

GEOMETRIC PROPERTIES ON (j, k) -SYMMETRIC FUNCTIONS RELATED TO STARLIKE AND CONVEX FUNCTION

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ABSTRACT. For $j = 0, 1, 2, \dots, k - 1$; $k \geq 2$; and $-1 \leq B < A \leq 1$, we have introduced the functions classes denoted by $\mathcal{ST}_{[j,k]}(A, B)$ and $\mathcal{K}_{[j,k]}(A, B)$, respectively, called the generalized (j, k) -symmetric starlike and convex functions. We first proved the sharp bounds on $|f(z)|$ and $|f'(z)|$. Various radii related problems, such as radius of (j, k) -symmetric starlikeness, convexity, strongly starlikeness and parabolic starlikeness are determined. The quantity $|a_3^2 - a_5|$, which provide the initial bound on Zalcman functional is obtained for the functions in the family $\mathcal{ST}_{[j,k]}$. Furthermore, the sharp pre-Schwarzian norm is also established for the case when f is a member of $\mathcal{K}_{[j,k]}(\alpha)$ for all $0 \leq \alpha < 1$.

1. Introduction and preliminaries

We denote by \mathcal{H} , the family of analytic functions in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ with the condition of normalization $f(0) = f'(0) - 1 = 0$. Thus, all functions in the family \mathcal{H} have the Taylor-Maclaurin series of the form:

$$(1.1) \quad f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad (a_1 = 1, z \in \mathbb{D}).$$

Let \mathcal{S} denote the subclass of \mathcal{H} that are univalent in \mathbb{D} . Suppose that $f, g \in \mathcal{H}$, then f is subordinate to g in \mathbb{D} (cf. [12]) denoted by $f \prec g$ if there exists a function $w \in \mathcal{H}$ satisfies the Schwarz condition $w(0) = 0$ and $|w(z)| < 1$ such that

$$f(z) = g(w(z)) \quad (z \in \mathbb{D}).$$

Received February 25, 2021; Accepted April 15, 2021.

2010 *Mathematics Subject Classification.* Primary 30C45.

Key words and phrases. Analytic and univalent function, subordination, (j, k) -symmetric starlike and convex functions, distortion, growth, radius of starlikeness, radius of convexity, radius of strong starlikeness, radius of parabolic starlikeness, Zalcman functional, pre-Schwarzian derivative.

The present work of the first author is supported by OSHEC, Government of Odisha, India. The second author acknowledges the support from INSPIRE Fellowship, DST, Government of India.

Let $\mathcal{P}[A, B]$, with $-1 \leq B < A \leq 1$, denote the class of functions p analytic in \mathbb{D} that satisfy the subordination relation $p(z) \prec \frac{1+Az}{1+Bz}$ and has the series of the form:

$$p(z) = 1 + p_1z + p_2z^2 + \cdots, \quad (z \in \mathbb{D}),$$

with $p(0) = 1, \Re(p(z)) > 0$. This class was introduced and studied by Janowski [8]. In particular, $\mathcal{P}[1, -1] := \mathcal{P}$ is the usual Caratheodory class. Most important basic subclasses of the family \mathcal{S} includes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$, respectively, the family of univalent starlike functions and convex functions of order α in \mathbb{D} for all $0 \leq \alpha < 1$. For fixed integer $k \geq 1$, corresponding k -fold symmetric function (cf. [16] also see [3, 22]), denoted by $f_k(z)$, is of the form

$$(1.2) \quad f_k(z) = \frac{1}{k} \sum_{\nu=0}^{k-1} \frac{f(\epsilon^\nu z)}{\epsilon^\nu}, \quad (z \in \mathbb{D}),$$

where $\epsilon = e^{2\pi i/k}$. The set \mathcal{S}^k denotes all such space of k -fold symmetric univalent functions. Similarly, (j, k) -symmetric functions is defined over a k -fold symmetric set Ω as $f : \Omega \rightarrow \mathbb{C}$ if $f(z) = \frac{1}{\epsilon^j} f(\epsilon z)$ for all $z \in \Omega, \epsilon = e^{\frac{2\pi i}{k}}$ and $j = 0, 1, 2, \dots, k-1; k \geq 2$. Here we have considered the domain Ω as \mathbb{D} . $\mathcal{S}^{(j,k)}$ denoted as the space of (j, k) -symmetrical functions. In particular, $\mathcal{S}^{(0,2)}, \mathcal{S}^{(1,2)}$ and $\mathcal{S}^{(1,k)}$ are known as the family of even, odd, and k -fold symmetric functions, respectively. The above family $\mathcal{S}^{(j,k)}$ is introduced and studied in [11]. For more recent work on the related class we refer [18] and the references therein. Every function $f_{j,k}$ is in the class $\mathcal{S}^{(j,k)}$ has the series representation:

$$f(z) = \sum_{j=0}^{k-1} f_{j,k}(z), \quad (z \in \mathbb{D}),$$

where

$$(1.3) \quad f_{j,k}(z) = \frac{1}{k} \sum_{\nu=0}^{k-1} \epsilon^{-\nu j} f(\epsilon^\nu z) = \frac{1}{k} \sum_{\nu=0}^{k-1} \epsilon^{-\nu j} \left(\sum_{n=1}^{\infty} a_n (\epsilon^\nu z)^n \right) \quad (z \in \mathbb{D}).$$

Further it is easy to see that

$$(1.4) \quad f_{j,k}(z) = \sum_{n=1}^{\infty} \psi_{n,j} a_n z^n, \quad (a_1 = 1, z \in \mathbb{D})$$

where $\psi_{n,j} = \frac{1}{k} \sum_{\nu=0}^{k-1} \epsilon^{(n-j)\nu} = \begin{cases} 1; & n = lk + j, \\ 0; & n \neq lk + j \end{cases}$ ($l \in \mathbb{N}_0$) and note that

$f_{0,2}(z) = f(z)$. Motivated by the above concept, we mainly considered the functions class $\mathcal{ST}_{[j,k]}(A, B)$ and $\mathcal{K}_{[j,k]}(A, B)$, respectively called the family of generalized (j, k) -symmetric starlike and convex functions, and defined in term of subordination as follows.

Definition 1. A function $f \in \mathcal{H}$ is said to be in the class of $\mathcal{ST}_{[j,k]}(A, B)$, $(-1 \leq B < A \leq 1)$ if the following subordination condition satisfied:

$$(1.5) \quad \frac{zf'(z)}{f_{j,k}(z)} \prec \frac{1 + Az^{j+k-1}}{1 + Bz^{j+k-1}}, \quad (z \in \mathbb{D}, j = 0, 1, 2, \dots, k-1; k \geq 1).$$

Definition 2. A function $f \in \mathcal{H}$ is said to be in the class of $\mathcal{K}_{[j,k]}(A, B)$, $(-1 \leq B < A \leq 1)$ if the following subordination condition satisfied:

$$(1.6) \quad \frac{(zf'(z))'}{f'_{j,k}(z)} \prec \frac{1 + Az^{j+k-1}}{1 + Bz^{j+k-1}}, \quad (z \in \mathbb{D}, j = 0, 1, 2, \dots, k-1; k \geq 1).$$

For the various choices of the parameters the above defined families unifies various classical as well as recently studied subclasses of \mathcal{S} . In Table 1, we provide some relevant connections for the family $\mathcal{ST}_{[j,k]}(A, B)$. Here, for an abbreviation, we set $p(z) = \frac{zf'(z)}{f_{j,k}(z)}$.

