Commun. Korean Math. Soc. **0** (0), No. 0, pp. 1–0
https://doi.org/10.4134/CKMS.c210039
pISSN: 1225-1763 / eISSN: 2234-3024

A NOTE ON DISCRETE SEMIGROUPS OF BOUNDED LINEAR OPERATORS ON NON-ARCHIMEDEAN BANACH SPACES

Aziz Blali, Abdelkhalek El Amrani, and Jawad Ettayb

Abstract. Let $A \in B(X)$ be a spectral operator on a non-archimedean Banach space over an algebraically closed field. In this note, we give a necessary and sufficient condition on the resolvent of A so that the discrete semigroup consisting of powers of A is uniformly-bounded.

1. Introduction and preliminaries

In the archimedean operator theory, necessary and sufficient conditions on the resolvent of a densely defined closed linear operator are given in order to be the infinitesimal generator of a strongly continuous semigroup $(T(s))_{s\in\mathbb{R}^+}$ such that there is $M \geq 1$, $\|T(s)\| \leq M$. For more details, we refer to [2, 4]. In particular, we have the following theorem and its corollary.

Theorem 1.1 ([6]). *A necessary and sufficient condition for a closed linear operator A with dense domain to be the infinitesimal generator of a strongly continuous semigroup $(T(s))_{s\in\mathbb{R}^+}$ such that for all $s \in \mathbb{R}^+$, $\|T(s)\| \leq M$ is that*

$$\|R_\lambda(A)^n\| \leq \frac{M}{\lambda^n}$$

for $\lambda > 0$ and $n \in \mathbb{N}$, where $R_\lambda(A) = (\lambda I - A)^{-1}$.

Corollary 1.2 ([6]). *A necessary and sufficient condition for a closed linear operator A with dense domain to be the infinitesimal generator of a strongly continuous semigroup $(T(s))_{s\in\mathbb{R}^+}$ such that for all $s \in \mathbb{R}^+$, $\|T(s)\| \leq 1$ is that*

$$\|R_\lambda(A)\| \leq \frac{M}{\lambda}$$

for $\lambda > 0$.

Received February 1, 2021; Revised April 26, 2021; Accepted May 25, 2021.
2010 *Mathematics Subject Classification.* Primary 47S10; Secondary 47A10, 47D03.
Key words and phrases. Non-archimedean Banach spaces, spectral operator, discrete semigroups.
Communicated by Choonkil Park.

Throughout this paper, X is a non-archimedean (n.a) Banach space over a (n.a) non trivially complete valued field $\mathbb{K}$ of characteristic zero which is also algebraically closed with valuation $|\cdot|$, $B(X)$ denotes the set of all bounded linear operators on X. $\mathbb{Q}_p$ is the field of p-adic numbers ($p \geq 2$ being a prime) equipped with p-adic valuation $|\cdot|_p$, $\mathbb{Z}_p$ denotes the ring of p-adic integers of $\mathbb{Q}_p$ and it is the unit ball of $\mathbb{Q}_p$. For more details and related issues, we refer to [5, 8]. We denote the completion of the algebraic closure of $\mathbb{Q}_p$ under the p-adic absolute value $|\cdot|_p$ by $\mathbb{C}_p$ (see [5]). Let $r > 0$ and Ω_r be the clopen ball of $\mathbb{K}$ centred at 0 with radius $r > 0$, that is $\Omega_r = \{t \in \mathbb{K} : |t| < r\}$. For more details on non-archimedean operators theory, we refer to [1, 2, 7].

Definition ([9])**.** For $A \in B(X)$, let $\nu(A) = \inf_n \|A^n\|^{\frac{1}{n}} = \lim_n \|A^n\|^{\frac{1}{n}}$. A is said to be a spectral operator if $\sup\{|\lambda| : \lambda \in \sigma(A)\} = \nu(A)$. For $A \in B(X)$, set

$$U_A = \{\lambda \in \mathbb{K} : (I - \lambda A)^{-1} \in B(X)\}$$

$\Big(U_A$ is open and $0 \in U_A\Big)$ and

$$C_A = \{\alpha \in \mathbb{K} : B(0, |\beta|) \subset U_A \text{ for some } \beta \in \mathbb{K},\ |\beta| > |\alpha|\}.$$

We have the following proposition.

