{\phi}\right)^2 \tau(\phi) = -\left(\nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i}\tau(\phi) - \nabla^{\phi}_{\nabla_{e_i} e_i}\tau(\phi)\right)$$

and

$$\Delta^k\tau(\phi) = \Delta\left(\Delta^{k-1}\tau(\phi)\right), \qquad \Delta^{-1} = 0.$$

There are several works concerning the construction of polyhamonic maps, see for example [4], [5], [8], [10] and [11]. Following Jiang [6], we define the stress-energy tensor associated with the functional E_k by taking variations of the domain metric. More precisely let $\phi : (M^m, g) \to (N^n, h)$ be a smooth map and, for any $X, Y \in \Gamma(TM)$, we have

(1) The stress-energy tensor for polyharmonic maps of even order is defined by

$$
\begin{aligned}
& S_{2s}(\phi)(X,Y) \\
= & \left(\frac{1}{2}\left|\Delta^{s-1}\tau(\phi)\right|^{2} - h\left(\Delta^{2s-2}\tau(\phi),\tau(\phi)\right)\right) g(X,Y) \\
& - \left(Tr_g h\left(\nabla\Delta^{2s-2}\tau(\phi), d\phi\right)\right) g(X,Y) \\
& + \left(\sum_{l=1}^{s-1} Tr_g h\left(\nabla\Delta^{s-l-1}\tau(\phi), \nabla\Delta^{s+l-2}\tau(\phi)\right)\right) g(X,Y) \\
& - \left(\sum_{l=1}^{s-1} h\left(\Delta^{s-l}\tau(\phi), \Delta^{s+l-2}\tau(\phi)\right)\right) g(X,Y) \\
& - \sum_{l=1}^{s-1} h\left(\nabla_X\Delta^{s-l-1}\tau(\phi), \nabla_Y\Delta^{s+l-2}\tau(\phi)\right) \\
& - \sum_{l=1}^{s-1} h\left(\nabla_Y\Delta^{s-l-1}\tau(\phi), \nabla_X\Delta^{s+l-2}\tau(\phi)\right) \\
& + h\left(\nabla_X\Delta^{2s-2}\tau(\phi), d\phi(Y)\right) + h\left(\nabla_Y\Delta^{2s-2}\tau(\phi), d\phi(X)\right).
\end{aligned} \tag{1.5}
$$

(2) The stress-energy tensor for polyharmonic maps of odd order is defined by

$$
\begin{aligned}
& S_{2s+1}(\phi)(X,Y) \\
= & \left(\frac{1}{2}\left|\nabla\Delta^{s-1}\tau(\phi)\right|^{2} - h\left(\Delta^{2s-1}\tau(\phi),\tau(\phi)\right)\right) g(X,Y) \\
& - \left(Tr_g h\left(\nabla\Delta^{2s-1}\tau(\phi), d\phi\right)\right) g(X,Y) \\
& + \left(\sum_{l=1}^{s-1} Tr_g h\left(\nabla\Delta^{s-l-1}\tau(\phi), \nabla\Delta^{s+l-1}\tau(\phi)\right)\right) g(X,Y) \\
& - \left(\sum_{l=1}^{s-1} h\left(\Delta^{s-l}\tau(\phi), \Delta^{s+l-1}\tau(\phi)\right)\right) g(X,Y) \\
& - \sum_{l=1}^{s-1} h\left(\nabla_X\Delta^{s-l-1}\tau(\phi), \nabla_Y\Delta^{s+l-1}\tau(\phi)\right) \\
& - \sum_{l=1}^{s-1} h\left(\nabla_Y\Delta^{s-l-1}\tau(\phi), \nabla_X\Delta^{s+l-1}\tau(\phi)\right) \\
& + h\left(\nabla_X\Delta^{2s-1}\tau(\phi), d\phi(Y)\right) + h\left(\nabla_Y\Delta^{2s-1}\tau(\phi), d\phi(X)\right) \\
& - h\left(\nabla_X\Delta^{s-1}\tau(\phi), \nabla_Y\Delta^{s-1}\tau(\phi)\right).
\end{aligned} \tag{1.6}
$$

Note that in both cases, the stress-energy tensor satisfies the following conservation law (see [3]):

$$divS_k(\phi) = -h(\tau_k(\phi), d\phi), \qquad k = 1, 2, 3, \ldots.$$

The harmonicity and the biharmonicity of the conformal map between equidimensional manifolds have been studied by several authors, see [1], [7] and [12] for more details. Recall that a mapping $\phi : (M^n, g) \to (N^n, h)$ is called conformal if there exists a smooth function $\lambda : M \to \mathbb{R}_+^*$ such that

$$h(d\phi(X), d\phi(Y)) = \lambda^2 g(X, Y)$$

for every $X, Y \in \Gamma(TM)$. The function λ is called the dilation for the conformal map ϕ. The tension field for a conformal map $\phi : (M^n, g) \to (N^n, h)$ of dilation λ is given by (see [2]):

$$\tau(\phi) = (2-n)\, d\phi(\text{grad } \ln\lambda).$$

Recall that in dimension 2 any conformal map is harmonic, whereas in other dimensions, it is harmonic if and only if its dilation is constant. Note that by a result in [2], any such map can have no critical points and so is a local conformal diffeomorphism. In [12], the authors calculated the bi-tension field and the stress bi-energy tensor for a conformal map and the following results are obtained:

Theorem 1.1 ([12])**.** *Let $\phi : (M^n, g) \to (N^n, h)$ $(n \geq 3)$ be a conformal map of dilation λ. The bi-tension field and the stress bi-energy tensor of ϕ are given by*

$$\tau_2(\phi) = (n-2)\, d\phi(H),$$

where

$$\begin{aligned} H = {} & grad\, \Delta \ln\lambda - \frac{n-6}{2} grad\, \left(|grad\, \ln\lambda|^2\right) - 2(\Delta\ln\lambda)\, grad\, \ln\lambda \\ & - (n-2)\, |grad\, \ln\lambda|^2\, grad\, \ln\lambda + 2Ricci^M(grad\, \ln\lambda) \end{aligned}$$

and

$$S_2(\phi) = (2-n)\lambda^2 \left\{ \left(\Delta\ln\lambda + \frac{n-2}{2}|grad\, \ln\lambda|^2\right) g - 2\nabla d\ln\lambda \right\}.$$

The biharmonicity of the conformal map is given by the following result (see [12]).

Theorem 1.2. *Let $\phi : (M^n, g) \to (N^n, h)$ $(n \geq 3)$ be a conformal map of dilation λ, ϕ is biharmonic if and only if*

$$\begin{aligned} & grad\, \Delta \ln\lambda - \frac{n-6}{2} grad\, \left(|grad\, \ln\lambda|^2\right) - 2(\Delta\ln\lambda)\, grad\, \ln\lambda \\ & - (n-2)\, |grad\, \ln\lambda|^2\, grad\, \ln\lambda + 2Ricci^M(grad\, \ln\lambda) = 0. \end{aligned}$$

Consequently, the biharmonicity of the conformal map $\phi : (\mathbb{R}^n, g) \to (N^n, h)$ $(n \geq 3)$ where the dilation λ is radial $(\ln\lambda = \alpha(r), r = |x| \text{ and } \alpha \in C^\infty(\mathbb{R}, \mathbb{R}))$ is equivalent to an ordinary differential equation. We have (see [12]):

Corollary 1.3. *Let* $\phi : (\mathbb{R}^n, g) \to (N^n, h)$ $(n \geq 3)$ *to be a conformal map of dilation* λ *when we suppose that* $\ln \lambda$ *is radial* $(\ln \lambda = \alpha(r), r = |x|)$. *Then* ϕ *is biharmonic if and only if* $\beta = \alpha'$ *satisfies the following ordinary differential equation*:

$$\beta'' - (n-4)\beta\beta' + \frac{n-1}{r}\beta' - \frac{n-1}{r^2}\beta - \frac{2(n-1)}{r}\beta^2 - (n-2)\beta^3 = 0. \tag{1.7}$$

Remark 1.4. If we consider particular solutions of the form $\beta = \frac{a}{r}$ $(a \in \mathbb{R}^*)$. By (1.7), a map $\phi : (\mathbb{R}^n \setminus \{0\}, g) \to (N^n, h)$ $(n \geq 3)$ is biharmonic if and only if a is a solution of the algebraic equation

$$(n-2)a^2 + (n+2)a + 2(n-2) = 0.$$

This equation has real solutions if and only if $n \in \{3, 4\}$.

(1) If $n = 3$, we find $a = \frac{-5+\sqrt{17}}{2}$ or $a = \frac{-5-\sqrt{17}}{2}$. Then any conformal map $\phi : (\mathbb{R}^3 \setminus \{0\}, g) \to (N^3, h)$ of dilation $\lambda = \frac{C}{r^{\frac{5-\sqrt{17}}{2}}}$ or $\lambda = \frac{C}{r^{\frac{5+\sqrt{17}}{2}}}$ $(C > 0)$ is biharmonic non-harmonic.

(2) If $n = 4$, the solutions are $a = -2$ and $a = -1$. The conformal map $\phi : (\mathbb{R}^4 \setminus \{0\}, g) \to (N^4, h)$ of dilation $\lambda = \frac{C}{r^2}$ or $\lambda = \frac{C}{r}$ $(C > 0)$ is biharmonic non-harmonic.

In this paper, we are interested in triharmonic maps, that are the solutions of the following equation:

$$\begin{aligned} \tau_3(\phi) &= \Delta^2\tau(\phi) - Tr_g R^N(\Delta\tau(\phi), d\phi)\, d\phi - Tr_g R^N(\nabla\tau(\phi), \tau(\phi))\, d\phi \\ &= 0. \end{aligned} \tag{1.8}$$

In this case, the triharmonic stress-energy tensor is given by:

$$\begin{aligned} & S_3(\phi)(X,Y) \\ = & \left(\frac{1}{2}|\tau(\phi)|^2 - h(\Delta\tau(\phi), \tau(\phi))\right) g(X,Y) \\ & - (Tr_g h(\nabla\Delta\tau(\phi), d\phi))\, g(X,Y) + h(\nabla_X \Delta\tau(\phi), d\phi(Y)) \\ & + h(\nabla_Y \Delta\tau(\phi), d\phi(X)) - h(\nabla_X \tau(\phi), \nabla_Y \tau(\phi)). \end{aligned} \tag{1.9}$$

As a first result, we calculate the expression of $\tau_3(\phi)$ where $\phi : (\mathbb{R}^n, g) \to (N^n, h)$ $(n \geq 3)$ is a conformal map of dilation λ. We show that a conformal mapping ϕ is triharmonic if and only if the gradient of its dilation, viewed as a section of the tangent bundle TM, satisfies a fourth-order elliptic partial differential equation and we give some basic examples of such mappings. We conclude our paper by calculating the trace of the stress-energy tensor for any conformal map $\phi : (\mathbb{R}^n, g) \to (N^n, h)$ $(n \geq 3)$ of dilation λ. As a consequence, we present some particular cases.