TABLE 1. For the certain values of j, k, A and B , the class $\mathcal{ST}_{[j,k]}(A, B)$ is represents in terms of subordination.

$\mathcal{ST}_{[j,k]}(A, B)$	Subordination relation	Relevant connections
$\mathcal{ST}_{[1,k]}(A, B) =: \mathcal{S}_k^*(A, B)$	$p(z) \prec \frac{1+Az^k}{1+Bz^k}$	[2]
$\mathcal{ST}_{[1,1]}(A, B) =: \mathcal{S}^*(A, B)$	$p(z) \prec \frac{1+Az}{1+Bz}$	[8]
$\mathcal{ST}_{[1,1]}(1, -1) =: \mathcal{S}^*$	$p(z) \prec \frac{1+z}{1-z}$	[13]
$\mathcal{ST}_{[j,k]}(1 - 2\alpha, -1) =: \mathcal{ST}_{[j,k]}(\alpha); \alpha \in [0, 1)$	$p(z) \prec \frac{1+(1-2\alpha)z^{j+k-1}}{1-z^{j+k-1}}$	[18]
$\mathcal{ST}_{[1,1]}(1 - 2\alpha, -1) =: \mathcal{S}^*(\alpha); \alpha \in [0, 1)$	$p(z) \prec \frac{1+(1-2\alpha)z}{1-z}$	[15]
$\mathcal{ST}_{[1,1]}(1, 0) =: \mathcal{S}^*(1, 0)$	$p(z) \prec 1 + z$	[19]
$\mathcal{ST}_{[1,1]}(e^{i\beta}(e^{i\beta} - 2\alpha \cos \beta), -1) =: \mathcal{S}_\beta(\alpha),$ $\alpha \in [0, 1)$ and $\beta \in (-\pi/2, \pi/2),$	$p(z) \prec \frac{1+e^{i\beta}(e^{i\beta}-2\alpha \cos \beta)z}{1-z}$	[20]
$\mathcal{ST}_{[1,1]}(A, B) =: \mathcal{S}^{(j,k)}(A, B)$	$p(z) \prec \frac{1+Az}{1+Bz}$	[17]

Organization of the paper is as follows. In Section 2, we first provide the sharp bounds on the growth and distortion problems. Various radii related problems for the functions class $\mathcal{ST}_{[j,k]}(A, B)$ and $\mathcal{K}_{[j,k]}(A, B)$ are discussed in Section 3. Sharp Zalcman-type bounds for $\mathcal{ST}_{[j,k]}$ and pre-Schwarzian norm for functions in $\mathcal{K}_{[j,k]}(\alpha)$ are illustrated in Sections 4 and 5, respectively.

2. Main results

Lemma 2.1. Suppose that $p(z)$ is analytic in \mathbb{D} . Then $p(z) \prec \frac{1+Az^{j+k-1}}{1+Bz^{j+k-1}}$ if and only if

$$(2.1) \quad \left| p(z) - \frac{1 - ABr^{2(j+k-1)}}{1 - B^2r^{2(j+k-1)}} \right| \leq \begin{cases} \frac{(A-B)r^{j+k-1}}{1 - B^2r^{2(j+k-1)}}; & B \neq 0; \\ Ar^{j+k-1}; & B = 0, (|z| \leq r < 1). \end{cases}$$

The estimate (2.1) is sharp bound for $|z| = r$.

Proof. Define $w(z)$ by

$$w(z) = \frac{1 + Az^{j+k-1}}{1 + Bz^{j+k-1}}, \quad j = 0, 1, 2, \dots; k \geq 1, z \in \mathbb{D}.$$

This implies that

$$w(z) + Bz^{j+k-1}w(z) = 1 + Az^{j+k-1}.$$

Taking modulus on both sides of the above equation and squaring the resulting equation yields

$$|w(z)|^2 - 2 \left[w(z) \left(\frac{1 - ABr^{2(j+k-1)}}{1 - B^2r^{2(j+k-1)}} \right) \right] \leq \frac{A^2r^{2(j+k-1)} - 1}{1 - B^2r^{2(j+k-1)}}.$$

Further computation gives

$$\begin{aligned} & |w(z)|^2 - 2 \left[w(z) \left(\frac{1 - ABr^{2(j+k-1)}}{1 - B^2r^{2(j+k-1)}} \right) \right] + \left| \frac{1 - ABr^{2(j+k-1)}}{1 - B^2r^{2(j+k-1)}} \right|^2 \\ & \leq \frac{A^2r^{2(j+k-1)} - 1}{1 - B^2r^{2(j+k-1)}} + \left| \frac{1 - ABr^{2(j+k-1)}}{1 - B^2r^{2(j+k-1)}} \right|^2. \end{aligned}$$

This implies that

$$\left| w(z) - \frac{1 - ABr^{2(j+k-1)}}{1 - B^2r^{2(j+k-1)}} \right| \leq \frac{(A - B)r^{j+k-1}}{1 - B^2r^{2(j+k-1)}}, \quad (|z| \leq r < 1).$$

Hence using the application of subordination relation we have

$$p(z) \prec \frac{1 + Az^{j+k-1}}{1 + Bz^{j+k-1}}$$

if and only if

$$\left| p(z) - \frac{1 - ABr^{2(j+k-1)}}{1 - B^2r^{2(j+k-1)}} \right| \leq \frac{(A - B)r^{j+k-1}}{1 - B^2r^{2(j+k-1)}}, \quad (B \neq 0, |z| \leq r < 1).$$

For the case $B = 0$, then

$$p(z) \prec 1 + Az^{j+k-1}$$

if and only if

$$|p(z) - 1| \leq Ar^{j+k-1}, \quad (|z| \leq r < 1).$$

This completes the proof of Lemma 2.1. \square

Remark 2.2. The family $\mathcal{ST}_{[j,k]}(A, B)$ is defined under the condition $-1 \leq B < A \leq 1$ such that the set of values of the functional $\frac{zf'(z)}{f_{j,k}(z)}$ lies in the right side of the half plane inside the disk with center $\frac{1 - ABr^{2(j+k-1)}}{1 - B^2r^{2(j+k-1)}}$ and radius $\frac{(A - B)r^{j+k-1}}{1 - B^2r^{2(j+k-1)}}$. So that $\mathcal{ST}_{[j,k]}(A, B) \subset \mathcal{S}^*$ whenever $-1 \leq B < A \leq 1$.

Remark 2.3. The family $\mathcal{K}_{[j,k]}(A, B)$ is defined under the condition $-1 \leq B < A \leq 1$ such that the set of values of the functional and $\frac{(zf'(z))'}{f'_{j,k}(z)}$ lies in the right side of the half plane inside the disk with center $\frac{(B-A)r^{2(j+k-1)}}{1-B^2r^{2(j+k-1)}}$ and radius $\frac{(A-B)r^{j+k-1}}{1-B^2r^{2(j+k-1)}}$. So that $\mathcal{K}_{[j,k]}(A, B) \subset \mathcal{K}$ whenever $-1 \leq B < A \leq 1$.