Proposition 1.3 ([9])**.** *Let $A \in B(X)$. Then the following are equivalent.*

(i) *A is a spectral operator.*
(ii) *For all $\lambda \in C_A$, $(I - \lambda A)^{-1} = \sum_{n=0}^{\infty} \lambda^n A^n$.*
(iii) *For each $\alpha \in C_A^*$, the function $\lambda \mapsto (I - \lambda A)^{-1}$ is analytic on $B(0, |\alpha|)$.*

We begin with the following definition.

Definition ([3])**.** Let X be a non-archimedean Banach space over $\mathbb{K}$. A family $(T(n))_{n\in\mathbb{N}}$ of bounded linear operators is said to be a discrete semigroup of bounded linear operators on X if

(i) $T(0) = I$, where I is the unit operator of X,
(ii) For all $m, n \in \mathbb{N}$, $T(m + n) = T(m)T(n)$.

Remark 1.4. Let $A \in B(X)$. Then, $T(n) = A^n$ is a discrete semigroup of bounded linear operators on X, and its generator is A.

Definition ([3])**.** Let X be a non-archimedean Banach space over $\mathbb{K}$. A discrete semigroup $(T(n))_{n\in\mathbb{N}}$ is said to be uniformly bounded if $\sup_{n\in\mathbb{N}} \|T(n)\|$ is finite.

Example 1.5 ([3])**.** Let $\mathbb{K} = \mathbb{Q}_p$. If

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

then A generates a discrete semigroup of bounded linear operators $(T(n))_{n\in\mathbb{N}}$ given by:

$$T(n) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix},\ \forall n \in \mathbb{N}.$$

We have the following definition.

Definition ([3])**.** Let $(T(n))_{n\in\mathbb{N}}$ be a discrete semigroup of bounded linear operators on X. $(T(n))_{n\in\mathbb{N}}$ is said to be a semigroup of contractions if $\|T(n)\| \leq 1$ for all $n \in \mathbb{N}$.

Definition ([1])**.** Let $\omega = (\omega_i)_i$ be a sequence of non-zero elements of $\mathbb{K}$. We define $\mathbb{E}_\omega$ by

$$\mathbb{E}_\omega = \{x = (x_i)_i : \forall i \in \mathbb{N},\ x_i \in \mathbb{K},\ \text{and } \lim_{i\to\infty} |\omega_i|^{\frac{1}{2}}|x_i| = 0\},$$

and it is equipped with the norm

$$\forall x \in \mathbb{E}_\omega : x = (x_i)_i,\ \|x\| = \sup_{i\in\mathbb{N}}(|\omega_i|^{\frac{1}{2}}|x_i|).$$

Remark 1.6 ([1])*.* The space $\big(\mathbb{E}_\omega, \|\cdot\|\big)$ is a non-archimedean Banach space.

Example 1.7. Let $X = \mathbb{E}_\omega$ with $\omega_i = p^i$ for all $i \in \mathbb{N}$. Let A be a unilateral shift given by

$$Ae_i = e_{i+1} \text{ for all } i \in \mathbb{N}.$$

Then $A^n e_i = e_{n+i}$ for all $n \in \mathbb{N}$, hence, $\frac{\|A^n e_i\|}{\|e_i\|} = p^{\frac{-n}{2}} \leq 1$ for all $i, n \in \mathbb{N}$. Consequently, $\|A^n\| \leq 1$ for all $n \in \mathbb{N}$. Moreover, $(A^n)_{n\in\mathbb{N}}$ is a discrete semigroup of contractions on $\mathbb{E}_\omega$.