2. The main results

According to the properties of conformal maps, we have the following results:

Proposition 2.1 ([2]). *Let $\phi : (M^n, g) \to (N^n, h)$ $(n \geq 3)$ be a conformal map of dilation λ. For any $X, Y \in \Gamma(TM)$, we have*

$$\begin{aligned} \nabla_Y d\phi(X) = {} & X(\ln\lambda)\, d\phi(Y) + Y(\ln\lambda)\, d\phi(X) \\ & - g(X, Y)\, d\phi(grad\ \ln\lambda) + d\phi(\nabla_Y X). \end{aligned} \tag{2.1}$$

In particular, for any $f \in C^\infty(M)$, we obtain

$$\begin{aligned} \nabla_Y d\phi(grad\ f) = {} & d\ln\lambda(grad\ f)\, d\phi(Y) + Y(\ln\lambda)\, d\phi(grad\ f) \\ & - Y(f)\, d\phi(grad\ \ln\lambda) + d\phi(\nabla_Y grad\ f), \end{aligned} \tag{2.2}$$

and if $f = \ln\lambda$, equation (2.2) *becomes*

$$\nabla_Y d\phi(grad\ \ln\lambda) = |grad\ \ln\lambda|^2\, d\phi(Y) + d\phi(\nabla_Y grad\ \ln\lambda). \tag{2.3}$$

Equations (2.2) and (2.3) lead us to the following remark:

Remark 2.2. Let $\phi : (M^n, g) \to (N^n, h)$ $(n \geq 3)$ be a conformal map of dilation λ. For any $X, Y \in \Gamma(TM)$ and for any $f \in C^\infty(M)$, we have

$$\begin{aligned} h(\nabla_X d\phi(\mathrm{grad}\ f), d\phi(Y)) = {} & \lambda^2 X(\ln\lambda)\, Y(f) - \lambda^2 X(f)\, Y(\ln\lambda) \\ & + \lambda^2 d\ln\lambda(\mathrm{grad}\ f)\, g(X, Y) \\ & + \lambda^2 g(\nabla_X \mathrm{grad}\ f, Y) \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} h(\nabla_X d\phi(\mathrm{grad}\ \ln\lambda), d\phi(Y)) & = h(\nabla_Y d\phi(\mathrm{grad}\ \ln\lambda), d\phi(X)) \\ & = \lambda^2 |\mathrm{grad}\ \ln\lambda|^2\, g(X, Y) \\ & \quad + \lambda^2 g(\nabla_X \mathrm{grad}\ \ln\lambda, Y). \end{aligned} \tag{2.5}$$

To study the triharmonicity of a conformal map, we need the following theorem:

Theorem 2.3. *Let $\phi : (M^n, g) \to (N^n, h)$ $(n \geq 3)$ be a conformal map of dilation λ. Then for any $X \in \Gamma(TM)$ and for any function $f \in C^\infty(M)$, we have*

$$\begin{aligned} Tr_g (\nabla^\phi)^2 d\phi(X) = {} & d\phi(Tr_g \nabla^2 X) + (2 - n)\, X(\ln\lambda)\, d\phi(grad\ \ln\lambda) \\ & + (\Delta\ln\lambda)\, d\phi(X) + 2d\phi(grad\ (X(\ln\lambda))) \\ & - 2(divX)\, d\phi(grad\ \ln\lambda) + 2d\phi(\nabla_{grad\ \ln\lambda} X) \\ & - 2d\phi(\nabla_X grad\ \ln\lambda), \end{aligned} \tag{2.6}$$

$$
\begin{aligned}
Tr_g\left(\nabla^{\phi}\right)^2 f d\phi(X) = {} & f d\phi\left(Tr_g \nabla^2 X\right) + (2-n) f X(\ln\lambda) d\phi(grad\ \ln\lambda) \\
& + f(\Delta \ln\lambda) d\phi(X) + 2 f d\phi(grad\ (X(\ln\lambda))) \\
& + (\Delta f) d\phi(X) - 2f(divX) d\phi(grad\ \ln\lambda) \\
& + 2 f d\phi(\nabla_{grad\ \ln\lambda} X) - 2 f d\phi(\nabla_X grad\ \ln\lambda) \\
& + 2X(\ln\lambda) d\phi(grad\ f) + 2 d\ln\lambda(grad\ f) d\phi(X) \\
& - 2X(f) d\phi(grad\ \ln\lambda) + 2 d\phi(\nabla_{grad\ f} X)
\end{aligned} \tag{2.7}
$$

and

$$
\begin{aligned}
R^N(d\phi(X), d\phi(Y)) d\phi(Z) = {} & (n-2) X(\ln\lambda) Y(\ln\lambda) d\phi(Z) \\
& - (n-2)|grad\ \ln\lambda|^2 g(X,Y) d\phi(Z) \\
& - (n-2) g(\nabla_X grad\ \ln\lambda, Y) d\phi(Z) \\
& - (\Delta \ln\lambda) g(X,Y) d\phi(Z) \\
& + d\phi\left(R^M(X,Y)Z\right).
\end{aligned} \tag{2.8}
$$

Proof. By definition, for any $X \in \Gamma(TM)$, we have

$$
Tr_g\left(\nabla^{\phi}\right)^2 d\phi(X) = \nabla_{e_i}\nabla_{e_i} d\phi(X) - \nabla_{\nabla_{e_i} e_i} d\phi(X). \tag{2.9}
$$

By (2.1), we obtain

$$
\begin{aligned}
\nabla_{e_i} d\phi(X) = {} & X(\ln\lambda) d\phi(e_i) + e_i(\ln\lambda) d\phi(X) \\
& - g(X, e_i) d\phi(\text{grad}\ \ln\lambda) + d\phi(\nabla_{e_i} X),
\end{aligned}
$$

then

$$
\begin{aligned}
\nabla_{e_i}\nabla_{e_i} d\phi(X) = {} & \nabla_{e_i} X(\ln\lambda) d\phi(e_i) + \nabla_{e_i} e_i(\ln\lambda) d\phi(X) \\
& - \nabla_{e_i} g(X, e_i) d\phi(\text{grad}\ \ln\lambda) + \nabla_{e_i} d\phi(\nabla_{e_i} X).
\end{aligned}
$$

The explicit form of the last equation is reported below term by term. We have

$$
\begin{aligned}
\nabla_{e_i} X(\ln\lambda) d\phi(e_i) &= X(\ln\lambda)\nabla_{e_i} d\phi(e_i) + e_i(X(\ln\lambda)) d\phi(e_i) \\
&= X(\ln\lambda)\nabla_{e_i} d\phi(e_i) + e_i(g(X, \text{grad}\ \ln\lambda)) d\phi(e_i) \\
&= X(\ln\lambda)\nabla_{e_i} d\phi(e_i) + g(\nabla_{e_i}\text{grad}\ \ln\lambda, X) d\phi(e_i) \\
&\quad + g(\text{grad}\ \ln\lambda, \nabla_{e_i} X) d\phi(e_i) \\
&= X(\ln\lambda)\nabla_{e_i} d\phi(e_i) + d\phi(\nabla_X \text{grad}\ \ln\lambda) \\
&\quad + g(\text{grad}\ \ln\lambda, \nabla_{e_i} X) d\phi(e_i) \\
&= X(\ln\lambda)\nabla_{e_i} d\phi(e_i) + d\phi(\text{grad}\ (X(\ln\lambda))),
\end{aligned}
$$

$$
\begin{aligned}
\nabla_{e_i} e_i(\ln\lambda) d\phi(X) &= e_i(\ln\lambda)\nabla_{e_i} d\phi(X) + e_i(e_i(\ln\lambda)) d\phi(X) \\
&= \nabla_{\text{grad}\ \ln\lambda} d\phi(X) + e_i(e_i(\ln\lambda)) d\phi(X) \\
&= |\text{grad}\ \ln\lambda|^2 d\phi(X) + d\phi(\nabla_{\text{grad}\ \ln\lambda} X) \\
&\quad + e_i(e_i(\ln\lambda)) d\phi(X),
\end{aligned}
$$

$$\begin{aligned}\nabla_{e_i} g(X,e_i)\, d\phi(\text{grad } \ln\lambda) &= \nabla_X d\phi(\text{grad } \ln\lambda) + e_i(g(X,e_i))\, d\phi(\text{grad } \ln\lambda) \\ &= |\text{grad } \ln\lambda|^2\, d\phi(X) + d\phi(\nabla_X \text{grad } \ln\lambda) \\ &\quad + g(\nabla_{e_i}X, e_i)\, d\phi(\text{grad } \ln\lambda) \\ &\quad + g(X, \nabla_{e_i}e_i)\, d\phi(\text{grad } \ln\lambda)\end{aligned}$$

and

$$\begin{aligned}\nabla_{e_i} d\phi(\nabla_{e_i}X) &= g(\text{grad } \ln\lambda, \nabla_{e_i}X)\, d\phi(e_i) + d\phi(\nabla_{\text{grad } \ln\lambda}X) \\ &\quad - g(\nabla_{e_i}X, e_i)\, d\phi(\text{grad } \ln\lambda) + d\phi(\nabla_{e_i}\nabla_{e_i}X) \\ &= d\phi(\text{grad } (X(\ln\lambda))) - d\phi(\nabla_X \text{grad } \ln\lambda) + d\phi(\nabla_{\text{grad } \ln\lambda}X) \\ &\quad - g(\nabla_{e_i}X, e_i)\, d\phi(\text{grad } \ln\lambda) + d\phi(\nabla_{e_i}\nabla_{e_i}X),\end{aligned}$$

which gives us

$$\begin{aligned}&\nabla_{e_i}\nabla_{e_i} d\phi(X) \\ &= d\phi(\nabla_{e_i}\nabla_{e_i}X) + X(\ln\lambda)\, \nabla^{\phi}_{e_i} d\phi(e_i) \\ &\quad + 2d\phi(\text{grad } (X(\ln\lambda))) + 2d\phi(\nabla_{\text{grad } \ln\lambda}X) \\ &\quad - 2d\phi(\nabla_X \text{grad } \ln\lambda) + e_i(e_i(\ln\lambda))\, d\phi(X) \\ &\quad - 2(divX)\, d\phi(\text{grad } \ln\lambda) - g(X, \nabla_{e_i}e_i)\, d\phi(\text{grad } \ln\lambda).\end{aligned} \tag{2.10}$$

Finally, it is clear that

$$\begin{aligned}\nabla_{\nabla_{e_i}e_i} d\phi(X) &= X(\ln\lambda)\, d\phi(\nabla_{e_i}e_i) + (\nabla_{e_i}e_i)(\ln\lambda)\, d\phi(X) \\ &\quad - g(X, \nabla_{e_i}e_i)\, d\phi(\text{grad } \ln\lambda) + d\phi\left(\nabla_{\nabla_{e_i}e_i}X\right).\end{aligned} \tag{2.11}$$

By replacing (2.10) and (2.11) in (2.9), we find that

$$\begin{aligned}Tr_g\left(\nabla^{\phi}\right)^2 d\phi(X) &= d\phi\left(Tr_g\nabla^2 X\right) + (2-n)\, X(\ln\lambda)\, d\phi(\text{grad } \ln\lambda) \\ &\quad + (\Delta\ln\lambda)\, d\phi(X) + 2d\phi(\text{grad } (X(\ln\lambda))) \\ &\quad - 2(divX)\, d\phi(\text{grad } \ln\lambda) + 2d\phi(\nabla_{\text{grad } \ln\lambda}X) \\ &\quad - 2d\phi(\nabla_X \text{grad } \ln\lambda).\end{aligned}$$