The next theorem based on growth and distortion theorems for the family $\mathcal{ST}_{[j,k]}(A, B)$ and $\mathcal{K}_{[j,k]}(A, B)$.

Theorem 2.4. *If $f(z) \in \mathcal{K}_{[j,k]}(A, B)$, then*

$$L(r) \leq |f'_{j,k}(r)| \leq H(r),$$

where

$$L(r) := (1 + Br^{j+k-1})^{\frac{-(1+B)(A-B)}{2B^2(j+k-1)}} (1 - Br^{j+k-1})^{\frac{-(1-B)(A-B)}{2B^2(j+k-1)}},$$

$$H(r) := (1 + Br^{j+k-1})^{\frac{(B-1)(A-B)}{2B^2(j+k-1)}} (1 - Br^{j+k-1})^{\frac{(-1-B)(A-B)}{2B^2(j+k-1)}}.$$

Proof. Suppose $f(z) \in \mathcal{K}_{[j,k]}(A, B)$. Then

$$(2.2) \quad \Re \left\{ \frac{(zf'(z))'}{f'_{j,k}(z)} \right\} \geq 0, \quad (z \in \mathbb{D}).$$

Therefore, upon replacing z by $\epsilon^\nu z$ in (2.2), we get

$$(2.3) \quad \Re \left\{ \frac{f'(\epsilon^\nu z) + z\epsilon^\nu f''(\epsilon^\nu z)}{f'_{j,k}(\epsilon^\nu z)} \right\} \geq 0, \quad (z \in \mathbb{D}, \nu = 0, 1, \dots, k-1).$$

From the relation $f'_{j,k}(\epsilon^\nu z) = \epsilon^{\nu j - \nu} f'_{j,k}(\epsilon^\nu z)$, (2.3) yields

$$(2.4) \quad \Re \left\{ \frac{\epsilon^{\nu - \nu j} f'(\epsilon^\nu z) + z\epsilon^{2\nu - \nu j} f''(\epsilon^\nu z)}{f'_{j,k}(z)} \right\} \geq 0, \quad (z \in \mathbb{D}).$$

Which implies that,

$$\Re \left\{ \frac{\sum_{\nu=0}^{k-1} \epsilon^{\nu - \nu j} f'(\epsilon^\nu z) + z \sum_{\nu=0}^{k-1} \epsilon^{2\nu - \nu j} f''(\epsilon^\nu z)}{f'_{j,k}(z)} \right\} \geq 0,$$

or,

$$\Re \left\{ \frac{f'_{j,k}(z) + zf''_{j,k}(z)}{f'_{j,k}(z)} \right\} \geq 0, \quad (z \in \mathbb{D}),$$

this implies $f_{j,k}(z) \in \mathcal{K}_{[j,k]}(A, B)$. Hence, application of subordination and ([7], Theorem 2), conclude that

$$g(|z| < r) \leq G(|z| \leq r^{j+k-1}), \quad r < 1.$$

The functional $g(z) = \frac{zf'_{j,k}(z)}{f'_{j,k}(z)}$ is a disc with center at $\frac{(A-B)r^{2(j+k-1)}}{1-B^2r^{2(j+k-1)}}$ and its radius is $\frac{(A-B)r^{j+k-1}}{1-B^2r^{2(j+k-1)}}$. Thus, for $z \in \mathbb{D}$ we have

$$(2.5) \quad \left| g(z) - \frac{(A-B)|z|^{2(j+k-1)}}{1-B^2|z|^{2(j+k-1)}} \right| \leq \frac{(A-B)|z|^{j+k-1}}{1-B^2|z|^{2(j+k-1)}}.$$

After simplification we get the result for $0 < |z| = r < 1$,

$$(2.6) \quad \left| \frac{f'_{j,k}(r)}{f'_{j,k}(r)} - \frac{(A-B)r^{2(j+k-1)-1}}{1-B^2r^{2(j+k-1)}} \right| \leq \frac{(A-B)r^{j+k-2}}{1-B^2r^{2(j+k-1)}}.$$

Upon integrating with respect to r (2.6), we get

$$\begin{aligned} & \left| \log f'_{j,k}(r) + \frac{(A-B)}{2B^2(j+k-1)} \log(1-B^2r^{2(j+k-1)}) \right| \\ & \leq \frac{(A-B)}{2B(j+k-1)} \log \left(\frac{1+Br^{j+k-1}}{1-Br^{j+k-1}} \right), \end{aligned}$$

or

$$\begin{aligned} & \frac{-(A-B)}{2B(j+k-1)} \log \left(\frac{1+Br^{j+k-1}}{1-Br^{j+k-1}} \right) \\ & \leq \log |f'_{j,k}(r)| + \frac{(A-B)}{2B^2(j+k-1)} \log(1-B^2r^{2(j+k-1)}) \\ & \leq \frac{(A-B)}{2B(j+k-1)} \log \left(\frac{1+Br^{j+k-1}}{1-Br^{j+k-1}} \right), \end{aligned}$$

or

$$\begin{aligned} & \frac{-(A-B)}{2B^2(j+k-1)} [\log(1+Br^{j+k-1})^{1+B}(1-Br^{j+k-1})^{1-B}] \\ & \leq \log |f'_{j,k}(r)| \leq \frac{(A-B)}{2B^2(j+k-1)} [\log(1+Br^{j+k-1})^{B-1}(1-Br^{j+k-1})^{-1-B}], \end{aligned}$$

or

$$\begin{aligned} & (1+Br^{j+k-1})^{\frac{-(A-B)(1+B)}{2B^2(j+k-1)}} (1-Br^{j+k-1})^{\frac{-(A-B)(1-B)}{2B^2(j+k-1)}} \\ & \leq |f'_{j,k}(r)| \leq (1+Br^{j+k-1})^{\frac{(A-B)(B-1)}{2B^2(j+k-1)}} (1-Br^{j+k-1})^{\frac{(A-B)(-1-B)}{2B^2(j+k-1)}}. \end{aligned}$$

This completes the proof of Theorem 2.4. \square

Theorem 2.5. *Suppose that $f(z) \in \mathcal{K}_{[j,k]}(A, B)$. Then*

$$\begin{aligned} & \frac{1}{r} \int_0^r \frac{1-Ax^{j+k-1}}{1-Bx^{j+k-1}} (1+Bx^{j+k-1})^{\frac{-(A-B)(1+B)}{2B^2(j+k-1)}} (1-Bx^{j+k-1})^{\frac{-(A-B)(1-B)}{2B^2(j+k-1)}} dx \\ & \leq |f'(z)| \\ & \leq \frac{1}{r} \int_0^r \frac{1+Ax^{j+k-1}}{1+Bx^{j+k-1}} (1+Bx^{j+k-1})^{\frac{(A-B)(B-1)}{2B^2(j+k-1)}} (1-Bx^{j+k-1})^{\frac{(A-B)(-1-B)}{2B^2(j+k-1)}} dx. \end{aligned}$$