Lemma 1.8 ([3])**.** *Let $(T(n))_{n\in\mathbb{N}}$ be a discrete semigroup on X such that $\sup_{n\in\mathbb{N}} \|T(n)\| \leq M$. Then there exists an equivalent norm on X such that T becomes a contraction.*

In the rest of this paper, we let $A \in B(X)$ be a spectral operator such that $\sup_{n\in\mathbb{N}} \|A^n\|$ is finite, and assume that $U_A = \Omega_1$ where $\Omega_1 = \{\lambda \in \mathbb{K} : |\lambda| < 1\}$, and for all $\lambda \in U_A$, $R(\lambda, A) = (I - \lambda A)^{-1}$.

Proposition 1.9 ([3])**.** *Let X be a non-archimedean Banach space over $\mathbb{K}$, and let A be a spectral operator for which there is $M \geq 1$ such that $\sup_{n\in\mathbb{N}} \|A^n\| \leq M$. Then*

$$\|R(\lambda,\ A)\| \leq M \quad \text{for all } \lambda \in C_A.$$

Proposition 1.10 ([3])**.** *Let $A \in B(X)$ be a spectral operator, and let $(A^n)_{n\in\mathbb{N}}$ be a discrete semigroup of bounded linear operators on X such that $\sup_{n\in\mathbb{N}} \|A^n\|$ is finite and $U_A = B(0, 1)$. Then, for all $\lambda, \mu \in C_A$,*

$$\lambda R(\lambda,\ A) - \mu R(\mu,\ A) = (\lambda - \mu) R(\lambda,\ A) R(\mu,\ A).$$

Proposition 1.11 ([3])**.** *Let $A \in B(X)$ be a spectral operator such that $U_A = \Omega_1$ and let $(A^n)_{n\in\mathbb{N}}$ be a discrete semigroup of contractions on X. Then for all $z \in C_A$, $\|R(\lambda, A) - I\| \leq |\lambda|$.*

As Proposition 2.12 of [3], we have the following proposition.

Proposition 1.12. *Let $A \in B(X)$ be a spectral operator such that for all $k \in \mathbb{N}$, $\|A^k\| \leq M$. Then for all $n \in \mathbb{N}$, $\alpha \in C_A^*$, $\lambda \in \Omega_{|\alpha|}$,*

$$R^{(n)}(\lambda, A) = \frac{n!(R(\lambda, A) - I)^n R(\lambda, A)}{\lambda^n}.$$

We have the following theorem.

Theorem 1.13 ([3])**.** *Let X be a non-archimedean Banach space over $\mathbb{C}_p$, and $A \in B(X)$ be a spectral operator. Then for all $k \in \mathbb{N}$, $\|A^k\| \leq 1$ if and only if*

$$\|\Big(R(\lambda, A) - I\Big)^n R(\lambda, A)\| \leq |\lambda|_p^n$$

for all $\lambda \in \Omega_{|\alpha|}$ and $n \in \mathbb{N}$ where $\alpha \in C_A^*$ and $R(\lambda, A) = (I - \lambda A)^{-1}$.

Remark 1.14 ([8]). Let $x \in \mathbb{K}$ and $n \in \mathbb{N}$, we define $\binom{x}{0} = 1$ and $\binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n!}$. If $k \in \mathbb{N}$ such that $k \geq n$, then $|\binom{k}{n}| \leq 1$.

2. Main results

We have the following theorem.

Theorem 2.1. *Let X be a non-archimedean Banach space over $\mathbb{K}$, and let $A \in B(X)$ be a spectral operator with $U_A = \Omega_1$. Then a necessary and sufficient condition that for all $k \in \mathbb{N}$, $\|A^k\| \leq M$ is that*

$$\|\Big(R(\lambda, A) - I\Big)^n R(\lambda, A)\| \leq M|\lambda|_p^n \tag{2.1}$$

for all $\lambda \in \Omega_{|\alpha|}$, $n \in \mathbb{N}$ where $\alpha \in C_A^*$ and $R(\lambda, A) = (I - \lambda A)^{-1}$.

