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ON A FIRST ORDER STRONG DIFFERENTIAL SUBORDINATION AND APPLICATION TO UNIVALENT FUNCTIONS

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Abstract. Using the concept of the strong differential subordination introduced in [2], we find conditions on the functions θ, φ, G, F such that the first order strong subordination

$$\theta(p(z)) + \frac{G(\xi)}{\xi} zp'(z)\varphi(p(z)) \prec\prec \theta(q(z)) + F(z)q'(z)\varphi(q(z),$$

implies

$$p(z) \prec q(z),$$

where $p(z), q(z)$ are analytic functions in the open unit disk $\mathbb{D}$ with $p(0) = q(0)$. Corollaries and examples of the main results are also considered, some of which extend and improve the results obtained in [1].

1. Introduction

Let $\mathcal{H} = \mathcal{H}(\mathbb{D})$ be the class of all analytic functions in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{A}$ denote the subclass of $\mathcal{H}$ consisting of functions $f(z)$ of the form

$$f(z) = z + a_2 z^2 + \cdots .$$

For two functions $f, g \in \mathcal{H}$, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists an analytic function $w(z)$ in $\mathbb{D}$ with

$$w(0) = 0 \text{ and } |w(z)| < 1$$

such that $f(z) = g(w(z))$. If g is univalent in $\mathbb{D}$, then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$ (see [7], p. 4).

Suppose that $F(z)$ is analytic and univalent in $\mathbb{D}$ with $F(0) = 0$. The class of F-convex functions denoted by FK is defined by

$$FK = \left\{ f \in \mathcal{A} : \operatorname{Re}\left(1 + F(z)\frac{f''(z)}{f'(z)}\right) > 0 \right\}.$$

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The classes of F-starlike functions FS^* and close-to-F-convex functions FC are defined as follows:

$$FS^* = \left\{ f \in \mathcal{A} : \mathrm{Re}\left(F(z)\frac{f'(z)}{f(z)} \right) > 0 \right\}$$

and

$$FC = \left\{ f \in \mathcal{A} : \exists\ g \in FS^*,\ \mathrm{Re}\left(F(z)\frac{f'(z)}{g(z)} \right) > 0 \right\}.$$

Some properties of these classes were studied by Antonino in [2]. If we set $F(z) = z$, then we obtain the usual convex, starlike and close-to-convex functions, respectively.

Let $g(z, \xi)$ be analytic in $\mathbb{D} \times \overline{\mathbb{D}}$ and let $f(z)$ be analytic and univalent in $\mathbb{D}$. We say that $g(z, \xi)$ is strongly subordinate to $f(z)$ and write $g(z, \xi) \prec\prec f(z)$, if for $\xi \in \overline{\mathbb{D}}$, $g(z, \xi)$ as a function of z is subordinate to $f(z)$. It is seen that

$$g(z, \xi) \prec\prec f(z) \Longleftrightarrow g(0, \xi) = f(0) \text{ and } g(\mathbb{D} \times \overline{\mathbb{D}}) \subseteq f(\mathbb{D}).$$

The concept of strong differential subordination has been studied extensively by many authors (see, for example [1, 2, 8, 9]).

A function $L : \mathbb{D} \times [0, \infty) \to \mathbb{C}$ is a subordination chain (or Loewner chain) if $L(z, t)$ is analytic and univalent in $\mathbb{D}$ for all $t \geq 0$, and $L(z, t_1) \prec L(z, t_2)$ when $0 \leq t_1 \leq t_2$ (see [7], p. 4).