Proof. Suppose $f(z) \in \mathcal{K}_{[j,k]}(A, B)$. The function $\frac{1+Az^{j+k-1}}{1+Bz^{j+k-1}}$ maps the disc $|z| \leq r$ on to the interior of the circle with the line segment

$$\left[\frac{1 - Ar^{j+k-1}}{1 - Br^{j+k-1}}, \frac{1 + Ar^{j+k-1}}{1 + Br^{j+k-1}} \right]$$

as diameter and $\frac{(zf'(z))'}{f'_{j,k}(z)} \prec \frac{1+Az^{j+k-1}}{1+Bz^{j+k-1}}$. From the above fact, we have

$$\frac{1 - Ar^{j+k-1}}{1 - Br^{j+k-1}} \leq \left| \frac{(zf'(z))'}{f'_{j,k}(z)} \right| \leq \frac{1 + Ar^{j+k-1}}{1 + Br^{j+k-1}}.$$

By Theorem 2.4, we get

$$\begin{aligned} & \frac{1 - Ar^{j+k-1}}{1 - Br^{j+k-1}} (1 + Br^{j+k-1})^{\frac{-(A-B)(1+B)}{2B^2(j+k-1)}} (1 - Br^{j+k-1})^{\frac{-(A-B)(1-B)}{2B^2(j+k-1)}} \\ & \leq |(zf'(z))'| \\ & \leq \frac{1 + Ar^{j+k-1}}{1 + Br^{j+k-1}} (1 + Br^{j+k-1})^{\frac{(A-B)(B-1)}{2B^2(j+k-1)}} (1 - Br^{j+k-1})^{\frac{(A-B)(-1-B)}{2B^2(j+k-1)}}. \end{aligned}$$

On integrating both sides of the above inequalities, we get

$$\begin{aligned} & \frac{1}{r} \int_0^r \frac{1 - Ax^{j+k-1}}{1 - Bx^{j+k-1}} (1 + Bx^{j+k-1})^{\frac{-(A-B)(1+B)}{2B^2(j+k-1)}} (1 - Bx^{j+k-1})^{\frac{-(A-B)(1-B)}{2B^2(j+k-1)}} dx \\ & \leq |f'(z)| \\ & \leq \frac{1}{r} \int_0^r \frac{1 + Ax^{j+k-1}}{1 + Bx^{j+k-1}} (1 + Bx^{j+k-1})^{\frac{(A-B)(B-1)}{2B^2(j+k-1)}} (1 - Bx^{j+k-1})^{\frac{(A-B)(-1-B)}{2B^2(j+k-1)}} dx. \end{aligned}$$

This completes the proof of Theorem 2.5. \square

Theorem 2.6. Let $f(z) \in \mathcal{ST}_{[j,k]}(A, B)$. Then

$$\begin{aligned} & \int_0^r \frac{1 - Ax^{j+k-1}}{1 - Bx^{j+k-1}} (1 + Bx^{j+k-1})^{\frac{-(A-B)(1+B)}{2B^2(j+k-1)}} (1 - Bx^{j+k-1})^{\frac{-(A-B)(1-B)}{2B^2(j+k-1)}} dx \\ & \leq |f(z)| \\ & \leq \int_0^r \frac{1 + Ax^{j+k-1}}{1 + Bx^{j+k-1}} (1 + Bx^{j+k-1})^{\frac{(A-B)(B-1)}{2B^2(j+k-1)}} (1 - Bx^{j+k-1})^{\frac{(A-B)(-1-B)}{2B^2(j+k-1)}} dx, \end{aligned}$$

Proof. Using the well-known result

$$f(z) \in \mathcal{K}_{[j,k]}(A, B) \Leftrightarrow zf'(z) \in \mathcal{ST}_{[j,k]}(A, B),$$

get the desired proof of Theorem 2.6. \square

3. Radius problems

In this section, we shall deal with various radii related problems associated with the above defined families. In general, our aim is to determine the largest number $R \in (0, 1)$ such that for any given two subfamilies \mathcal{E} and \mathcal{F} of \mathcal{H} , with each $f \in \mathcal{E}$, we have $r^{-1}f(rz) \in \mathcal{F}$, where $r \leq R$. We call R as the \mathcal{F} -radius of

\mathcal{E} . The number R is best possible if there exists a function $f_0 \in \mathcal{F}$ such that $r^{-1}f_0(rz) \notin \mathcal{E}$, where $r > R$.

3.1. $\mathcal{ST}_{[j,k]}(\alpha)$ -radius for the family $\mathcal{ST}_{[j,k]}(A, B)$

Theorem 3.1. For $0 \leq \alpha < 1$, $\mathcal{ST}_{[j,k]}(\alpha)$ -radius of functions in the family $\mathcal{ST}_{[j,k]}(A, B)$ is given by

$$(3.1) \quad R_1(\alpha) := \min \left\{ \frac{2(1-\alpha)}{(A-B) + \sqrt{(A-B)^2 + 4(AB - \alpha B^2)(1-\alpha)}}, 1 \right\}.$$

Following function shows that the estimate is best possible.

$$(3.2) \quad f(z) = \begin{cases} z(1 + Bz^{j+k-1})^{\frac{A-B}{B(j+k-1)}}; & B \neq 0, \\ z \exp\left(\frac{Az^{j+k-1}}{j+k-1}\right); & B = 0. \end{cases}$$

Proof. Let $f \in \mathcal{ST}_{[j,k]}(\alpha)$. Then

$$(3.3) \quad \Re \left\{ \frac{zf'(z)}{f_{j,k}(z)} \right\} > \alpha \quad (0 \leq \alpha < 1, z \in \mathbb{D}).$$

Definition 1 together with Lemma 2.1, gives

$$(3.4) \quad \Re \left(\frac{zf'(z)}{f_{j,k}(z)} \right) \geq \frac{1 - AB r^{2(j+k-1)}}{1 - B^2 r^{2(j+k-1)}} - \frac{(B-A)r^{j+k-1}}{1 - B^2 r^{2(j+k-1)}}.$$

From (3.3) and the inequality (3.4), we see that

$$\Re \left(\frac{zf'(z)}{f_{j,k}(z)} \right) \geq \frac{1 - AB r^{2(j+k-1)}}{1 - B^2 r^{2(j+k-1)}} - \frac{(B-A)r^{j+k-1}}{1 - B^2 r^{2(j+k-1)}} \geq \alpha$$

is true if

$$(AB - \alpha B^2)r^{2(j+k-1)} + (B-A)r^{j+k-1} + (\alpha - 1) \leq 0.$$

Further simplification yields

$$R_1(\alpha) = \frac{2(1-\alpha)}{(A-B) + \sqrt{(A-B)^2 + 4(AB - \alpha B^2)(1-\alpha)}}.$$

Indeed, for the sharpness, we consider

$$\frac{zf'(z)}{f_{j,k}(z)} = \frac{1 + Az^{j+k-1}}{1 + Bz^{j+k-1}} = 1 + \frac{(A-B)z^{j+k-1}}{1 + Bz^{j+k-1}}.$$

The case when $B \neq 0$, we have

$$\frac{f'(z)}{f_{j,k}(z)} - \frac{1}{z} = \frac{(A-B)z^{j+k-2}}{1 + Bz^{j+k-1}},$$

or

$$\log \left(\frac{f}{z} \right) = \frac{(A-B)}{B(j+k-1)} \log(1 + Bz^{j+k-1}).$$

This implies that

$$f(z) = z(1 + Bz^{j+k-1})^{\frac{(A-B)}{B(j+k-1)}}.$$

The case when $B = 0$, we have

$$\frac{f'(z)}{f_{j,k}(z)} - \frac{1}{z} = Az^{j+k-2}.$$

On integration, we get

$$f(z) = z \exp\left(\frac{Az^{j+k-1}}{(j+k-1)}\right). \quad \square$$

If we put $\alpha = 0$ in Theorem 3.1, then the following result obtain:

Theorem 3.2. *The $ST_{[j,k]}$ -radius of functions in the family of $ST_{[j,k]}(A, B)$ is given by*

$$(3.5) \quad R_2 = \frac{2}{(A-B) + \sqrt{(A-B)^2 + 4AB}}.$$

The radius is sharp for the function defined in (3.2).