For the term $Tr_g\left(\nabla^{\phi}\right)^2 f d\phi(X)$, a simple calculation gives us

$$Tr_g\left(\nabla^{\phi}\right)^2 f d\phi(X) = f Tr_g\left(\nabla^{\phi}\right)^2 d\phi(X) + 2\nabla^{\phi}_{\text{grad } f} d\phi(X) + (\Delta f)\, d\phi(X).$$

Using equation (2.6) and the fact that

$$\begin{aligned}\nabla^{\phi}_{\text{grad } f} d\phi(X) &= X(\ln\lambda)\, d\phi(\text{grad } f) + d\ln\lambda(\text{grad } f)\, d\phi(X) \\ &\quad - X(f)\, d\phi(\text{grad } \ln\lambda) + d\phi(\nabla_{\text{grad } f}X),\end{aligned}$$

we conclude that

$$\begin{aligned} Tr_g\left(\nabla^{\phi}\right)^2 f d\phi\left(X\right) = & \, f d\phi\left(Tr_g \nabla^2 X\right) + \left(2-n\right) f X\left(\ln\lambda\right) d\phi\left(\text{grad } \ln\lambda\right) \\ & + f\left(\Delta \ln\lambda\right) d\phi\left(X\right) + 2 f d\phi\left(\text{grad } \left(X\left(\ln\lambda\right)\right)\right) \\ & + \left(\Delta f\right) d\phi\left(X\right) - 2f\left(div X\right) d\phi\left(\text{grad } \ln\lambda\right) \\ & + 2 f d\phi\left(\nabla_{\text{grad } \ln\lambda} X\right) - 2 f d\phi\left(\nabla_X \text{grad } \ln\lambda\right) \\ & + 2X\left(\ln\lambda\right) d\phi\left(\text{grad } f\right) + 2 d\ln\lambda\left(\text{grad } f\right) d\phi\left(X\right) \\ & - 2X\left(f\right) d\phi\left(\text{grad } \ln\lambda\right) + 2 d\phi\left(\nabla_{\text{grad } f} X\right). \end{aligned}$$

Finally, the proof of equation (2.8) is based on the following result (see [2])

$$\begin{aligned} & Ric^N\left(d\phi\left(X\right), d\phi\left(Y\right)\right) \\ = & \left(n-2\right) X\left(\ln\lambda\right) Y\left(\ln\lambda\right) - \left(n-2\right) g\left(\nabla_X \text{grad } \ln\lambda, Y\right) \\ & - \left(\Delta\ln\lambda\right) g\left(X,Y\right) - \left(n-2\right)\left|\text{grad } \ln\lambda\right|^2 g\left(X,Y\right) + Ric^M\left(X,Y\right). \end{aligned}$$

□

Theorem 2.4. *Let $\phi : (\mathbb{R}^n, g) \to (N^n, h)$ $(n \geq 3)$ be a conformal map of dilation λ. Then the tri-tension field of ϕ is given by*

$$\tau_3\left(\phi\right) = -\left(n-2\right) d\phi\left(H\left(\lambda\right)\right),$$

where

$$\begin{aligned} & H\left(\lambda\right) \\ = & \, grad \, \left(\Delta^2 \ln\lambda\right) + 2 grad \, \left(\Delta\left(\left|grad \, \ln\lambda\right|^2\right)\right) - \left(\Delta\ln\lambda\right) grad \, \Delta\ln\lambda \\ & - 2\left(n-1\right)\left|grad \, \ln\lambda\right|^2 grad \, \Delta\ln\lambda - 2\left(\Delta\ln\lambda\right) grad \, \left(\left|grad \, \ln\lambda\right|^2\right) \\ & - 6\left(n-2\right)\left|grad \, \ln\lambda\right|^2 grad \, \left(\left|grad \, \ln\lambda\right|^2\right) - 3\left(\Delta^2\ln\lambda\right) grad \, \ln\lambda \\ & + 2\left(n-2\right)\left(\Delta\ln\lambda\right)\left|grad \, \ln\lambda\right|^2 grad \, \ln\lambda + \left(n-2\right)^2\left|grad \, \ln\lambda\right|^4 grad \, \ln\lambda \\ & + 2\left(\Delta\ln\lambda\right)^2 grad \, \ln\lambda - \left(n+2\right)\left(\Delta\left(\left|grad \, \ln\lambda\right|^2\right)\right) grad \, \ln\lambda \\ & - n\nabla_{grad \, \Delta\ln\lambda} grad \, \ln\lambda - \frac{\left(n-2\right)\left(n+6\right)}{2}\nabla_{grad \, \left(\left|grad \, \ln\lambda\right|^2\right)} grad \, \ln\lambda \\ & + 4\nabla_{grad \, \ln\lambda} grad \, \Delta\ln\lambda + 8\nabla_{grad \, \ln\lambda} grad \, \left(\left|grad \, \ln\lambda\right|^2\right). \end{aligned}$$

Proof. We prove this theorem in three steps. By definition, we have

$$\tau_3\left(\phi\right) = \Delta^2\tau\left(\phi\right) - Tr_g R^N\left(\Delta\tau\left(\phi\right), d\phi\right) d\phi - Tr_g R^N\left(\nabla\tau\left(\phi\right), \tau\left(\phi\right)\right) d\phi. \tag{2.12}$$

Step 1. In the first step, we look at the first term of the equation (2.12). Since

$$\begin{aligned} \Delta\tau\left(\phi\right) = & -Tr_g\left(\nabla^{\phi}\right)^2\tau\left(\phi\right) = \left(n-2\right) Tr_g\left(\nabla^{\phi}\right)^2 d\phi\left(\text{grad } \ln\lambda\right) \\ = & \left(n-2\right) d\phi\left(\text{grad } \Delta\ln\lambda\right) + 2\left(n-2\right) d\phi\left(\text{grad } \left(\left|\text{grad } \ln\lambda\right|^2\right)\right) \\ & - \left(n-2\right)\left(\Delta\ln\lambda\right) d\phi\left(\text{grad } \ln\lambda\right) - \left(n-2\right)^2\left|\text{grad } \ln\lambda\right|^2 d\phi\left(\text{grad } \ln\lambda\right), \end{aligned}$$

then

$$\begin{aligned}\Delta^2\tau(\phi) &= \Delta(\Delta\tau(\phi))\\ &= (n-2)\,\Delta\left(d\phi\left(\operatorname{grad}\Delta\ln\lambda\right)\right)\\ &\quad + 2(n-2)\,\Delta\left(d\phi\left(\operatorname{grad}\left(|\operatorname{grad}\ln\lambda|^2\right)\right)\right)\\ &\quad - (n-2)\,\Delta\left((\Delta\ln\lambda)\,d\phi\left(\operatorname{grad}\ln\lambda\right)\right)\\ &\quad - (n-2)^2\,\Delta\left(|\operatorname{grad}\ln\lambda|^2\,d\phi\left(\operatorname{grad}\ln\lambda\right)\right).\end{aligned}$$

For the two terms $\Delta\left(d\phi\left(\operatorname{grad}\Delta\ln\lambda\right)\right)$ and $\Delta\left(d\phi\left(\operatorname{grad}\left(|\operatorname{grad}\ln\lambda|^2\right)\right)\right)$ and using the equation (2.6), we obtain

$$\begin{aligned}&\Delta\left(d\phi\left(\operatorname{grad}\Delta\ln\lambda\right)\right)\\ &= -Tr_g\left(\nabla^{\phi}\right)^2 d\phi\left(\operatorname{grad}\Delta\ln\lambda\right)\\ &= -d\phi\left(\operatorname{grad}\Delta^2\ln\lambda\right) - (\Delta\ln\lambda)\,d\phi\left(\operatorname{grad}\Delta\ln\lambda\right)\\ &\quad + (n-2)\,d\ln\lambda\left(\operatorname{grad}\Delta\ln\lambda\right)d\phi\left(\operatorname{grad}\ln\lambda\right)\\ &\quad + 2\left(\Delta^2\ln\lambda\right)d\phi\left(\operatorname{grad}\ln\lambda\right) - 4d\phi\left(\nabla_{\operatorname{grad}\ln\lambda}\operatorname{grad}\Delta\ln\lambda\right)\end{aligned}$$

and

$$\begin{aligned}&\Delta\left(d\phi\left(\operatorname{grad}\left(|\operatorname{grad}\ln\lambda|^2\right)\right)\right)\\ &= -Tr_g\left(\nabla^{\phi}\right)^2 d\phi\left(\operatorname{grad}\left(|\operatorname{grad}\ln\lambda|^2\right)\right)\\ &= -d\phi\left(\operatorname{grad}\Delta\left(|\operatorname{grad}\ln\lambda|^2\right)\right) - (\Delta\ln\lambda)\,d\phi\left(\operatorname{grad}\left(|\operatorname{grad}\ln\lambda|^2\right)\right)\\ &\quad + 2\left(\Delta\left(|\operatorname{grad}\ln\lambda|^2\right)\right)d\phi\left(\operatorname{grad}\ln\lambda\right)\\ &\quad - 4d\phi\left(\nabla_{\operatorname{grad}\ln\lambda}\operatorname{grad}\left(|\operatorname{grad}\ln\lambda|^2\right)\right)\\ &\quad + (n-2)\,d\ln\lambda\left(\operatorname{grad}\left(|\operatorname{grad}\ln\lambda|^2\right)\right)d\phi\left(\operatorname{grad}\ln\lambda\right).\end{aligned}$$

For the term $\Delta\left((\Delta\ln\lambda)\,d\phi\left(\operatorname{grad}\ln\lambda\right)\right)$, we have

$$\begin{aligned}&\Delta\left((\Delta\ln\lambda)\,d\phi\left(\operatorname{grad}\ln\lambda\right)\right)\\ &= -Tr_g\left(\nabla^{\phi}\right)^2(\Delta\ln\lambda)\,d\phi\left(\operatorname{grad}\ln\lambda\right)\\ &= -(\Delta\ln\lambda)\,Tr_g\left(\nabla^{\phi}\right)^2 d\phi\left(\operatorname{grad}\ln\lambda\right) - \left(\Delta^2\ln\lambda\right)d\phi\left(\operatorname{grad}\ln\lambda\right)\\ &\quad - 2\nabla^{\phi}_{\operatorname{grad}\Delta\ln\lambda}d\phi\left(\operatorname{grad}\ln\lambda\right)\\ &= -(\Delta\ln\lambda)\,d\phi\left(\operatorname{grad}\Delta\ln\lambda\right) - 2\,|\operatorname{grad}\ln\lambda|^2\,d\phi\left(\operatorname{grad}\Delta\ln\lambda\right)\\ &\quad - 2(\Delta\ln\lambda)\,d\phi\left(\operatorname{grad}\left(|\operatorname{grad}\ln\lambda|^2\right)\right) + (\Delta\ln\lambda)^2\,d\phi\left(\operatorname{grad}\ln\lambda\right)\\ &\quad - \left(\Delta^2\ln\lambda\right)d\phi\left(\operatorname{grad}\ln\lambda\right) - 2d\phi\left(\nabla_{\operatorname{grad}\Delta\ln\lambda}\operatorname{grad}\ln\lambda\right)\\ &\quad + (n-2)\,(\Delta\ln\lambda)\,|\operatorname{grad}\ln\lambda|^2\,d\phi\left(\operatorname{grad}\ln\lambda\right).\end{aligned}$$