In the present paper we aim to find conditions on θ, φ, F, G, p and q such that

$$\theta(p(z)) + \frac{G(\xi)}{\xi} zp'(z)\varphi(p(z)) \prec\prec \theta(q(z)) + F(z)q'(z)\varphi(q(z))$$

implies

$$p(z) \prec q(z).$$

Such result for the case of Briot-Bouquet differential subordination has already been done by Antonino in [1]. So our results extend the results obtained in [1]. This implication has many applications in the geometric function theory and so its investigation seems to be very important. Also Bulboacă in [3–5] generalized differential subordination $\theta[p(z)] + zp'(z) \cdot \varphi[p(z)] \prec h(z)$ and studied differential subordinations of the following types:

$$\theta[p(z)] + \psi[zp'(z)] \cdot \varphi[p(z)] \prec h(z),$$
$$\theta[zp'(z)] + \psi[zp'(z)] \cdot \varphi[p(z)] \prec h(z), \text{ and}$$
$$\theta[p(z)] + zp'(z) \cdot \varphi[p(z)] \prec \psi[h(z)].$$

Dominants and best dominants for these subordinations were obtained with appropriately defined conditions on θ, φ and ψ. To prove our main results, we shall need the following lemmas.

Lemma 1.1 ([7], p. 24)**.** *Let $p(z) = a + a_n z^n + \cdots$ be analytic in $\mathbb{D}$ with $p(z) \not\equiv a$ and $n \geq 1$, and let $q(z)$ be analytic and univalent in $\overline{\mathbb{D}}$ with $q(0) = a$. If p is not subordinate to q, then there exist points $z_0 \in \mathbb{D}$, $\xi_0 \in \partial\mathbb{D}$ and an $m \geq n \geq 1$ for which $p(\{z \in \mathbb{C} : |z| < |z_0|\}) \subseteq q(\mathbb{D})$,*

(i) $p(z_0) = q(\xi_0)$, *and*
(ii) $z_0 p'(z_0) = m\xi_0 q'(\xi_0)$.

Lemma 1.2 ([10], p. 159)**.** *Let* $L(z,t) = a_1(t)z + a_2(t)z^2 + \cdots$ *with* $a_1(t) \neq 0$ *for all* $t \geq 0$ *and* $\lim_{t\to+\infty} |a_1(t)| = +\infty$. *Suppose that* $L(z,t)$ *as a function of* z *is analytic in* $\mathbb{D}$ *and continuously differentiable function of* t *on* $[0,+\infty)$ *for all* $z \in \mathbb{D}$. *If* $L(z,t)$ *satisfies*

$$\mathrm{Re}\left(\frac{z\partial L/\partial z}{\partial L/\partial t}\right) > 0,\ (z \in \mathbb{D},\ t \geq 0),$$

and

$$|L(z,t)| \leq k_0 |a_1(t)|,\ (|z| < r_0 < 1,\ t \geq 0)$$

for some positive constants k_0 *and* r_0, *then* $L(z,t)$ *is a subordination chain.*

Lemma 1.2 was widely used in many articles, giving some interesting results. Unfortunately, many authors only check the first condition and leave the second one. For example, let us consider the function

$$L(z,t) := \exp[(1+t)\pi z] - 1 = \frac{(1+t)\pi z}{1!} + \frac{(1+t)^2\pi^2 z^2}{2!} + \cdots,\ (z \in \mathbb{D}, t \geq 0).$$

It is easy to see that $\lim_{t\to\infty} |a_1(t)| = \infty$ and

$$\mathrm{Re}\left(\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right) = 1 + t > 0,\ (z \in \mathbb{D}, t \geq 0),$$

hence the above defined function L satisfies the first condition of Lemma 1.2, and so L is a subordination chain.

But this conclusion isn't true, because the function $L(z,t_0)$, where $t_0 > 1$ is an arbitrary fixed number, is not univalent since the largest open subsets of $\mathbb{C}$, where the exponential is univalent, are the "open stripe domains" parallel with the real axis with maximum width less or equal than 2π. In our case,

$$|z| < 1 \Longrightarrow |(1+t_0)\pi z| < |(1+t_0)\pi| = (1+t_0)\pi,$$

and the disk $\Omega = \{\xi \in \mathbb{C} : |\xi| < (1+t_0)\pi\}$ overlaps the stripe $\Delta = \{\xi \in \mathbb{C} : |\mathrm{Im}\xi| < \pi\}$, where the exponential function is univalent.