3.2. $\mathcal{K}_{[j,k]}(\alpha)$ -radius for the family $\mathcal{K}_{[j,k]}(A, B)$

Theorem 3.3. *For $0 \leq \alpha < 1$, the $\mathcal{K}_{[j,k]}(\alpha)$ -radius of functions in the family of $\mathcal{K}_{[j,k]}(A, B)$ is given by*

$$(3.6) \quad R_3(\alpha) := \min \left\{ \frac{2(\alpha-1)}{(A-B) + \sqrt{(A-B)^2 + 4(-A+B+B^2-\alpha B^2)(\alpha-1)}}, 1 \right\}.$$

The bound is sharp for the function defined in (3.2).

Proof. From (2.5) we have

$$(3.7) \quad \Re \left(\frac{zf''_{j,k}(z)}{f'_{j,k}(z)} \right) \geq \frac{(A-B+B^2)r^{2(j+k-1)} - (A-B)r^{j+k-1} - 1}{1 - B^2r^{2(j+k-1)}}.$$

If $f \in \mathcal{K}_{[j,k]}(\alpha)$, then

$$(3.8) \quad \Re \left(1 + \frac{zf''_{j,k}(z)}{f'_{j,k}(z)} \right) > \alpha.$$

From (3.7) and (3.8), we see that

$$\Re \left(1 + \frac{zf''_{j,k}(z)}{f'_{j,k}(z)} \right) \geq \frac{(A-B+B^2)r^{2(j+k-1)} - (A-B)r^{j+k-1} - 1}{1 - B^2r^{2(j+k-1)}} \geq \alpha$$

is true if

$$(-A+B+B^2-\alpha B^2)r^{2(j+k-1)} + (A-B)r^{j+k-1} + (\alpha-1) \leq 0.$$

Upon simplification we get the required result (3.6). \square

If we put $\alpha = 0$ in Theorem 3.3, then the following result obtain:

Theorem 3.4. *The $\mathcal{K}_{[j,k]}$ -radius of functions in the family of $\mathcal{K}_{[j,k]}(A, B)$ is*

$$(3.9) \quad R_4 = \frac{2}{(A - B) + \sqrt{(A - B)^2 + 4(A - B + B^2)}}.$$

The radius is sharp for the function defined in (3.2).

3.3. Radius of (j, k) -symmetric strongly starlikeness of order γ

A function $f \in \mathcal{H}$ is said to be (j, k) -symmetric strongly starlike of order γ ; $0 < \gamma \leq 1$ if it satisfies the subordination relation

$$\frac{zf'(z)}{f_{j,k}(z)} \prec \left(\frac{1 + z^{j+k-1}}{1 - z^{j+k-1}} \right)^\gamma \quad (j = 0, 1, 2, \dots, k-1; k \geq 1, z \in \mathbb{D}).$$

Or, equivalently, we have

$$\left| \arg \frac{zf'(z)}{f_{j,k}(z)} \right| \leq \frac{\pi}{2} \gamma.$$

For our requirement we recall the following important result due to Gangadharan et al.

Lemma 3.5 ([6]). *If c is any point in $|\arg w| \leq \frac{\pi}{2} \gamma$ and if $R_c \leq \Re[c] \sin \frac{\pi}{2} \gamma - \Im[c] \cos \frac{\pi}{2} \gamma$, $\Im[c] \geq 0$. The disk $|w - c| \leq R_c$ is contained in the sector $|\arg w| \leq \frac{\pi}{2} \gamma$, $0 < \gamma \leq 1$. In particular when $\Im[c] = 0$, the condition becomes $R_c \leq c \sin \frac{\pi}{2} \gamma$.*

In this subsection we compute radius of (j, k) -symmetric strongly starlikeness for the class $\mathcal{ST}_{[j,k]}(A, B)$.

Theorem 3.6. *Let $f \in \mathcal{ST}_{[j,k]}(A, B)$ and $0 < \gamma \leq 1$. Then the radius of (j, k) -symmetric strongly starlike of order γ in $|z| < R(\gamma)$ given by*

$$R_5(\gamma) = \frac{2 \sin \frac{\pi}{2} \gamma}{(B - A) + \sqrt{(B - A)^2 + 4AB \sin^2 \frac{\pi}{2} \gamma}}.$$

Proof. From Lemma 2.1, we have

$$\left| \frac{zf'(z)}{f_{j,k}(z)} - \frac{1 - ABr^{2(j+k-1)}}{1 - B^2r^{2(j+k-1)}} \right| \leq \frac{(B - A)r^{j+k-1}}{1 - B^2r^{2(j+k-1)}}.$$

Set

$$c = \frac{1 - ABr^{2(j+k-1)}}{1 - B^2r^{2(j+k-1)}} \quad \text{and} \quad R_c = \frac{(B - A)r^{j+k-1}}{1 - B^2r^{2(j+k-1)}}.$$

Since $\Im(AB) = 0$, Lemma 3.5 yields

$$\frac{(B - A)r^{j+k-1}}{1 - B^2r^{2(j+k-1)}} \leq \frac{1 - ABr^{2(j+k-1)}}{1 - B^2r^{2(j+k-1)}} \sin \left(\frac{\pi}{2} \gamma \right)$$

or

$$-ABr^{2(j+k-1)} \sin \frac{\pi}{2} \gamma - (B - A)r^{j+k-1} + \sin \frac{\pi}{2} \gamma \geq 0.$$

Since $\sin \frac{\pi}{2} \gamma \geq 0$, therefore, the above inequality gives the required radius $R_5(\gamma)$. This completes the proof of Theorem 3.6. \square

3.4. Radius of (j, k) -symmetric parabolic starlikeness of order β

A function $f \in \mathcal{H}$ is said to be (j, k) -symmetric parabolic starlike of order β , denoted by $SP_{[j,k]}(\beta)$, if it satisfy the following inequality

$$(3.10) \quad \Re \left(\frac{zf'(z)}{f_{j,k}(z)} \right) > \left| \frac{zf'(z)}{f_{j,k}(z)} - 1 \right| + \beta, \quad (\beta \in [-1, 1), z \in \mathbb{D}).$$

Geometrically, the values of the functional $\frac{zf'(z)}{f_{j,k}(z)}$ lies in the parabolic region

$$(3.11) \quad \Theta = \left\{ w = u + iv : v^2 < 2(1 - \beta) \left(u - \frac{1 + \beta}{2} \right) \right\}.$$

The $SP_{[j,k]}(\beta)$ -radius of functions in the family of $\mathcal{ST}_{[j,k]}(A, B)$ are determined.