Finally, for the term $\Delta\left(|\text{grad } \ln\lambda|^2 d\phi(\text{grad } \ln\lambda)\right)$, a similar calculation yields

$$\begin{aligned}
&\Delta\left(|\text{grad } \ln\lambda|^2 d\phi(\text{grad } \ln\lambda)\right)\\
=\;& -Tr_g\left(\nabla^\phi\right)^2 |\text{grad } \ln\lambda|^2 d\phi(\text{grad } \ln\lambda)\\
=\;& -|\text{grad } \ln\lambda|^2 Tr_g\left(\nabla^\phi\right)^2 d\phi(\text{grad } \ln\lambda)\\
& -\left(\Delta\left(|\text{grad } \ln\lambda|^2\right)\right) d\phi(\text{grad } \ln\lambda)\\
& -2\nabla^\phi_{\text{grad }\left(|\text{grad } \ln\lambda|^2\right)} d\phi(\text{grad } \ln\lambda)\\
=\;& -|\text{grad } \ln\lambda|^2 d\phi(\text{grad } \Delta\ln\lambda) - 4|\text{grad } \ln\lambda|^2 d\phi\left(\text{grad }\left(|\text{grad } \ln\lambda|^2\right)\right)\\
& +(\Delta\ln\lambda)|\text{grad } \ln\lambda|^2 d\phi(\text{grad } \ln\lambda) + (n-2)|\text{grad } \ln\lambda|^4 d\phi(\text{grad } \ln\lambda)\\
& -\left(\Delta\left(|\text{grad } \ln\lambda|^2\right)\right) d\phi(\text{grad } \ln\lambda) - 2d\phi\left(\nabla_{\text{grad }\left(|\text{grad } \ln\lambda|^2\right)}\text{grad } \ln\lambda\right).
\end{aligned}$$

We deduce that the final result of this first step is given by the following equation

$$\begin{aligned}
\Delta^2\tau(\phi) =\;& -(n-2)\,d\phi\left(\text{ grad } \Delta^2\ln\lambda\right)\\
& +n(n-2)|\text{ grad } \ln\lambda|^2 d\phi(\text{grad } \Delta\ln\lambda)\\
& -2(n-2)\,d\phi\left(\text{grad } \Delta\left(|\text{ grad } \ln\lambda|^2\right)\right)\\
& -(n-2)^3|\text{grad } \ln\lambda|^4 d\phi(\text{grad } \ln\lambda)\\
& +4(n-2)^2|\text{grad } \ln\lambda|^2 d\phi\left(\text{grad }\left(|\text{grad } \ln\lambda|^2\right)\right)\\
& +3(n-2)\left(\Delta^2\ln\lambda\right) d\phi(\text{grad } \ln\lambda)\\
& -(n-2)(\Delta\ln\lambda)^2 d\phi(\text{grad } \ln\lambda)\\
& +(n-2)(n+2)\left(\Delta\left(|\text{ grad } \ln\lambda|^2\right)\right) d\phi(\text{grad } \ln\lambda)\\
& -2(n-2)^2(\Delta\ln\lambda)|\text{ grad } \ln\lambda|^2 d\phi(\text{grad } \ln\lambda)\\
& +(n-2)^2 d\ln\lambda(\text{grad } \Delta\ln\lambda)\, d\phi(\text{grad } \ln\lambda)\\
& +2(n-2)^2 d\ln\lambda\left(\text{grad }\left(|\text{grad } \ln\lambda|^2\right)\right) d\phi(\text{ grad } \ln\lambda)\\
& -4(n-2)\,d\phi\left(\nabla_{\text{grad } \ln\lambda}\text{ grad } \Delta\ln\lambda\right)\\
& +2(n-2)\,d\phi\left(\nabla_{\text{grad } \Delta\ln\lambda}\text{grad } \ln\lambda\right)\\
& -8(n-2)\,d\phi\left(\nabla_{\text{grad } \ln\lambda}\text{ grad }\left(|\text{grad } \ln\lambda|^2\right)\right)\\
& +2(n-2)^2 d\phi\left(\nabla_{\text{grad }\left(|\text{grad } \ln\lambda|^2\right)}\text{grad } \ln\lambda\right).
\end{aligned} \tag{2.13}$$

Step 2. In this step, simplify the term $Tr_gR^N(\Delta\tau(\phi), d\phi)\,d\phi$. Using the expression of $\Delta\tau(\phi)$, we have

$$\begin{aligned}
& Tr_gR^N(\Delta\tau(\phi), d\phi)\,d\phi \\
= \; & (n-2)\,Tr_gR^N(d\phi(\text{grad }\Delta\ln\lambda), d\phi)\,d\phi \\
& + 2(n-2)\,Tr_gR^N\left(d\phi\left(\text{grad }\left(|\text{grad }\ln\lambda|^2\right)\right), d\phi\right)d\phi \\
& - (n-2)(\Delta\ln\lambda)\,Tr_gR^N(d\phi(\text{grad }\ln\lambda), d\phi)\,d\phi \\
& - (n-2)^2\,|\text{grad }\ln\lambda|^2\,Tr_gR^N(d\phi(\text{grad }\ln\lambda), d\phi)\,d\phi.
\end{aligned}$$

From equation (2.8), we obtain

$$\begin{aligned}
& Tr_gR^N(d\phi(\text{grad }\Delta\ln\lambda), d\phi)\,d\phi \\
= \; & -(\Delta\ln\lambda)\,d\phi(\text{grad }\Delta\ln\lambda) \\
& - (n-2)\,|\text{grad }\ln\lambda|^2\,d\phi(\text{grad }\Delta\ln\lambda) \\
& + (n-2)\,d\ln\lambda(\text{grad }\Delta\ln\lambda)\,d\phi(\text{grad }\ln\lambda) \\
& - (n-2)\,d\phi(\nabla_{\text{grad }\Delta\ln\lambda}\text{grad }\ln\lambda),
\end{aligned}$$

$$\begin{aligned}
& Tr_gR^N\left(d\phi\left(\text{grad }\left(|\text{grad }\ln\lambda|^2\right)\right), d\phi\right)d\phi \\
= \; & -(\Delta\ln\lambda)\,d\phi\left(\text{grad }\left(|\text{grad }\ln\lambda|^2\right)\right) \\
& - (n-2)\,|\text{grad }\ln\lambda|^2\,d\phi\left(\text{grad }\left(|\text{grad }\ln\lambda|^2\right)\right) \\
& + (n-2)\,d\ln\lambda\left(\text{grad }\left(|\text{grad }\ln\lambda|^2\right)\right)d\phi(\text{grad }\ln\lambda) \\
& - (n-2)\,d\phi\left(\nabla_{\text{grad }\left(|\text{grad }\ln\lambda|^2\right)}\text{grad }\ln\lambda\right)
\end{aligned}$$

and

$$\begin{aligned}
& Tr_gR^N(d\phi(\text{grad }\ln\lambda), d\phi)\,d\phi \\
= \; & -(\Delta\ln\lambda)\,d\phi(\text{grad }\ln\lambda) - \frac{(n-2)}{2}d\phi\left(\text{grad }\left(|\text{grad }\ln\lambda|^2\right)\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
& Tr_gR^N(\Delta\tau(\phi), d\phi)\,d\phi \\
= \; & -(n-2)(\Delta\ln\lambda)\,d\phi(\text{grad }\Delta\ln\lambda) \\
& - (n-2)^2\,|\text{grad }\ln\lambda|^2\,d\phi(\text{grad }\Delta\ln\lambda) \\
& + \frac{(n-2)(n-6)}{2}(\Delta\ln\lambda)\,d\phi\left(\text{grad }\left(|\text{grad }\ln\lambda|^2\right)\right) \\
& + \frac{(n-2)^2(n-6)}{2}|\text{grad }\ln\lambda|^2\,d\phi\left(\text{grad }\left(|\text{grad }\ln\lambda|^2\right)\right) \\
& + (n-2)^2\,d\ln\lambda(\text{grad }\Delta\ln\lambda)\,d\phi(\text{grad }\ln\lambda)
\end{aligned}$$

$$
\begin{aligned}
&+2(n-2)^2 d\ln\lambda\left(\operatorname{grad}\left(|\operatorname{grad}\ \ln\lambda|^2\right)\right) d\phi(\operatorname{grad}\ \ln\lambda)\\
&+(n-2)^2(\Delta\ln\lambda)|\operatorname{grad}\ \ln\lambda|^2 d\phi(\operatorname{grad}\ \ln\lambda)\\
&+(n-2)(\Delta\ln\lambda)^2 d\phi(\operatorname{grad}\ \ln\lambda)\\
&-(n-2)^2 d\phi\left(\nabla_{\operatorname{grad}\,\Delta\ln\lambda}\operatorname{grad}\ \ln\lambda\right)\\
&-2(n-2)^2 d\phi\left(\nabla_{\operatorname{grad}\,\left(|\operatorname{grad}\ \ln\lambda|^2\right)}\operatorname{grad}\ \ln\lambda\right).
\end{aligned}
$$

Step 3. To complete the proof, we develop the term $Tr_g R^N(\nabla\tau(\phi),\tau(\phi))\,d\phi$; we have

$$
\begin{aligned}
&Tr_g R^N(\nabla\tau(\phi),\tau(\phi))\,d\phi\\
&= R^N(\nabla_{e_i}\tau(\phi),\tau(\phi))\,d\phi(e_i)\\
&= (n-2)^2 R^N(\nabla_{e_i}d\phi(\operatorname{grad}\ \ln\lambda), d\phi(\operatorname{grad}\ \ln\lambda))\,d\phi(e_i).
\end{aligned}
$$

Using the fact that

$$
\nabla_{e_i}d\phi(\operatorname{grad}\ \ln\lambda)=|\operatorname{grad}\ \ln\lambda|^2 d\phi(e_i)+d\phi(\nabla_{e_i}\operatorname{grad}\ \ln\lambda),
$$

we obtain

$$
\begin{aligned}
&Tr_g R^N(\nabla\tau(\phi),\tau(\phi))\,d\phi\\
&= (n-2)^2|\operatorname{grad}\ \ln\lambda|^2 R^N(d\phi(e_i), d\phi(\operatorname{grad}\ \ln\lambda))\,d\phi(e_i)\\
&\quad+(n-2)^2 R^N(d\phi(\nabla_{e_i}\operatorname{grad}\ \ln\lambda), d\phi(\operatorname{grad}\ \ln\lambda))\,d\phi(e_i).
\end{aligned}
$$