Throughout this paper, we will assume that $F(z), p(z)$ are analytic in $\mathbb{D}$ with $F(0) = 0$, $q(z)$ is analytic and univalent in $\mathbb{D}$ with $p(0) = q(0)$ and that θ and φ are analytic in a domain D containing $q(\mathbb{D})$ and $p(\mathbb{D})$ with $\varphi(w) \neq 0$, when $w \in q(\mathbb{D})$. In addition, we suppose that $G(z)$ is analytic in $\overline{\mathbb{D}}$ and $G(0) = 0$, unless it is explicitly stated. We define the analytic function $g(z,\xi)$ in $\mathbb{D} \times \overline{\mathbb{D}}$ by

$$g(z,\xi) = \theta(p(z)) + \frac{G(\xi)}{\xi} zp'(z)\varphi(p(z)). \tag{1.1}$$

2. Main results

Theorem 2.1. *Let $q(z)$ be an analytic and univalent solution in $\mathbb{D}$ of the differential equation*

$$h(z) = \theta(q(z)) + F(z)q'(z)\varphi(q(z)).$$

Suppose that $h(z)$ is a convex (univalent) function in $\mathbb{D}$. If $g(z,\xi)$ given by (1.1) *satisfies*

(i) $g(z,\xi) \prec\prec h(z)$,

and in addition,

(ii) $\mathrm{Re}\frac{G(z)q'(z)}{zh'(z)}\varphi(q(z)) > 0$ *and* $\mathrm{Re}\left(\frac{G(z)}{z} - \frac{F(z)}{z}\right)\frac{q'(z)}{h'(z)}\varphi(q(z)) \geq 0, \ (z \in \mathbb{D})$,

then $p(z) \prec q(z)$ and when $F(z) = z$ and $G(z) = z$, q is the best dominant.

Proof. Without loss of generality we can assume that $p(z)$, $q(z)$, $h(z)$, $F(z)$, $G(z)$ and $g(z,\xi)$ satisfy the conditions of the theorem on $\overline{\mathbb{D}}$ (or $\overline{\mathbb{D}} \times \overline{\mathbb{D}}$). If not, we can replace $p(z)$, $q(z)$, $h(z)$, $G(z)$ and $F(z)$ with $p_r(z) = p(rz)$, $q_r(z) = q(rz)$, $F_r(z) = F(rz)$, $G_r(z) = G(rz)$, $g_r(z,\xi) = g(rz,\xi)$ and $h_r(z) = h(rz)$, respectively, where $0 < r < 1$. Now suppose that all conditions of the theorem are satisfied, but p is not subordinate to q. By Lemma 1.1 there exist points $z_0 \in \mathbb{D}$, $\xi_0 \in \partial\mathbb{D}$ and $m \geq 1$ such that $p(z_0) = q(\xi_0)$ and $z_0p'(z_0) = m\xi_0q'(\xi_0)$. Using these results, we obtain

$$\begin{aligned} g(z_0,\xi_0) &= \theta(p(z_0)) + \frac{G(\xi_0)}{\xi_0}z_0p'(z_0)\varphi(p(z_0)) \\ &= \theta(q(\xi_0)) + m\frac{G(\xi_0)}{\xi_0}\xi_0q'(\xi_0)\varphi(q(\xi_0)) \\ &= h(\xi_0) + (mG(\xi_0) - F(\xi_0))q'(\xi_0)\varphi(q(\xi_0)). \end{aligned}$$

If we let

$$\lambda := \frac{g(z_0,\xi_0) - h(\xi_0)}{\xi_0h'(\xi_0)} = \frac{(mG(\xi_0) - F(\xi_0))q'(\xi_0)\varphi(q(\xi_0))}{\xi_0h'(\xi_0)},$$

then since $m \geq 1$, from (ii), we obtain

$$\mathrm{Re}\lambda = (m-1)\mathrm{Re}\frac{G(\xi_0)q'(\xi_0)\varphi(q(\xi_0))}{\xi_0h'(\xi_0)} + \mathrm{Re}\frac{(G(\xi_0) - F(\xi_0))q'(\xi_0)\varphi(q(\xi_0))}{\xi_0h'(\xi_0)} \geq 0,$$

or equivalently $|\arg\lambda| \leq \frac{\pi}{2}$. Using this, together with the fact that $\xi_0h'(\xi_0)$ is the outer normal to the boundary of the convex domain $h(\mathbb{D})$ at $h(\xi_0)$, we conclude that $g(z_0,\xi_0) \notin h(\mathbb{D})$. This contradicts $g(z,\xi) \prec\prec h(z)$ and so we have $p \prec q$. □