Theorem 3.7. For $\beta < 1$, $A, B \in \mathbb{R}$, $A < B$ and $|B| \leq 1$. Let R_6 be given by

$$R_6 := \min \left\{ 1, \frac{2(1 - \beta)}{(B - A) + \sqrt{(B - A)^2 + 4B^2(1 - \alpha)^2}} \right\},$$

R_7 is defined in $(0, 1]$ such that $1 \geq (B(1 + \beta) - 2A)r^{j+k-1} + \beta$ for all $r \in [0, R_7]$ and R_8 is defined in $(0, 1]$ such that $A + B(1 - 2\beta) \geq 2B^2(1 - \beta)r^{2(j+k-1)}$ for all $r \in [0, R_8]$. If $f \in \mathcal{ST}_{[j,k]}(A, B)$, then the $SP_{[j,k]}(\beta)$ -radius is given by

$$R_9 = \begin{cases} R_7; & R_7 \leq R_6, \\ R_8; & R_7 > R_6. \end{cases}$$

Proof. Since

$$\frac{zf'(z)}{f_{j,k}(z)} \prec \frac{1 + Az^{j+k-1}}{1 + Bz^{j+k-1}},$$

it follows from Lemma 2.1 that

$$(3.12) \quad \left| \frac{zf'(z)}{f_{j,k}(z)} - \frac{1 - ABr^{2(j+k-1)}}{1 - B^2r^{2(j+k-1)}} \right| \leq \frac{(B - A)r^{j+k-1}}{1 - B^2r^{2(j+k-1)}}, \quad (|z| \leq r < 1).$$

By letting $w(z) = \frac{zf'(z)}{f_{j,k}(z)} = u + iv$, the points lies the boundary of the disk in (3.12) are given by,

$$w(z) = \frac{1 - ABr^{2(j+k-1)}}{1 - B^2r^{2(j+k-1)}} + \frac{(B - A)r^{j+k-1}}{1 - B^2r^{2(j+k-1)}} e^{i\theta}$$

and hence,

$$(3.13) \quad \begin{aligned} \Re(w(z)) &= \frac{1 - ABr^{2(j+k-1)} + (B - A)r^{j+k-1} \cos \theta}{1 - B^2r^{2(j+k-1)}}, \\ \Im(w(z)) &= \frac{(B - A)r^{j+k-1}}{1 - B^2r^{2(j+k-1)}} \sin \theta. \end{aligned}$$

For $f \in SP_{[j,k]}(\beta)$, we have

$$\Re \left(\frac{zf'(z)}{f_{j,k}(z)} \right) > \left| \frac{zf'(z)}{f_{j,k}(z)} - 1 \right| + \beta, \quad \beta \in [-1, 1), z \in \mathbb{D}.$$

Or, equivalently

$$(3.14) \quad u > |(u + v) - 1| + \beta.$$

Squaring both side of (3.14) yields

$$v^2 < 2u(1 - \beta) + \beta^2 - 1.$$

Or

$$(3.15) \quad \Im(w(z))^2 < 2(1 - \beta) \left(\Re w(z) - \frac{1 + \beta}{2} \right).$$

Putting (3.13) in (3.15) we get,

$$(3.16) \quad \left(\frac{(B - A)r^{j+k-1}}{1 - B^2r^{2(j+k-1)}} \sin \theta \right)^2 < 2(1 - \beta) \left(\frac{(1 - ABr^{2(j+k-1)}) + (B - A)r^{j+k-1} \cos \theta}{1 - B^2r^{2(j+k-1)}} \right) + \beta^2 - 1.$$

Which leads to

$$\begin{aligned} L(x) := & (B - A)^2 r^{2(j+k-1)} x^2 + 2(1 - \beta)(B - A)(1 - B^2 r^{2(j+k-1)}) r^{j+k-1} x \\ & + 2(1 - \beta)(1 - AB r^{2(j+k-1)})(1 - B^2 r^{2(j+k-1)}) \\ & + (\beta^2 - 1)(1 - B^2 r^{2(j+k-1)})^2 - (B - A)^2 r^{2(j+k-1)} \geq 0, \end{aligned}$$

where we set $x = \cos \theta$. It is suffices to find $r(= R)$ such that $L(x) \geq 0$ for all $x \in [-1, 1]$. Note that

$$L'(x) = 2(B - A)^2 r^{2(j+k-1)} x + 2(1 - \beta)(B - A)(1 - B^2 r^{2(j+k-1)}) r^{j+k-1},$$

which vanishes for

$$x = x_0 = \frac{-(1 - \beta)(1 - B^2 r^{2(j+k-1)})}{(B - A) r^{j+k-1}}.$$

With $\beta < 1, A < B$ and $|B| \leq 1$, we have $x_0 \leq 0$. If $x_0 \leq -1$, we need $L(-1) \geq 0$ and if $-1 < x_0 < 0$, we need $L(x_0) \geq 0$.

Indeed, if $x_0 \leq -1$, then we have

$$(\beta B^2 - B^2) r^{2(j+k-1)} - (B - A) r^{j+k-1} + (1 - \beta) \geq 0$$

which is equivalent to

$$R_6 = r \leq \frac{2(1 - \beta)}{(B - A) + \sqrt{(B - A)^2 + 4B^2(1 - \alpha)^2}}.$$

Therefore, the condition $L(-1) \geq 0$ is equivalent to

$$2(1 - AB r^{2(j+k-1)}) - B r^{j+k-1} + A r^{j+k-1} \geq (1 + \beta)(1 - B^2 r^{2(j+k-1)}).$$

That is

$$1 \geq (B(1 + \beta) - 2A) r^{j+k-1} + \beta.$$

Also, we have $L(x_0) \geq 0$ implies that

$$\begin{aligned} & (B-A)^2 r^{2(j+k-1)} \left(\frac{(1-\beta)(1-B^2 r^{2(j+k-1)})}{(B-A)r^{j+k-1}} \right)^2 - 2(1-\beta)(B-A) \\ & (1-B^2 r^{2(j+k-1)}) r^{j+k-1} \times \left(\frac{(1-\beta)(1-B^2 r^{2(j+k-1)})}{(B-A)r^{j+k-1}} \right) + 2(1-\beta) \\ & (1-AB r^{2(j+k-1)})(1-B^2 r^{2(j+k-1)}) + (\beta^2 - 1)(1-B^2 r^{2(j+k-1)})^2 \\ & - (B-A)^2 r^{2(j+k-1)} \geq 0. \end{aligned}$$

This leads to

$$A + B(1-2\beta) \geq 2B^2(1-\beta)r^{2(j+k-1)}.$$

If $R_7 \leq R_6$, then (3.12) will be lies in parabolic region (3.11) if and only if $r \leq R_7$. If $R_7 > R_6$, then (3.12) will be lies in parabolic region (3.11) if and only if $r \leq R_8$. This completes the proof. \square

4. Zalcman conjecture for the family of $\mathcal{ST}_{[j,k]}$

In 1960, Lawrence Zalcman conjectured that the coefficients of the family \mathcal{S} satisfy the quantity $|a_n^2 - a_{2n-1}| \leq (n-1)^2$, for each $n \geq 2$ with sharp bound for Koebe function and its rotation. Note that this conjecture implies the classical Fekete-Szegő inequality, namely, $|a_2^2 - a_3| \leq 1$. Though the conjecture is settled for certain subfamilies of \mathcal{S} , the original problem is still open for $n > 6$. For recent development of the problem we refer [10,14,21] and the reference therein. In this section, we considered the family of (j, k) -symmetric starlike function $\mathcal{ST}_{[j,k]}$ to evaluate the Zalcman functional for $n = 3$. The problem is open for larger n .