Equation (2.8) leads us to

$$
\begin{aligned}
&R^N(d\phi(e_i), d\phi(\operatorname{grad}\ \ln\lambda))\,d\phi(e_i)\\
&= -(\Delta\ln\lambda)\,d\phi(\operatorname{grad}\ \ln\lambda)-\frac{(n-2)}{2}d\phi\left(\operatorname{grad}\left(|\operatorname{grad}\ \ln\lambda|^2\right)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&R^N(d\phi(\nabla_{e_i}\operatorname{grad}\ \ln\lambda), d\phi(\operatorname{grad}\ \ln\lambda))\,d\phi(e_i)\\
&= -\frac{(n-2)}{2}d\phi\left(\nabla_{\operatorname{grad}\,\left(|\operatorname{grad}\ \ln\lambda|^2\right)}\operatorname{grad}\ \ln\lambda\right)\\
&\quad-\frac{1}{2}(\Delta\ln\lambda)\,d\phi\left(\operatorname{grad}\left(|\operatorname{grad}\ \ln\lambda|^2\right)\right),
\end{aligned}
$$

then

$$
\begin{aligned}
&Tr_g R^N(\nabla\tau(\phi),\tau(\phi))\,d\phi\\
&= -\frac{(n-2)^2}{2}(\Delta\ln\lambda)\,d\phi\left(\operatorname{grad}\left(|\operatorname{grad}\ \ln\lambda|^2\right)\right)\\
&\quad-\frac{(n-2)^3}{2}|\operatorname{grad}\ \ln\lambda|^2 d\phi\left(\operatorname{grad}\left(|\operatorname{grad}\ \ln\lambda|^2\right)\right)\\
&\quad-(n-2)^2(\Delta\ln\lambda)|\operatorname{grad}\ \ln\lambda|^2 d\phi(\operatorname{grad}\ \ln\lambda)\\
&\quad-\frac{(n-2)^3}{2}d\phi\left(\nabla_{\operatorname{grad}\,\left(|\operatorname{grad}\ \ln\lambda|^2\right)}\operatorname{grad}\ \ln\lambda\right).
\end{aligned}
$$

The results of the three steps and equation (2.12) allow us to conclude that

$$\tau_3(\phi) = -(n-2)\, d\phi(H(\lambda)),$$

where

$$\begin{aligned} H(\lambda) = {} & \operatorname{grad}\left(\Delta^2 \ln\lambda\right) + 2\operatorname{grad}\left(\Delta\left(|\operatorname{grad}\ \ln\lambda|^2\right)\right) \\ & - (\Delta\ln\lambda)\operatorname{grad}\Delta\ln\lambda - 2(n-1)|\operatorname{grad}\ \ln\lambda|^2\operatorname{grad}\Delta\ln\lambda \\ & - 2(\Delta\ln\lambda)\operatorname{grad}\left(|\operatorname{grad}\ \ln\lambda|^2\right) \\ & - 6(n-2)|\operatorname{grad}\ \ln\lambda|^2\operatorname{grad}\left(|\operatorname{grad}\ \ln\lambda|^2\right) - 3\left(\Delta^2\ln\lambda\right)\operatorname{grad}\ \ln\lambda \\ & + 2(n-2)(\Delta\ln\lambda)|\operatorname{grad}\ \ln\lambda|^2\operatorname{grad}\ \ln\lambda \\ & + (n-2)^2|\operatorname{grad}\ \ln\lambda|^4\operatorname{grad}\ \ln\lambda \\ & + 2(\Delta\ln\lambda)^2\operatorname{grad}\ \ln\lambda - (n+2)\left(\Delta\left(|\operatorname{grad}\ \ln\lambda|^2\right)\right)\operatorname{grad}\ \ln\lambda \\ & - n\nabla_{\operatorname{grad}\ \Delta\ln\lambda}\operatorname{grad}\ \ln\lambda - \frac{(n-2)(n+6)}{2}\nabla_{\operatorname{grad}\ \left(|\operatorname{grad}\ \ln\lambda|^2\right)}\operatorname{grad}\ \ln\lambda \\ & + 4\nabla_{\operatorname{grad}\ \ln\lambda}\operatorname{grad}\Delta\ln\lambda + 8\nabla_{\operatorname{grad}\ \ln\lambda}\operatorname{grad}\left(|\operatorname{grad}\ \ln\lambda|^2\right). \end{aligned}$$

□

A necessary and sufficient condition of the triharmonicity of a conformal map is given in the following theorem.

Theorem 2.5. *Let $\phi : (\mathbb{R}^n, g) \to (N^n, h)$ be a conformal map of dilation λ between manifolds of the same dimension $n \geq 3$. Then ϕ is triharmonic if and only if the vector field grad $\ln\lambda$ satisfies the fourth order elliptic partial differential equation*:

$$\begin{aligned} & grad\left(\Delta^2\ln\lambda\right) + 2grad\left(\Delta\left(|grad\ \ln\lambda|^2\right)\right) - (\Delta\ln\lambda)\, grad\,\Delta\ln\lambda \\ & - 2(n-1)|grad\ \ln\lambda|^2 grad\,\Delta\ln\lambda - 2(\Delta\ln\lambda)\, grad\left(|grad\ \ln\lambda|^2\right) \\ & - 6(n-2)|grad\ \ln\lambda|^2 grad\left(|grad\ \ln\lambda|^2\right) - 3\left(\Delta^2\ln\lambda\right) grad\ \ln\lambda \\ & + 2(n-2)(\Delta\ln\lambda)|grad\ \ln\lambda|^2 grad\ \ln\lambda + (n-2)^2|grad\ \ln\lambda|^4 grad\ \ln\lambda \\ & + 2(\Delta\ln\lambda)^2 grad\ \ln\lambda - (n+2)\left(\Delta\left(|grad\ \ln\lambda|^2\right)\right) grad\ \ln\lambda \\ & - n\nabla_{grad\ \Delta\ln\lambda} grad\ \ln\lambda - \frac{(n-2)(n+6)}{2}\nabla_{grad\ \left(|grad\ \ln\lambda|^2\right)} grad\ \ln\lambda \\ & + 4\nabla_{grad\ \ln\lambda} grad\,\Delta\ln\lambda + 8\nabla_{grad\ \ln\lambda} grad\left(|grad\ \ln\lambda|^2\right) = 0. \end{aligned}$$

A case of particular interest is when the dilation is radial, we prove that the triharmonicity of the conformal map $\phi : (\mathbb{R}^n, g) \to (N^n, h)$ $(n \geq 3)$ is equivalent to an ordinary differential equation of the fourth order. More precisely, we have:

Corollary 2.6. *Let $\phi : (\mathbb{R}^n, g) \to (N^n, h)$ $(n \geq 3)$ be a conformal map of dilation λ where we suppose that $\ln \lambda$ is radial $(\ln \lambda = \alpha(r), r = |x|)$. Then ϕ is triharmonic if and only if $\beta = \alpha'$ satisfies the following ordinary differential equation:*

$$
\begin{aligned}
&\beta^{(4)} + \frac{2(n-1)}{r}\beta^{(3)} + 5\beta\beta^{(3)} + \frac{(n-1)(n-5)}{r^2}\beta'' - (n-11)\beta'\beta'' \\
&+ \frac{(n-1)}{r}\beta\beta'' - (4n-14)\beta^2\beta'' - \left(n^2+6n-22\right)\beta(\beta')^2 \\
&- \frac{(n-3)(n-1)}{r}(\beta')^2 - \frac{3(n-1)(n-3)}{r^3}\beta' - 10(n-2)\beta^3\beta' \\
&- \frac{2(n-1)(2n+1)}{r}\beta^2\beta' - \frac{(3n+1)(n-1)}{r^2}\beta\beta' + \frac{3(n-1)(n-3)}{r^4}\beta \\
&+ \frac{(n-1)(4n-2)}{r^3}\beta^2 + \frac{4(n-1)^2}{r^2}\beta^3 + \frac{2(n-2)(n-1)}{r}\beta^4 \\
&+ (n-2)^2\beta^5 = 0.
\end{aligned} \tag{2.14}
$$

Proof. Let $\phi : (\mathbb{R}^n, g) \to (N^n, h)$ $(n \geq 3)$ be a conformal map of dilation λ where we suppose that $\ln \lambda$ is radial $(\ln \lambda = \alpha(r), r = |x|)$. By Theorem 2.5, ϕ is triharmonic if and only if

$$
\begin{aligned}
&\text{grad } \left(\Delta^2 \ln \lambda\right) + 2\text{grad } \left(\Delta\left(|\text{grad } \ln \lambda|^2\right)\right) - (\Delta \ln \lambda)\,\text{grad } \Delta \ln \lambda \\
&- 2(n-1)\,|\text{grad } \ln \lambda|^2 \,\text{grad } \Delta \ln \lambda - 2(\Delta \ln \lambda)\ \text{grad } \left(|\text{grad } \ln \lambda|^2\right) \\
&- 6(n-2)\,|\text{grad } \ln \lambda|^2 \,\text{grad } \left(|\text{grad } \ln \lambda|^2\right) - 3\left(\Delta^2 \ln \lambda\right)\text{grad } \ln \lambda \\
&+ 2(n-2)(\Delta \ln \lambda)\,|\text{grad } \ln \lambda|^2 \,\text{grad } \ln \lambda + (n-2)^2\,|\text{grad } \ln \lambda|^4 \,\text{grad } \ln \lambda \\
&+ 2(\Delta \ln \lambda)^2 \,\text{grad } \ln \lambda - (n+2)\left(\Delta\left(|\text{grad } \ln \lambda|^2\right)\right)\text{grad } \ln \lambda \\
&+ 4\nabla_{\text{grad } \ln \lambda}\text{grad } \Delta \ln \lambda + 8\nabla_{\text{grad } \ln \lambda}\text{grad } \left(|\text{grad } \ln \lambda|^2\right) \\
&- n\nabla_{\text{grad } \Delta \ln \lambda}\text{grad } \ln \lambda - \frac{(n-2)(n+6)}{2}\nabla_{\text{grad } \left(|\text{grad } \ln \lambda|^2\right)}\text{grad } \ln \lambda = 0.
\end{aligned}
$$