We remark that by putting $F(z) = G(z) = z$ we obtain the famous result of Miller and Mocanu (see [6]).

Example 2.2. Let $f \in \mathcal{A}$, $p(z) = f'(z)$, $q(z) = 1 + z$, $F(z) = Az$, $G(z) = z - z^2$, $\theta(z) = (1 - Az)e^{z-1}$, $\varphi(z) = e^{z-1}$, where $A < 0$ and $z \in \mathbb{D}$. Therefore we obtain

$$h(z) = \theta(q(z)) + F(z)q'(z)\varphi(q(z)) = (1-A)e^z$$

and that $h(z)$ is convex (univalent) in $\mathbb{D}$. Also, we have

$$\begin{aligned} g(z,\xi) &= \theta(f'(z)) + \frac{G(\xi)}{\xi} z f''(z)\varphi(f'(z)) \\ &= (1 - Af'(z) + (1-\xi)zf''(z))e^{f'(z)-1}. \end{aligned}$$

Moreover, the condition (ii) of Theorem 2.1 is

$$\begin{aligned} \mathrm{Re}\left[\left(\frac{G(z)}{z} - \frac{F(z)}{z}\right)\frac{q'(z)}{h'(z)}\varphi(q(z))\right] &= \frac{1}{1-A}\mathrm{Re}(1-A-z) \\ &> \frac{-A}{1-A} > 0, \end{aligned}$$

and

$$\mathrm{Re}\frac{G(z)q'(z)}{zh'(z)}\varphi(q(z)) = \frac{1}{1-A}\mathrm{Re}(1-z) > 0.$$

Hence, if

$$(1 - Af'(z) + (1-\xi)zf''(z))e^{f'(z)-1} \prec\prec (1-A)e^z, \ (z \in \mathbb{D}, \xi \in \overline{\mathbb{D}}),$$

then by Theorem 2.1 we obtain

$$f'(z) \prec 1 + z$$

which implies $Re(f'(z)) > 0$ and so f is close-to-convex (univalent) (see [10], p. 51).

In the case that $h(z)$ is univalent but is not convex, then using the conditions analogous to the previous theorem, we obtain the same result.

Theorem 2.3. *Let $q(z)$ be an analytic and univalent function in $\mathbb{D}$ and $F(z)$ be analytic in $\overline{\mathbb{D}}$ with $F(0) = 0$. Set*

$$h(z) = \theta(q(z)) + F(z)q'(z)\varphi(q(z)),$$

and suppose that $h(z)$ is analytic and univalent in $\mathbb{D}$. If $g(z,\xi)$ given by (1.1) *satisfies*

(i) *$g(z,\xi) \prec\prec h(z)$,*

and in addition,

(ii) *$Q(z) = zq'(z)\varphi(q(z))$ is starlike,*

(iii) Re $\left(\frac{G(\xi)}{\xi}\frac{Q(z)}{zh'(z)}\right) > 0$, *and* Re $\left[\left(\frac{G(\xi)}{\xi} - \frac{F(\xi)}{\xi}\right)\frac{Q(z)}{zh'(z)}\right] > 0, \ (z \in \mathbb{D}, \xi \in \overline{\mathbb{D}})$,

then $p(z) \prec q(z)$.