Theorem 4.1. *If $f \in \mathcal{ST}_{[j,k]}$, then*

$$\begin{aligned} & (2 - \psi_{2,j})^2(3 - \psi_{3,j})^2(4 - \psi_{4,j})(5 - \psi_{5,j})|a_3^2 - a_5| \\ & \leq (4 - \psi_{4,j})(5 - \psi_{5,j})(16\psi_{2,j}^2 + 4(2 - \psi_{2,j})^2 + 16\psi_{2,j}(2 - \psi_{2,j})) \\ & - (3 - \psi_{3,j})(2 - \psi_{2,j})(16\psi_{2,j}\psi_{3,j}\psi_{4,j} + 8[\psi_{4,j}\psi_{3,j}(2 - \psi_{2,j}) \\ & + \psi_{4,j}\psi_{2,j}(3 - \psi_{3,j}) + \psi_{3,j}\psi_{2,j}(4 - \psi_{4,j})] \\ & + 4(3 - \psi_{3,j})[\psi_{4,j}(2 - \psi_{2,j}) + \psi_{2,j}(4 - \psi_{4,j})] + 4\psi_{3,j}(4 - \psi_{4,j}) \\ & + 2(4 - \psi_{4,j})(3 - \psi_{3,j})(2 - \psi_{2,j})). \end{aligned}$$

Proof. It is known that $f \in \mathcal{ST}_{[j,k]}$ if and only if $\frac{zf'(z)}{f_{j,k}(z)} \in \mathcal{P}$. By (1.4) we have

$$\frac{zf'(z)}{f_{j,k}(z)} = p(z),$$

which implies that

$$zf'(z) = \left(1 + \sum_{n=1}^{\infty} p_n z^n \right) \times f_{j,k}(z).$$

Using (1.1) and (1.4), we have

$$(4.1) \quad a_n = \frac{1}{n - \psi_{n,j}} \sum_{m=1}^{n-1} p_m \psi_{n-m,j} a_{n-m}; \quad \psi_{1,j} = 1, n \geq 2.$$

From (4.1) we have

$$\begin{aligned} a_3 &= \frac{1}{(2 - \psi_{2,j})(3 - \psi_{3,j})} (p_1^2 \psi_{2,j} + p_2(2 - \psi_{2,j})), \\ a_5 &= \frac{1}{(2 - \psi_{2,j})(3 - \psi_{3,j})(4 - \psi_{4,j})(5 - \psi_{5,j})} (p_1^4 \psi_{2,j} \psi_{3,j} \psi_{4,j} \\ &\quad + p_1^2 p_2 [\psi_{4,j} \psi_{3,j} (2 - \psi_{2,j}) + \psi_{4,j} \psi_{2,j} (3 - \psi_{3,j}) + \psi_{3,j} \psi_{2,j} (4 - \psi_{4,j})] \\ &\quad + p_1 p_3 (3 - \psi_{3,j}) [\psi_{4,j} (2 - \psi_{2,j}) + \psi_{2,j} (4 - \psi_{4,j})] \\ &\quad + p_2^2 \psi_{3,j} (4 - \psi_{4,j}) + p_4 (4 - \psi_{4,j}) (3 - \psi_{3,j}) (2 - \psi_{2,j})), \end{aligned}$$

and hence, we have

$$\begin{aligned} &(2 - \psi_{2,j})^2 (3 - \psi_{3,j})^2 (4 - \psi_{4,j}) (5 - \psi_{5,j}) (a_3^2 - a_5) \\ &= (4 - \psi_{4,j}) (5 - \psi_{5,j}) (p_1^4 \psi_{2,j}^2 + p_2^2 (2 - \psi_{2,j})^2 + 2p_1^2 p_2 \psi_{2,j} (2 - \psi_{2,j})) \\ &\quad - (3 - \psi_{3,j}) (2 - \psi_{2,j}) (p_1^4 \psi_{2,j} \psi_{3,j} \psi_{4,j} + p_1^2 p_2 [\psi_{4,j} \psi_{3,j} (2 - \psi_{2,j}) \\ &\quad + \psi_{4,j} \psi_{2,j} (3 - \psi_{3,j}) + \psi_{3,j} \psi_{2,j} (4 - \psi_{4,j})] \\ &\quad + p_1 p_3 (3 - \psi_{3,j}) [\psi_{4,j} (2 - \psi_{2,j}) + \psi_{2,j} (4 - \psi_{4,j})] \\ &\quad + p_2^2 \psi_{3,j} (4 - \psi_{4,j}) + p_4 (4 - \psi_{4,j}) (3 - \psi_{3,j}) (2 - \psi_{2,j})). \end{aligned}$$

The inequality $|p_n| \leq 2$, which follows the required result. \square

Remark 4.2. If we put $j = k = 1$, then we have the result of [5].

5. Pre-Schwarzian norm estimate for the family $\mathcal{K}_{[j,k]}(\alpha)$

Pre-Schwarzian derivative and its norm have wide applications in the theory of Teichmüller space (cf. [9, 25]). We recall here that, the norm of Pre-Schwarzian derivative is given by $T_f = f''/f'$, where $\|T_f\| = \sup_{|z| < 1} (1 - |z|^2) |T_f(z)|$. For $f \in \mathcal{S}$, we have $\|T_f\| \leq 6$ and for $f \in \mathcal{K}$, we have $\|T_f\| \leq 4$. Converse part of the above results which follows from Beckers theorem [4] shows that if $f \in \mathcal{H}$ and $\|T_f\| \leq 1$, then $f \in \mathcal{S}$. Yamashita [24] proved that if $f \in \mathcal{K}(\alpha)$, then $\|f\| = 4(1 - \alpha)$ and for $f \in \mathcal{S}^*(\alpha)$ then $\|f\| = 6 - 4\alpha$. In a recent paper, Aghalary and Orouji [1] have estimated the sharp bound $\|T_f\|$ for functions in the class of α -spiral like functions of order ρ . Wherein the authors have also pointed various connections with other subfamilies of \mathcal{S} .