A rigorous calculation gives us

$$
\begin{aligned}
&\text{grad } \alpha = \alpha' \frac{\partial}{\partial r}, \quad |\text{grad } \alpha|^2 = (\alpha')^2, \quad \text{grad } \left(|\text{grad } \alpha|^2\right) = 2\alpha'\alpha'' \frac{\partial}{\partial r}, \\
&\Delta\alpha = \alpha'' + \frac{n-1}{r}\alpha', \quad \text{grad } \Delta\alpha = \left(\alpha^{(3)} + \frac{n-1}{r}\alpha'' - \frac{n-1}{r^2}\alpha'\right)\frac{\partial}{\partial r}, \\
&\Delta^2\alpha = \alpha^{(4)} + \frac{2(n-1)}{r}\alpha^{(3)} + \frac{(n-1)(n-3)}{r^2}\alpha'' - \frac{(n-1)(n-3)}{r^3}\alpha',
\end{aligned}
$$

$$
\text{grad } \Delta^2\alpha = \left(\alpha^{(5)} + \frac{2(n-1)}{r}\alpha^{(4)} + \frac{(n-1)(n-5)}{r^2}\alpha^{(3)}\right)\frac{\partial}{\partial r}
$$

$$-\left(\frac{3\left(n-1\right)\left(n-3\right)}{r^{3}}\alpha''+\frac{3\left(n-1\right)\left(n-3\right)}{r^{4}}\alpha'\right)\frac{\partial}{\partial r},$$

$$\Delta\left(\left|\text{grad }\alpha\right|^{2}\right)=2\alpha'\alpha^{(3)}+2\left(\alpha''\right)^{2}+\frac{2\left(n-1\right)}{r}\alpha'\alpha''$$

and

$$\begin{aligned}\text{grad }\left(\Delta\left(\left|\text{grad }\alpha\right|^{2}\right)\right)=&\left(2\alpha'\alpha^{(4)}+6\alpha''\alpha^{(3)}+\frac{2\left(n-1\right)}{r}\alpha'\alpha^{(3)}\right)\frac{\partial}{\partial r}\\&+\left(\frac{2\left(n-1\right)}{r}\left(\alpha''\right)^{2}-\frac{2\left(n-1\right)}{r^{2}}\alpha'\alpha''\right)\frac{\partial}{\partial r}.\end{aligned}$$

Hence, ϕ is triharmonic if and only if the function α satisfies the following differential equation

$$\begin{aligned}&\alpha^{(5)}+\frac{2\left(n-1\right)}{r}\alpha^{(4)}+5\alpha'\alpha^{(4)}+\frac{\left(n-1\right)\left(n-5\right)}{r^{2}}\alpha^{(3)}-\left(n-11\right)\alpha''\alpha^{(3)}\\&+\frac{\left(n-1\right)}{r}\alpha'\alpha^{(3)}-\left(4n-14\right)\left(\alpha'\right)^{2}\alpha^{(3)}-\frac{3\left(n-1\right)\left(n-3\right)}{r^{3}}\alpha''\\&-10\left(n-2\right)\left(\alpha'\right)^{3}\alpha''-\frac{2\left(n-1\right)\left(2n+1\right)}{r}\left(\alpha'\right)^{2}\alpha''-\frac{\left(3n+1\right)\left(n-1\right)}{r^{2}}\alpha'\alpha''\\&-\left(n^{2}+6n-22\right)\alpha'\left(\alpha''\right)^{2}-\frac{\left(n-3\right)\left(n-1\right)}{r}\left(\alpha''\right)^{2}+\frac{3\left(n-1\right)\left(n-3\right)}{r^{4}}\alpha'\\&+\frac{\left(n-1\right)\left(4n-2\right)}{r^{3}}\left(\alpha'\right)^{2}+\frac{4\left(n-1\right)^{2}}{r^{2}}\left(\alpha'\right)^{3}+\frac{2\left(n-2\right)\left(n-1\right)}{r}\left(\alpha'\right)^{4}\\&+\left(n-2\right)^{2}\left(\alpha'\right)^{5}=0.\end{aligned}$$

Let $\beta=\alpha'$. Then the last equation is equivalent to the fourth-order differential equation

$$\begin{aligned}&\beta^{(4)}+\frac{2\left(n-1\right)}{r}\beta^{(3)}+5\beta\beta^{(3)}+\frac{\left(n-1\right)\left(n-5\right)}{r^{2}}\beta''-\left(n-11\right)\beta'\beta''\\&+\frac{\left(n-1\right)}{r}\beta\beta''-\left(4n-14\right)\beta^{2}\beta''-\left(n^{2}+6n-22\right)\beta\left(\beta'\right)^{2}\\&-\frac{\left(n-3\right)\left(n-1\right)}{r}\left(\beta'\right)^{2}-\frac{3\left(n-1\right)\left(n-3\right)}{r^{3}}\beta'-10\left(n-2\right)\beta^{3}\beta'\\&-\frac{2\left(n-1\right)\left(2n+1\right)}{r}\beta^{2}\beta'-\frac{\left(3n+1\right)\left(n-1\right)}{r^{2}}\beta\beta'+\frac{3\left(n-1\right)\left(n-3\right)}{r^{4}}\beta\\&+\frac{\left(n-1\right)\left(4n-2\right)}{r^{3}}\beta^{2}+\frac{4\left(n-1\right)^{2}}{r^{2}}\beta^{3}+\frac{2\left(n-2\right)\left(n-1\right)}{r}\beta^{4}\\&+\left(n-2\right)^{2}\beta^{5}=0.\end{aligned}$$

□

As a consequence of Corollary 2.6, we present some remarks that give us a particular solutions of equation (2.14) and that allows us to construct a triharmonic maps.

Remark 2.7. We are looking for particular solutions of the type $\beta = \frac{a}{r}$ $(a \in \mathbb{R}^*)$. By (2.14), we deduce that $\phi : (\mathbb{R}^n \backslash \{0\}, g) \to (N^n, h)$ $(n \geq 3)$ is triharmonic if and only if a is solution of the algebraic equation

$$(n-2)^2 a^4 + 2(n-2)(n+4) a^3 + \left(7n^2 - 24n + 52\right) a^2 + \left(6n^2 - 56\right) a + 8(n-2)(n-4) = 0.$$

This equation has real solutions if and only if $n \in \{3, 4, 5, 6\}$.

(1) If $n = 3$, we find $a = \frac{-5+\sqrt{17}}{2}$, $a = \frac{-5-\sqrt{17}}{2}$, $a = \frac{-9+\sqrt{97}}{2}$ and $a = \frac{-9-\sqrt{97}}{2}$. Note that for $a = \frac{-5+\sqrt{17}}{2}$ or $a = \frac{-5-\sqrt{17}}{2}$ any conformal map $\phi : (\mathbb{R}^3 \backslash \{0\}, g) \to (N^3, h)$ of dilation $\lambda = \frac{C}{r^{\frac{5-\sqrt{17}}{2}}}$ or $\lambda = \frac{C}{r^{\frac{5+\sqrt{17}}{2}}}$ $(C > 0)$ is biharmonic and triharmonic. For $\lambda = \frac{C}{r^{\frac{9-\sqrt{97}}{2}}}$ or $\lambda = \frac{C}{r^{\frac{9+\sqrt{97}}{2}}}$, the conformal map $\phi : (\mathbb{R}^3 \backslash \{0\}, g) \to (N^3, h)$ of dilation $\lambda = Cr^{\frac{-9+\sqrt{97}}{2}}$ or $\lambda = \frac{C}{r^{\frac{9+\sqrt{97}}{2}}}$ is triharmonic non-biharmonic.

(2) If $n = 4$, the solutions are $a = -5$, $a = -2$ and $a = -1$. For $a = -2$ and $a = -1$, the conformal map $\phi : (\mathbb{R}^4 \backslash \{0\}, g) \to (N^4, h)$ of dilation $\lambda = \frac{C}{r^2}$ or $\lambda = \frac{C}{r}$ $(C > 0)$ is biharmonic and triharmonic. For $a = -5$, the conformal map $\phi : (\mathbb{R}^4 \backslash \{0\}, g) \to (N^4, h)$ of dilation $\lambda = \frac{C}{r^5}$ is triharmonic non-biharmonic.

(3) If $n = 5$, we find $a = \frac{-11+\sqrt{73}}{6}$ and $a = \frac{-11-\sqrt{73}}{6}$ so $\lambda = \frac{C}{r^{\frac{11-\sqrt{73}}{6}}}$ or $\lambda = \frac{C}{r^{\frac{11+\sqrt{73}}{6}}}$ $(C > 0)$, it follows that any conformal map $\phi : (\mathbb{R}^5 \backslash \{0\}, g) \to (N^5, h)$ of dilation $\lambda = \frac{C}{r^{\frac{11+\sqrt{73}}{6}}}$ or $\lambda = \frac{C}{r^{\frac{11-\sqrt{73}}{6}}}$ is triharmonic non-biharmonic.

(4) The case where $n = 6$ gives us $a = -2$ and $a = -1$, so $\lambda = \frac{C}{r^2}$ and $\lambda = \frac{C}{r}$ $(C > 0)$. Then, in this case any conformal map $\phi : (\mathbb{R}^6 \backslash \{0\}, g) \to (N^6, h)$ of dilation $\lambda = \frac{C}{r^2}$ or $\lambda = \frac{C}{r}$ is triharmonic non-biharmonic.

Example 2.8. Using Remark 2.7, we give two examples:

(1) For $a = -2$, we can cite the example of the inversion $\phi : \mathbb{R}^n \backslash \{0\} \longrightarrow \mathbb{R}^n \backslash \{0\}$ defined by $\phi(x) = \frac{x}{|x|^2}$; it is a conformal map of dilation $\lambda = \frac{1}{r^2}$. We deduce that the inversion is
   - harmonic if and only if $n = 2$.
   - triharmonic, biharmonic and non-harmonic if and only if $n = 4$.
   - triharmonic non- biharmonic if and only if $n = 6$.

(2) The case $a = -1$ gives dilation $\lambda = \frac{1}{r}$. In this case we consider the conformal map $\phi : \mathbb{R}^n \backslash \{0\} \longrightarrow \mathbb{R} \times \mathbb{S}^{n-1}$, given in polar coordinates by $\phi(r\theta) = (\ln r, \theta)$ for $r > 0$, $\theta \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$. This map is
   - harmonic if and only if $n = 2$.
   - triharmonic, biharmonic and non-harmonic if and only if $n = 4$.
   - triharmonic non-biharmonic if and only if $n = 6$.

We consider $\phi : (\mathbb{R}^n, g) \to (N^n, h)$ $(n \geq 3)$ a conformal map of dilation λ. To calculate the trace of the stress-energy tensor $S_3(\phi)$, we need some lemmas. In the first lemma, we consider a Riemannian manifold (M^n, g) and we give a relation between $\Delta |\text{grad } f|^2$ and $|\nabla \text{grad } f|^2$ for any function $f \in C^\infty(M)$.