Proof. As we have done before, without loss of generality we can assume that p, q, h satisfy the conditions of the theorem on the unit disk $\overline{\mathbb{D}}$. The function

$$L(z,t;\xi,s) = h(z) + t\left(\frac{sG(\xi)}{\xi} - \frac{F(\xi)}{\xi}\right)Q(z) \tag{2.1}$$

is analytic in $\mathbb{D}$ for all $s \geq 1$, $t \geq 0$, $\xi \in \overline{\mathbb{D}}$, and is a continuously differentiable function of t on $[0, +\infty)$ for all $z \in \mathbb{D}$, $\xi \in \overline{\mathbb{D}}$ and $s \geq 1$. We have

$$\begin{aligned} a_1(t) = \left.\frac{\partial L}{\partial z}\right|_{z=0} &= h'(0) + t\left(\frac{sG(\xi)}{\xi} - \frac{F(\xi)}{\xi}\right)Q'(0) \\ &= h'(0)\left[1 + t\left(\frac{sG(\xi)}{\xi} - \frac{F(\xi)}{\xi}\right)\frac{q'(0)\varphi(q(0))}{h'(0)}\right]. \end{aligned}$$

Since $t \geq 0$, from (iii) we deduce that $a_1(t) \neq 0$ and $\lim_{t\to+\infty}|a_1(t)| = +\infty$ for all $s \geq 1$ and $\xi \in \overline{\mathbb{D}}$. A simple calculation combined with (ii) and (iii) shows that

$$\begin{aligned} \mathrm{Re}\left(\frac{z\partial L/\partial z}{\partial L/\partial t}\right) &= \mathrm{Re}\left(\frac{z(\xi h'(z) + t(sG(\xi) - F(\xi))Q'(z))}{(sG(\xi) - F(\xi))Q(z)}\right) \\ &= \mathrm{Re}\left(\frac{\xi z h'(z)}{Q(z)(sG(\xi) - F(\xi))}\right) + t\mathrm{Re}\left(\frac{zQ'(z)}{Q(z)}\right) > 0 \end{aligned}$$

for $s \geq 1$ and $\xi \in \overline{\mathbb{D}}$. Now

$$\begin{aligned} \frac{|L(z,t;\xi,s)|}{|a_1(t)|} &= \frac{\left|h(z) + t\left(\frac{sG(\xi)}{\xi} - \frac{F(\xi)}{\xi}\right)Q(z)\right|}{\left|h'(0)\left[1 + t\left(\frac{sG(\xi)}{\xi} - \frac{F(\xi)}{\xi}\right)\frac{q'(0)\varphi(q(0))}{h'(0)}\right]\right|} \\ &\leq \frac{|h(z)| + t\left|\left(\frac{sG(\xi)}{\xi} - \frac{F(\xi)}{\xi}\right)Q(z)\right|}{|h'(0)|\left[1 + t\mathrm{Re}\left(\frac{sG(\xi)}{\xi} - \frac{F(\xi)}{\xi}\right)\frac{q'(0)\varphi(q(0))}{h'(0)}\right]} \\ &\leq \frac{|h(z)|}{|h'(0)|} + \frac{t\left|\left[(s-1)\left(\frac{G(\xi)}{\xi}\right) + \left(\frac{G(\xi)}{\xi} - \frac{F(\xi)}{\xi}\right)\right]Q(z)\right|}{|h'(0)|\left[1 + t\mathrm{Re}\left(\frac{sG(\xi)}{\xi} - \frac{F(\xi)}{\xi}\right)\frac{q'(0)\varphi(q(0))}{h'(0)}\right]}. \end{aligned} \tag{2.2}$$