In this section we considered the functions class $\mathcal{K}_{[j,k]}(A, B)$, which is defined by subordination as follows. For $-1 \leq B < A \leq 1$ with $A \neq B$, every $f \in \mathcal{H}$ is said to be in $\mathcal{K}_{[j,k]}(A, B)$ if it satisfying the subordination relation:

$$1 + \frac{z f''(z)}{f'_{j,k}(z)} \prec \frac{1 + Az}{1 + Bz}, \quad z \in \mathbb{D}.$$

If we take $A = 1 - 2\alpha$ and $B = -1$, then the class is defined by:

$$\mathcal{K}_{[j,k]}(\alpha) := \left\{ f \in \mathcal{S} : 1 + \frac{zf''(z)}{f'_{j,k}(z)} \prec \frac{1 + (1 - 2\alpha)z}{1 - z} \right\}.$$

Consider the function $\Phi(z) : \mathbb{D} \rightarrow \mathbb{C}$ given by

$$\Phi(z) = \begin{cases} \frac{1 - (1-z)^{2\alpha-1}}{2\alpha-1}; & \alpha \neq 1/2, \\ \log \frac{1}{1-z}; & \alpha = 1/2, \end{cases}$$

for which $1 + \frac{z\Phi''(z)}{\Phi'(z)} = \frac{1 + (1-2\alpha)z}{1-z}$, then $\Phi(z) \in \mathcal{K}_{[j,k]}(\alpha)$. Note that Φ treat as an extremal function for the family $\mathcal{K}_{[j,k]}(\alpha)$. For $f \in \mathcal{K}_{[j,k]}(\alpha)$, we have $|a_2| \leq 1 - \alpha$ and equality holds if and only if

$$(5.1) \quad f(z) = \bar{\mu}\Phi(\mu z),$$

μ is unimodular constant that is μ is complex number with $|\mu|^2 = \mu\bar{\mu} = 1$.

Thus, we have the following theorem.

Theorem 5.1. *For $0 \leq \alpha < 1$, the following two statements holds true.*

- i) *Suppose that $f \in \mathcal{K}_{[j,k]}(\alpha)$, then $\|T_f\| = 2(1 + \eta)$ if and only if f is of the form (5.1).*
- ii) *If $f \in \mathcal{K}_{[j,k]}(\alpha)$ is not of the form (5.1), then*

$$(5.2) \quad \|T_f\| \leq 2(1 + \eta) \frac{1 + C + D}{D - C + 3} \text{ for } \eta = 1 - 2\alpha,$$

where

$$(5.3) \quad 0 \leq C = \frac{2|a_2|}{1 + \eta} \leq 1,$$

$$(5.4) \quad 0 \leq D = \frac{|2a_3(3 + 3\eta) - 4a_2^2(2 + \eta)|}{(1 + \eta)(1 + \eta - 2|a_2|)} \leq 1 + C < 2,$$

so that

$$\frac{1}{3} \leq \frac{1 + C + D}{D - C + 3} \leq \frac{1 + C}{2} < 1.$$

Proof. Let us consider the function

$$(5.5) \quad F(z) \equiv F_\alpha(z) = \frac{1 + \eta z}{1 - z}, \quad \text{where } \eta = 1 - 2\alpha, \quad z \in \mathbb{D}.$$

Clearly F is univalent in \mathbb{D} with $F'(0) = \eta + 1$ and $F''(z) = 2(\eta + 1)$. Geometrically, F maps conformally \mathbb{D} onto $\Re(F(z)) > \alpha$. For $f \in \mathcal{K}_{[j,k]}(\alpha)$ then we set

$$g(z) = 1 + \frac{zf''(z)}{f'_{j,k}(z)}, \quad z \in \mathbb{D}.$$

Therefore, the composed function, $\phi \equiv F^{-1} \circ g : \mathbb{D} \rightarrow \mathbb{D}$, is analytic in \mathbb{D} with $\phi(0) = 0$ and $g = F \circ \phi$ which means g is subordinate to F . Since

$g'(0) = 2a_2, g''(0) = 12a_3 - 8a_2^2$. It is clear that $\phi(z) = \frac{g(z)-1}{g(z)+\eta}$ which follows that

$$(5.6) \quad \phi'(0) = \frac{2a_2}{1+\eta} \text{ and } \phi''(0) = \frac{4a_3(3+3\eta) - 8a_2^2(2+\eta)}{(1+\eta)^2}.$$

Hence, the function $\phi(z)$ satisfy the Schwarz lemma shows that

$$C =: |\phi'(0)| = \frac{|2a_2|}{1+\eta} \leq 1,$$

and further $C = 1$, i.e., equality holds if and only if

$$(5.7) \quad \phi(z) \equiv \mu z$$

with $|\mu| = 1$ or f is of the form (5.1). Further, it follows from $g = F \circ \phi$ that

$$(5.8) \quad \frac{f''(z)}{f'_{j,k}(z)} = \frac{\phi(z)(1+\eta)}{z(1-\phi(z))}$$

is analytic in \mathbb{D} . We need to prove that $\|f\| = 2(1+\eta)$ for which f of the form (5.1). In fact, from (5.8) we have

$$\begin{aligned} (1-|z|^2) \left| \frac{f''(z)}{f'_{j,k}(z)} \right| &= (1+\eta) \frac{(1-|z|^2)|\phi(z)|}{|z|(1-|\phi(z)|)} \\ &= \frac{(1+\eta)(1-|z|^2)}{|1-\mu z|} = (1+\eta)(1+x) \leq 2(1+\eta). \end{aligned}$$

Hence, the required result of (i) holds true for $z = \bar{\mu}x$, as $x \rightarrow 1$.

Next to proof of (ii), where $\phi(z)$ is not of the form of (5.1). It follows from [23] that

$$(5.9) \quad |\phi(z)| \leq |z|Q(|z|), \quad z \in \mathbb{D},$$

where $Q(x) = \frac{x^2+Dx+C}{Cx^2+Dx+1}$, $0 \leq x \leq 1$. Here $D = \frac{|\phi''(0)|}{2(1-|\phi'(0)|)}$, which together with (5.6) provides the expression of D in terms of a_2 and a_3 . By the application of Schwarz-Pick inequality at origin to $\chi(z) = \frac{\phi(z)}{z}$, where $|\chi| < 1$, we observe that

$$\frac{D}{1+|\phi'(0)|} = \frac{|\chi'(0)|}{1-|\chi(0)|^2} \leq 1.$$

Thus, $D \leq 1 + C = 1 + |\phi'(0)| = 1 + \frac{|2a_2|}{1+\eta} < 2$, by $|\phi'(0)| = C < 1$. Combining (5.8) and (5.9), we have

$$(5.10) \quad (1-|z|^2) \left| \frac{f''(z)}{f'_{j,k}(z)} \right| \leq (1+\eta) \frac{|\phi(z)|(1-|z|^2)}{|z|(1-|\phi(z)|)} = (1+\eta)G(|z|),$$

where $G(x) = \frac{(1+x)(x^2+Dx+C)}{x^2+x(D-C+1)+1}$, $0 \leq x \leq 1$. To prove that

$$(5.11) \quad G(x) \leq G(1) = \frac{2(C+D+1)}{D-C+3}.$$

Let $H(x)$ be the enumerator of the $G'(x)$. Then $H(0), H'(0), H''(0)$ are positive. Hence $H(x) \geq 0$ or $G(x)$ is increasing in $0 \leq x \leq 1$, which yields the condition (5.11). Combining (5.10) with (5.11), finally we get the result,

$$\|T_f\| \leq 2(1 + \eta) \frac{1 + C + D}{D - C + 3}.$$

This completes the proof of Theorem 5.1. \square

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