Lemma 2.9. *Let (M^n, g) be a Riemannian manifold. Then for any function $f \in C^\infty(M)$, we have*

$$\Delta |grad\ f|^2 = 2df(grad\ \Delta f) + 2|\nabla grad\ f|^2 + 2df(Ricci(grad\ f)). \tag{2.15}$$

Proof. Let $(e_i)_{1\leq i\leq n}$ be an orthonormal frame on M. By definition, we have

$$\Delta |\text{grad } f|^2 = e_i\left(e_i\left(|\text{grad } f|^2\right)\right) - (\nabla_{e_i} e_i)\left(|\text{grad } f|^2\right).$$

Then

$$\begin{aligned}
\Delta |\text{grad } f|^2 &= e_i(e_i(g(\text{grad } f, \text{grad } f))) - (\nabla_{e_i} e_i)(g(\text{grad } f, \text{grad } f)) \\
&= 2e_i(g(\nabla_{e_i}\text{grad } f, \text{grad } f)) - 2\left(g\left(\nabla_{\nabla_{e_i} e_i}\text{grad } f, \text{grad } f\right)\right) \\
&= 2g(\nabla_{e_i}\nabla_{e_i}\text{grad } f, \text{grad } f) + 2g(\nabla_{e_i}\text{grad } f, \nabla_{e_i}\text{grad } f) \\
&\quad - 2\left(g\left(\nabla_{\nabla_{e_i} e_i}\text{grad } f, \text{grad } f\right)\right) \\
&= 2g\left(Tr_g\nabla^2\text{grad } f, \text{grad } f\right) + 2|\nabla\text{grad } f|^2,
\end{aligned}$$

which gives us

$$\Delta |\text{grad } f|^2 = 2df(\text{grad } \Delta f) + 2|\nabla\text{grad } f|^2 + 2df(Ricci(\text{grad } f)). \qquad \square$$

Lemma 2.10. *Let $\phi : (M^n, g) \to (N^n, h)$ $(n \geq 3)$ be a conformal map of dilation λ. Then*

$$\begin{aligned}
&|\nabla\tau(\phi)|^2 \\
&= n(n-2)^2\lambda^2|grad\ \ln\lambda|^4 + 2(n-2)^2\lambda^2(\Delta\ln\lambda)|grad\ \ln\lambda|^2 \\
&\quad + \frac{(n-2)^2}{2}\lambda^2\Delta\left(|grad\ \ln\lambda|^2\right) - (n-2)^2\lambda^2 d\ln\lambda(grad\ \Delta\ln\lambda) \\
&\quad - (n-2)^2\lambda^2 d\ln\lambda(Ricci(grad\ \ln\lambda))
\end{aligned} \tag{2.16}$$

and

$$\begin{aligned}
\langle d\phi, \nabla\Delta\tau(\phi)\rangle = &\ (n-2)(n-1)\lambda^2 d\ln\lambda(grad\ \Delta\ln\lambda) \\
&+ (n-2)(n+2)\lambda^2 d\ln\lambda\left(grad\ \left(|grad\ \ln\lambda|^2\right)\right) \\
&- 2(n-2)(n-1)\lambda^2(\Delta\ln\lambda)|grad\ \ln\lambda|^2 \\
&+ 2(n-2)\lambda^2\Delta\left(|grad\ \ln\lambda|^2\right) + (n-2)\lambda^2\left(\Delta^2\ln\lambda\right) \\
&- (n-2)\lambda^2(\Delta\ln\lambda)^2 - n(n-2)^2\lambda^2|grad\ \ln\lambda|^4 \\
&+ n(n-2)\lambda^2 d\ln\lambda(Ricci(grad\ \ln\lambda)) \\
&+ (n-2)\lambda^2 div(Ricci(grad\ \ln\lambda)).
\end{aligned} \tag{2.17}$$

Proof. For the term $|\nabla\tau(\phi)|^2$, we have

$$\begin{aligned}|\nabla\tau(\phi)|^2 &= h\left(\nabla_{e_i}\tau(\phi), \nabla_{e_i}\tau(\phi)\right)\\ &= (n-2)^2\, h\left(\nabla_{e_i}d\phi\left(\text{grad }\ln\lambda\right), \nabla_{e_i}d\phi\left(\text{grad }\ln\lambda\right)\right).\end{aligned}$$

Using the fact that

$$\nabla_{e_i}d\phi\left(\text{grad }\ln\lambda\right) = |\text{grad }\ln\lambda|^2\, d\phi(e_i) + d\phi\left(\nabla_{e_i}\text{grad }\ln\lambda\right),$$

we deduce that

$$\begin{aligned}|\nabla\tau(\phi)|^2 =\ & (n-2)^2\, h\left(|\text{grad }\ln\lambda|^2\, d\phi(e_i), |\text{grad }\ln\lambda|^2\, d\phi(e_i)\right)\\ & + 2(n-2)^2\, h\left(|\text{grad }\ln\lambda|^2\, d\phi(e_i), d\phi\left(\nabla_{e_i}\text{grad }\ln\lambda\right)\right)\\ & + (n-2)^2\, h\left(d\phi\left(\nabla_{e_i}\text{grad }\ln\lambda\right), d\phi\left(\nabla_{e_i}\text{grad }\ln\lambda\right)\right)\\ =\ & n(n-2)^2\lambda^2\, |\text{grad }\ln\lambda|^4 + (n-2)^2\lambda^2\, |\nabla\text{grad }\ln\lambda|^2\\ & + 2(n-2)^2\lambda^2\,(\Delta\ln\lambda)\,|\text{grad }\ln\lambda|^2.\end{aligned}$$

By Lemma 2.9, we have

$$\begin{aligned}|\nabla\text{grad }\ln\lambda|^2 = \frac{1}{2}\Delta\left(|\text{grad }\ln\lambda|^2\right) - d\ln\lambda\left(\text{grad }\Delta\ln\lambda\right)\\ - d\ln\lambda\left(Ricci\left(\text{grad }\ln\lambda\right)\right).\end{aligned}$$

It follows that

$$\begin{aligned}|\nabla\tau(\phi)|^2 =\ & n(n-2)^2\lambda^2\,|\text{grad }\ln\lambda|^4 + 2(n-2)^2\lambda^2(\Delta\ln\lambda)\,|\text{grad }\ln\lambda|^2\\ & + \frac{(n-2)^2}{2}\lambda^2\Delta\left(|\text{grad }\ln\lambda|^2\right) - (n-2)^2\lambda^2 d\ln\lambda\left(\text{grad }\Delta\ln\lambda\right)\\ & - (n-2)^2\lambda^2 d\ln\lambda\left(Ricci\left(\text{grad }\ln\lambda\right)\right).\end{aligned}$$

Now let's look at the term $\langle d\phi, \nabla\Delta\tau(\phi)\rangle$, we have

$$\langle d\phi, \nabla\Delta\tau(\phi)\rangle = h\left(d\phi(e_i), \nabla_{e_i}\Delta\tau(\phi)\right).$$

By Theorem 2.3, we obtain

$$\begin{aligned}\Delta\tau(\phi) =\ & (n-2)\, d\phi\left(\text{grad }\Delta\ln\lambda\right) + 2(n-2)\, d\phi\left(\text{grad }\left(|\text{grad }\ln\lambda|^2\right)\right)\\ & - (n-2)(\Delta\ln\lambda)\, d\phi\left(\text{grad }\ln\lambda\right)\\ & - (n-2)^2\,|\text{grad }\ln\lambda|^2\, d\phi\left(\text{grad }\ln\lambda\right)\\ & + (n-2)\, d\phi\left(Ricci\left(\text{grad }\ln\lambda\right)\right),\end{aligned}$$

then

$$\begin{aligned}\langle d\phi, \nabla\Delta\tau(\phi)\rangle =\ & (n-2)\, h\left(d\phi(e_i), \nabla_{e_i}d\phi\left(\text{grad }\Delta\ln\lambda\right)\right)\\ & + 2(n-2)\, h\left(d\phi(e_i), \nabla_{e_i}d\phi\left(\text{grad }\left(|\text{grad }\ln\lambda|^2\right)\right)\right)\\ & - (n-2)\, h\left(d\phi(e_i), \nabla_{e_i}(\Delta\ln\lambda)\, d\phi\left(\text{grad }\ln\lambda\right)\right)\end{aligned}$$

$$
\begin{aligned}
&- (n-2)^2 h\left(d\phi(e_i), \nabla_{e_i} |\text{grad } \ln\lambda|^2 d\phi(\text{grad } \ln\lambda)\right) \\
&+ (n-2) h\left(d\phi(e_i), \nabla_{e_i} d\phi(Ricci(\text{grad } \ln\lambda))\right).
\end{aligned}
$$

The development of each term of this last equation gives

$$
h\left(d\phi(e_i), \nabla_{e_i} d\phi(\text{grad } \Delta\ln\lambda)\right) = n\lambda^2 d\ln\lambda(\text{grad } \Delta\ln\lambda) + \lambda^2\left(\Delta^2\ln\lambda\right),
$$

$$
\begin{aligned}
h\left(d\phi(e_i), \nabla_{e_i} d\phi\left(\text{grad } \left(|\text{grad } \ln\lambda|^2\right)\right)\right) &= n\lambda^2 d\ln\lambda\left(\text{grad } \left(|\text{grad } \ln\lambda|^2\right)\right) \\
&\quad + \lambda^2\Delta\left(|\text{grad } \ln\lambda|^2\right),
\end{aligned}
$$

$$
\begin{aligned}
& h\left(d\phi(e_i), \nabla_{e_i}(\Delta\ln\lambda) d\phi(\text{grad } \ln\lambda)\right) \\
&= (\Delta\ln\lambda) h\left(d\phi(e_i), \nabla_{e_i} d\phi(\text{grad } \ln\lambda)\right) \\
&\quad + h\left(d\phi(e_i), e_i(\Delta\ln\lambda) d\phi(\text{grad } \ln\lambda)\right) \\
&= n\lambda^2(\Delta\ln\lambda)|\text{grad } \ln\lambda|^2 + \lambda^2(\Delta\ln\lambda)^2 \\
&\quad + \lambda^2 d\ln\lambda(\text{grad } \Delta\ln\lambda),
\end{aligned}
$$

$$
\begin{aligned}
& h\left(d\phi(e_i), \nabla_{e_i} |\text{grad } \ln\lambda|^2 d\phi(\text{grad } \ln\lambda)\right) \\
&= |\text{grad } \ln\lambda|^2 h\left(d\phi(e_i), \nabla_{e_i} d\phi(\text{grad } \ln\lambda)\right) \\
&\quad + h\left(d\phi(e_i), e_i\left(|\text{grad } \ln\lambda|^2\right) d\phi(\text{grad } \ln\lambda)\right) \\
&= n\lambda^2 |\text{grad } \ln\lambda|^4 + \lambda^2(\Delta\ln\lambda)|\text{grad } \ln\lambda|^2 \\
&\quad + \lambda^2 d\ln\lambda\left(\text{grad } \left(|\text{grad } \ln\lambda|^2\right)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
h\left(d\phi(e_i), \nabla_{e_i} d\phi(Ricci(\text{grad } \ln\lambda))\right) &= n\lambda^2 d\ln\lambda(Ricci(\text{grad } \ln\lambda)) \\
&\quad + \lambda^2(div(Ricci(\text{grad } \ln\lambda))).
\end{aligned}
$$

We conclude that

$$
\begin{aligned}
\langle d\phi, \nabla\Delta\tau(\phi)\rangle = &\ (n-2)(n-1)\lambda^2 d\ln\lambda(\text{grad } \Delta\ln\lambda) \\
&+ (n-2)(n+2)\lambda^2 d\ln\lambda\left(\text{grad } \left(|\text{grad } \ln\lambda|^2\right)\right) \\
&- 2(n-2)(n-1)\lambda^2(\Delta\ln\lambda)|\text{grad } \ln\lambda|^2 \\
&+ 2(n-2)\lambda^2\Delta\left(|\text{grad } \ln\lambda|^2\right) + (n-2)\lambda^2\left(\Delta^2\ln\lambda\right) \\
&- (n-2)\lambda^2(\Delta\ln\lambda)^2 - n(n-2)^2\lambda^2|\text{grad } \ln\lambda|^4 \\
&+ n(n-2)\lambda^2 d\ln\lambda(Ricci(\text{grad } \ln\lambda)) \\
&+ (n-2)\lambda^2 div(Ricci(\text{grad } \ln\lambda)).
\end{aligned}
$$