We know that the function $\left[\frac{G(\xi)}{\xi}\right]\frac{q'(0)\varphi(q(0))}{h'(0)}$ is analytic in $\overline{\mathbb{D}}$ and so the function $\mathrm{Re}\left[\frac{G(\xi)}{\xi}\right]\frac{q'(0)\varphi(q(0))}{h'(0)}$ is harmonic. Since $\mathrm{Re}\left[\frac{G(\xi)}{\xi}\right]\frac{q'(0)\varphi(q(0))}{h'(0)} > 0$ for all $\xi \in \overline{\mathbb{D}}$ and $\overline{\mathbb{D}}$ is compact, so there exists a positive number δ_0 so that $\mathrm{Re}\left[\frac{G(\xi)}{\xi}\right]\frac{q'(0)\varphi(q(0))}{h'(0)} > \delta_0$ for all $\xi \in \overline{\mathbb{D}}$. Also, similar the previous argument shows that there exists a positive number δ_1 so that $\mathrm{Re}\left[\frac{G(\xi)}{\xi} - \frac{F(\xi)}{\xi}\right]\frac{q'(0)\varphi(q(0))}{h'(0)} > \delta_1$ for all $\xi \in \overline{\mathbb{D}}$. Further in view of analyticity of $\frac{G(\xi)}{\xi}$ and $\frac{G(\xi)}{\xi} - \frac{F(\xi)}{\xi}$ on the unit disc $\overline{\mathbb{D}}$ there are numbers M_1 and M_2 such that

$$\left|\frac{G(\xi)}{\xi}\right| < M_1 \text{ and } \left|\frac{G(\xi)}{\xi} - \frac{F(\xi)}{\xi}\right| < M_2, \quad (\xi \in \overline{\mathbb{D}}).$$

Now from (2.2) we have

$$\frac{|L(z,t;\xi,s)|}{|a_1(t)|} \leq \frac{|h(z)|}{|h'(0)|} + \frac{(s-1)M_1 + M_2}{|h'(0)|[(s-1)\delta_0 + \delta_1]}|Q(z)|, \ (s \geq 1, z \in \mathbb{D}). \tag{2.3}$$

Now let $x = s-1$ and define $f(x) = \frac{xM_1+M_2}{x\delta_0+\delta_1}, (x \geq 0)$, then $f'(x) = \frac{M_1\delta_1-M_2\delta_0}{(x\delta_0+\delta_1)^2}$. We consider two cases:

If $M_1\delta_1 - M_2\delta_0 > 0$, then the function f is increasing in terms of x and so $f(x) \leq \frac{M_1}{\delta_0}$ for all $x \geq 0$ and in other case, i.e., $M_1\delta_1 - M_2\delta_0 < 0$, we have $f(x) \leq \frac{M_2}{\delta_1}$. Let $M = \max\{\frac{M_2}{\delta_1}, \frac{M_1}{\delta_0}\}$, then (2.3) implies that

$$\frac{|L(z,t;\xi,s)|}{|a_1(t)|} \leq \frac{|h(z)|}{|h'(0)|} + \frac{M}{|h'(0)|}|Q(z)|.$$

But we know that h is univalent and $Q(z)$ is starlike on $\mathbb{D}$, so in the every disk $|z| \leq r_0$ these functions will be bounded. Therefore, we conclude that the function $\frac{|L(z,t;\xi,s)|}{|a_1(t)|}$ is bounded on the $|z| \leq r_0$.

Hence by Lemma 1.2, $L(z,t;\xi,s)$ is a subordination chain for fixed $s \geq 1$ and $\xi \in \overline{\mathbb{D}}$. Now, suppose that p is not subordinate to q. Using the same argument as in the proof of Theorem 2.1, we see that there are points $z_0 \in \mathbb{D}$, $\xi_0 \in \partial\mathbb{D}$ and an $m \geq 1$ such that

$$g(z_0,\xi_0) = h(\xi_0) + \left(\frac{mG(\xi_0)}{\xi_0} - \frac{F(\xi_0)}{\xi_0}\right) Q(\xi_0).$$

From (2.1) we have that $g(z_0,\xi_0) = L(\xi_0,1;\xi_0,m)$. Since

$$h(z) = L(z,0;\xi_0,m) \prec L(z,t;\xi_0,m);\ (t \geq 0)$$

hence we must have $L(\xi_0,t;\xi_0,m) \notin h(\mathbb{D})$ for all $t \geq 0$. So, we have $g(z_0,\xi_0) \notin h(\mathbb{D})$. But this contradicts the condition (i) of the theorem and we have $p \prec q$. □

Example 2.4. Let A, B be positive real numbers, $C \in \mathbb{R} \setminus \{0\}$ and $M > 1$, such that $C + AM > 0$ and $B > A + \left(\frac{M+1}{M-1}\right)$. Suppose that

$$q(z) = \frac{C}{M-z},\ F(z) = Az,\ G(z) = Bz + z^2,\ \theta(z) = z \text{ and } \varphi(z) = \frac{1}{z}.$$

From this we obtain

$$h(z) = \theta(q(z)) + F(z)q'(z)\varphi(q(z)) = \frac{C+Az}{M-z}.$$

We are going to investigate the conditions of Theorem 2.3. It is clear that $q(z)$ is univalent in $\mathbb{D}$. We have

$$Q(z) = zq'(z)\varphi(q(z)) = \frac{z}{M-z} \in S^* \text{ (or starlike)},$$

because

$$\mathrm{Re}\frac{zQ'(z)}{Q(z)} = \mathrm{Re}\frac{M}{M-z} > \frac{M}{M+1} > 0,$$

and (ii) is then satisfied. It is easy to see that

$$\mathrm{Re}\left[\left(\frac{G(\xi)}{\xi} - \frac{F(\xi)}{\xi}\right)\frac{Q(z)}{zh'(z)}\right] = \frac{1}{C+AM}\mathrm{Re}[((B+\xi)-A)(M-z)]$$

and

$$\operatorname{Re}\left[\left(\frac{G(\xi)}{\xi}\right)\frac{Q(z)}{zh'(z)}\right]=\frac{1}{C+AM}\operatorname{Re}[(B+\xi)(M-z)].$$

In order that (iii) is satisfied, it is sufficient to show that

$$\operatorname{Re}[((B+\xi)-A)(M-z)]>0,\ \text{ and } \operatorname{Re}[(B+\xi)(M-z)]>0.$$

But

$$\begin{aligned}\operatorname{Re}[((B+\xi)-A)(M-z)] &= M(B+\operatorname{Re}(\xi))-(B\operatorname{Re}(z)+\operatorname{Re}(\xi z))+\\ &\quad + A\operatorname{Re}(z)-AM\\ &> (B-A)M-(B-A)-M-1\end{aligned}$$

and

$$\begin{aligned}\operatorname{Re}[(B+\xi)(M-z)] &= M(B+\operatorname{Re}(\xi))-(B\operatorname{Re}(z)+\operatorname{Re}(\xi z))\\ &> B(M-1)-(M+1).\end{aligned}$$

Since $B>A+\left(\frac{M+1}{M-1}\right)$, the last inequalities show that the condition (iii) is also true. Therefore we conclude that

$$p(z)+(B+\xi)\frac{zp'(z)}{p(z)}\prec\prec\frac{C+Az}{M-z}\Longrightarrow p(z)\prec\frac{C}{M-z}.$$

For example if $C=M$ and $f\in\mathcal{A}$, then

$$e^{f(z)}+(B+\xi)zf'(z)\prec\prec\frac{M+Az}{M-z}\Longrightarrow e^{f(z)}\prec\frac{M}{M-z}.$$

Corollary 2.5. *Let $q(z)$ be analytic and univalent in $\mathbb{D}$. Set*

$$\theta(q(z))+F(z)q'(z)\varphi(q(z))=h(z),\ z\in\mathbb{D}.$$

Suppose that $G(z)$ is an analytic function in $\overline{\mathbb{D}}$ with $G(0)=0$ and that $f(z)$ is an analytic function in $\mathbb{D}$ with $f'(0)\neq 0, f(z)\neq 0$ in $\mathbb{D}\setminus\{0\}$, $\frac{Gf'}{f}(\mathbb{D})\subseteq D$ and $\frac{G(0)f'(0)}{f(0)}=q(0)$. In addition, assume that $H_\xi(z)$ is an analytic function in $\mathbb{D}$ with $H_\xi(0)=1$ given by

$$H_\xi(z)=\exp\left[\int_0^z\left(\frac{\xi\theta(\frac{Gf'}{f}(t))}{tG(\xi)}+\left(\frac{Gf'}{f}(t)\right)'\varphi\left(\frac{Gf'}{f}(t)\right)\right)dt\right]. \tag{2.4}$$