□

Theorem 2.11. *Let $\phi : (\mathbb{R}^n, g) \to (N^n, h)$ $(n \geq 3)$ be a conformal map of dilation λ. Then*

$$\begin{aligned}
Tr_g S_3(\phi) = & -\frac{(n-2)^2(n-4)}{2}\lambda^2 d\ln\lambda\,(grad\;\Delta\ln\lambda) - (n-2)^2\lambda^2\left(\Delta^2\ln\lambda\right) \\
& + (n-2)^3\lambda^2 d\ln\lambda\left(grad\;\left(|grad\;\ln\lambda|^2\right)\right) + (n-2)^2\lambda^2(\Delta\ln\lambda)^2 \\
& + \frac{(n-2)^2(n-10)}{4}\lambda^2\Delta\left(|grad\;\ln\lambda|^2\right) + \frac{n(n-2)^3}{2}\lambda^2|grad\;\ln\lambda|^4 \\
& + 2(n-2)^3\lambda^2(\Delta\ln\lambda)|grad\;\ln\lambda|^2 .
\end{aligned}$$

Proof. By definition, for any vectors field X, Y, we have

$$\begin{aligned}
S_3(\phi)(X,Y) = & \left(\frac{1}{2}|\nabla\tau(\phi)|^2 - h(\tau(\phi), \Delta\tau(\phi)) - \langle d\phi, \nabla\Delta\tau(\phi)\rangle\right) g(X,Y) \\
& + h(d\phi(X), \nabla_Y\Delta\tau(\phi)) + h(d\phi(Y), \nabla_X\Delta\tau(\phi)) \\
& - h(\nabla_X\tau(\phi), \nabla_Y\tau(\phi)),
\end{aligned}$$

then

$$\begin{aligned}
Tr_g S_3(\phi) = & \, n\left(\frac{1}{2}|\nabla\tau(\phi)|^2 - h(\tau(\phi), \Delta\tau(\phi)) - \langle d\phi, \nabla\Delta\tau(\phi)\rangle\right) \\
& + 2Tr_h h(d\phi, \nabla\Delta\tau(\phi)) - |\nabla\tau(\phi)|^2,
\end{aligned}$$

which gives us

$$\begin{aligned}
& Tr_g S_3(\phi) \\
& = \frac{n-2}{2}|\nabla\tau(\phi)|^2 - nh(\tau(\phi), \Delta\tau(\phi)) - (n-2)\langle d\phi, \nabla\Delta\tau(\phi)\rangle .
\end{aligned} \tag{2.18}$$

The terms $|\nabla\tau(\phi)|^2$ and $\langle d\phi, \nabla\Delta\tau(\phi)\rangle$ are calculated in Lemma 2.10. For the term $h(\tau(\phi), \Delta\tau(\phi))$, we have

$$\tau(\phi) = (2-n)\,d\phi(\text{grad}\;\ln\lambda)$$

and

$$\begin{aligned}
\Delta\tau(\phi) = & \,(n-2)\,d\phi(\text{grad}\;\Delta\ln\lambda) + 2(n-2)\,d\phi\left(\text{grad}\;\left(|\text{grad}\;\ln\lambda|^2\right)\right) \\
& - (n-2)(\Delta\ln\lambda)\,d\phi(\text{grad}\;\ln\lambda) \\
& - (n-2)^2|\text{grad}\;\ln\lambda|^2\,d\phi(\text{grad}\;\ln\lambda).
\end{aligned}$$

It follows that

$$\begin{aligned}
h(\tau(\phi), \Delta\tau(\phi)) = & -2(n-2)^2\lambda^2 d\ln\lambda\left(\text{grad}\;\left(|\text{grad}\;\ln\lambda|^2\right)\right) \\
& - (n-2)^2\lambda^2 d\ln\lambda\,(\text{grad}\;\Delta\ln\lambda) \\
& + (n-2)^2\lambda^2(\Delta\ln\lambda)|\text{grad}\;\ln\lambda|^2 \\
& + (n-2)^3\lambda^2|\text{grad}\;\ln\lambda|^4 .
\end{aligned} \tag{2.19}$$

By equations (2.16), (2.17) and (2.19), we deduce that

$$
\begin{aligned}
&Tr_g S_3(\phi) \\
= &-\frac{(n-2)^2(n-4)}{2}\lambda^2 d\ln\lambda\left(\operatorname{grad}\Delta\ln\lambda\right) - (n-2)^2\lambda^2\left(\Delta^2\ln\lambda\right) \\
&+(n-2)^3\lambda^2 d\ln\lambda\left(\operatorname{grad}\left(|\operatorname{grad}\ \ln\lambda|^2\right)\right) + (n-2)^2\lambda^2(\Delta\ln\lambda)^2 \\
&+\frac{(n-2)^2(n-10)}{4}\lambda^2\Delta\left(|\operatorname{grad}\ \ln\lambda|^2\right) + \frac{n(n-2)^3}{2}\lambda^2|\operatorname{grad}\ \ln\lambda|^4 \\
&+2(n-2)^3\lambda^2(\Delta\ln\lambda)|\operatorname{grad}\ \ln\lambda|^2. \qquad \square
\end{aligned}
$$

Corollary 2.12. *Let $\phi : (\mathbb{R}^n, g) \to (N^n, h)$ $(n \geq 3)$ be a conformal map of dilation λ. Then $Tr_g S_3(\phi) = 0$ if and only if*

$$
\begin{aligned}
&\Delta^2\ln\lambda - \frac{(n-10)}{4}\Delta\left(|grad\ \ln\lambda|^2\right) - 2(n-2)(\Delta\ln\lambda)|grad\ \ln\lambda|^2 \\
&-(\Delta\ln\lambda)^2 - \frac{n(n-2)}{2}|grad\ \ln\lambda|^4 + \frac{(n-4)}{2}d\ln\lambda\left(grad\ \Delta\ln\lambda\right) \\
&-(n-2)\,d\ln\lambda\left(grad\left(|grad\ \ln\lambda|^2\right)\right) = 0.
\end{aligned}
$$

Moreover, if we suppose that $\ln\lambda$ is radial $(\ln\lambda = \alpha(r), r = |x|)$, we deduce that $Tr_g S_3(\phi) = 0$ if and only if $\beta = \alpha'$ satisfies the following ordinary differential equation:

$$
\begin{aligned}
&\beta^{(3)} + \frac{2(n-1)}{r}\beta^{(2)} + 3\beta\beta^{(2)} + \frac{(n-1)(n-3)}{r^2}\beta' + \frac{(n-1)}{r}\beta\beta' \\
&-\frac{(n-8)}{2}(\beta')^2 - 4(n-2)\beta^2\beta' - \frac{(n-1)(n-3)}{r^3}\beta \\
&-\frac{3(n-2)(n-1)}{2r^2}\beta^2 - \frac{2(n-1)(n-2)}{r}\beta^3 - \frac{n(n-2)}{2}\beta^4 = 0.
\end{aligned} \tag{2.20}
$$

Remark 2.13. Let us look for particular solutions of equation (2.20), and are written in the form $\beta = \frac{a}{r}$ $(a \in \mathbb{R}^*)$, we find the following algebraic equation

$$a^3n^2 - 2a^3n + 4a^2n^2 - 20a^2n + 24a^2 + 3an^2 - 6an - 16a + 4n^2 - 24n + 32 = 0.$$

Example 2.14. Using Remark 2.13, we give two examples:

(1) For $a = -2$, we can cite the example of the inversion
$$\phi : \mathbb{R}^n \backslash \{0\} \longrightarrow \mathbb{R}^n \backslash \{0\}$$
defined by $\phi(x) = \frac{x}{|x|^2}$, it is a conformal map of dilation $\lambda = \frac{1}{r^2}$.
We conclude that $Tr_g S_3(\phi) = 0$ if and only if $n = 10$.

(2) The case $a = -1$ gives dilation $\lambda = \frac{1}{r}$, in this case we consider the conformal map $\phi : \mathbb{R}^n \backslash \{0\} \longrightarrow \mathbb{R} \times \mathbb{S}^{n-1}$, given in polar coordinates by $\phi(r\theta) = (\ln r, \theta)$, for $r > 0, \theta \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$.
We deduce that $Tr_g S_3(\phi) = 0$ if and only if $n = 3$ or $n = 6$.

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References

[1] P. Baird, A. Fardoun, and S. Ouakkas, *Conformal and semi-conformal biharmonic maps*, Ann. Global Anal. Geom. **34** (2008), no. 4, 403–414. https://doi.org/10.1007/s10455-008-9118-8

[2] P. Baird and J. C. Wood, *Harmonic morphisms between Riemannian manifolds*, London Mathematical Society Monographs. New Series, 29, The Clarendon Press, Oxford University Press, Oxford, 2003. https://doi.org/10.1093/acprof:oso/9780198503620.001.0001

[3] V. Branding, *The stress-energy tensor for polyharmonic maps*, Nonlinear Anal. **190** (2020), 111616, 17 pp. https://doi.org/10.1016/j.na.2019.111616

[4] V. Branding, *A structure theorem for polyharmonic maps between Riemannian manifolds*, J. Differential Equations **273** (2021), 14–39. https://doi.org/10.1016/j.jde.2020.11.046

[5] P. Goldstein, P. Strzelecki, and A. Zatorska-Goldstein, *On polyharmonic maps into spheres in the critical dimension*, Ann. Inst. H. Poincaré Anal. Non Linéaire **26** (2009), no. 4, 1387–1405. https://doi.org/10.1016/j.anihpc.2008.10.008

[6] G. Y. Jiang, 2*-harmonic maps and their first and second variational formulas*, Chinese Ann. Math. Ser. A **7** (1986), no. 4, 389–402.

[7] E. Loubeau and Y.-L. Ou, *Biharmonic maps and morphisms from conformal mappings*, Tohoku Math. J. (2) **62** (2010), no. 1, 55–73. https://doi.org/10.2748/tmj/1270041027

[8] S. Maeta, *k-harmonic maps into a Riemannian manifold with constant sectional curvature*, Proc. Amer. Math. Soc. **140** (2012), no. 5, 1835–1847. https://doi.org/10.1090/S0002-9939-2011-11049-9

[9] S. Maeta, *The second variational formula of the k-energy and k-harmonic curves*, Osaka J. Math. **49** (2012), no. 4, 1035–1063. http://projecteuclid.org/euclid.ojm/1355926886

[10] S. Maeta, *Construction of triharmonic maps*, Houston J. Math. **41** (2015), no. 2, 433–444.

[11] N. Nakauchi and H. Urakawa, *Polyharmonic maps into the Euclidean space*, Note Mat. **38** (2018), no. 1, 89–100. https://doi.org/10.5840/ancientphil20183815

[12] S. Ouakkas and D. Djebbouri, *Conformal maps, biharmonic maps, and the warped product*, Mathematics **15** (2016), no. 4. https://doi.org/10.3390/math4010015

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