Consider $g(z,\xi)=\frac{G(\xi)}{\xi}z\frac{H'_\xi(z)}{H_\xi(z)}$. If the conditions (i) *and* (ii) *of Theorem 2.1, or the conditions* (i), (ii) *and* (iii) *of Theorem 2.3 are satisfied, then $G(z)\frac{f'(z)}{f(z)}\prec q(z)$. In the special case, if $\operatorname{Re}(q(z))>0$ and $G(z)$ is univalent in $\mathbb{D}$, then $f(z)$ is G-starlike.*

Proof. Let $p(z)=G(z)\frac{f'(z)}{f(z)}$; $(z\in\mathbb{D})$. Then $p(z)$ is analytic in $\mathbb{D}$ such that $p(0)=q(0)$. From (2.4) we obtain that

$$g(z,\xi)=\theta(p(z))+\frac{G(\xi)}{\xi}zp'(z)\varphi(p(z)).$$

Therefore $p(z)$ satisfies all conditions of Theorem 2.1 or all conditions of Theorem 2.3, hence we have $p(z) = G(z)\frac{f'(z)}{f(z)} \prec q(z)$. This completes the proof. $\square$

If we set

$$J(\theta, \varphi, \frac{z}{\xi}G(\xi); f(z), G(z)) = \theta\left(\frac{Gf'}{f}\right) + \frac{zG(\xi)}{\xi}\left(\frac{Gf'}{f}\right)'\varphi\left(\frac{Gf'}{f}\right),$$

then Corollary 2.5 can be written now in the following form:

If the condition (ii) of Theorem 2.1, or the conditions (ii) and (iii) of Theorem 2.3 are satisfied, then

$$J(\theta, \varphi, \frac{z}{\xi}G(\xi); f(z), G(z)) \prec\prec h(z) \Longrightarrow G(z)\frac{f'(z)}{f(z)} \prec q(z).$$

With this notation, we rewrite the final result as follows:

Corollary 2.6. *Suppose that the condition* (ii) *of Theorem 2.1, or the conditions* (ii) *and* (iii) *of Theorem 2.3 are satisfied. Let* $q(0) = 0$*. Also let* $G(z)$ *be an analytic function in* $\mathbb{D}$ *with* $G(0) = 0$*. In addition assume that* $f(z)$ *is analytic in* $\mathbb{D}$ *such that* $f(z) \neq 0$ *for all* $z \in \mathbb{D}$ *and* $\frac{Gf'}{f}(\mathbb{D}) \subseteq D$*. If*

$$g(z, \xi) = J(\theta, \varphi, \frac{z}{\xi}G(\xi); f(z), G(z))$$

and $\frac{Gk'}{k}(\mathbb{D}) \subseteq D$ *with* $k(z) = \exp(\int_0^z \frac{q(t)}{G(t)}dt)$*, then*

$$J(\theta, \varphi, \frac{z}{\xi}G(\xi); f(z), G(z)) \prec\prec J(\theta, \varphi, F(z); k(z), G(z)) \Longrightarrow G\frac{f'}{f} \prec G\frac{k'}{k}.$$

Proof. From the definition of $k(z)$ we see that $q(z) = G(z)\frac{k'(z)}{k(z)}$. Substituting $q(z)$ in $\theta(q(z)) + F(z)q'(z)\varphi(q(z)) = h(z)$, we obtain that $h(z) = J(\theta, \varphi, F(z); k(z), G(z))$. Using Theorem 2.1 or Theorem 2.3 with $p(z) = G(z)\frac{f'(z)}{f(z)}$ we conclude that $G(z)\frac{f'(z)}{f(z)} \prec G(z)\frac{k'(z)}{k(z)}$. $\square$

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