Proof. Assume that for all $k \in \mathbb{N}$, $\|A^k\| \leq M$, and let $\alpha \in C_A^*$. Then by Proposition 1.3, $R(\lambda, A) = (I - \lambda A)^{-1} = \sum_{k=0}^{\infty} \lambda^k A^k$ is analytic on $\Omega_{|\alpha|}$. Using Proposition 1.12, for all $n \in \mathbb{N}$, $\lambda \in \Omega_{|\alpha|}$

$$R^{(n)}(\lambda, A) = \frac{n!(R(\lambda, A) - I)^n R(\lambda, A)}{\lambda^n}, \tag{2.2}$$

and

$$R^{(n)}(\lambda, A) = \sum_{k=n}^{\infty} k(k-1)\cdots(k-n+1)\lambda^{k-n} A^k = \sum_{k=n}^{\infty} n! \binom{k}{n} \lambda^{k-n} A^k,$$

then for all $n \in \mathbb{N}$ and $\lambda \in \Omega_{|\alpha|}$,

$$\begin{aligned} \Big\|\frac{R^{(n)}(\lambda, A)}{n!}\Big\| &= \|\sum_{k=n}^{\infty} \binom{k}{n} \lambda^{k-n} A^k\| \\ &\leq \sup_{k \geq n} \Big|\binom{k}{n}\Big| |\lambda|^{k-n} \|A^k\| \\ &\leq \sup_{k \geq n} |\lambda|^{k-n} \|A^k\| \\ &\leq M. \end{aligned}$$

Thus, for all $n \in \mathbb{N}$ and $\lambda \in \Omega_{|\alpha|}$,

$$\Big\| \frac{R^{(n)}(\lambda, A)}{n!} \Big\| \leq M. \tag{2.3}$$

From (2.2) and (2.3), we have for all $n \in \mathbb{N}$, $\lambda \in \Omega_{|\alpha|}$,

$$\|(R(\lambda, A) - I)^n R(\lambda, A)\| \leq M|\lambda|_p^n. \tag{2.4}$$

Conversely, let $A \in B(X)$ be a spectral operator, we assume that (2.1) holds, then for all $\lambda \in \Omega_{|\alpha|}$, $R(\lambda, A) = \sum_{n=0}^{\infty} \lambda^n A^n$. Set for all $\lambda \in \Omega_{|\alpha|}$, $k \in \mathbb{N}$, $S_k(\lambda) = \lambda^{-k}(R(\lambda, A) - I)^k R(\lambda, A)$, then for all $\lambda \in \Omega_{|\alpha|}$, $k \in \mathbb{N}$, $\|S_k(\lambda)\| \leq M$. Since A and $R(\lambda, A)$ commute, we have

$$\begin{aligned} S_k(\lambda) &= \lambda^{-k}\Big(\big(I - (I - \lambda A)\big) R(\lambda, A)\Big)^k R(\lambda, A) \\ &= \lambda^{-k}(\lambda A R(\lambda, A))^k R(\lambda, A) \\ &= A^k R(\lambda, A)^{k+1}. \end{aligned}$$

Then for all $\lambda \in \Omega_{|\alpha|}$, $k \in \mathbb{N}$,

$$\begin{aligned} \|A^k\| &= \|(I - \lambda A)^{k+1} S_k(\lambda)\| \\ &\leq \|(I - \lambda A)^{k+1}\| \|S_k(\lambda)\| \\ &\leq M \Big\| \sum_{j=0}^{k+1} \binom{k+1}{j} (-\lambda A)^j \Big\| \\ &\leq M \max\{1, \|\lambda A\|, \|\lambda^2 A^2\|, \dots, \|\lambda^{k+1} A^{k+1}\|\} \end{aligned}$$

for $\lambda \to 0$, we have for all $k \in \mathbb{N}$, $\|A^k\| \leq M$. □

Remark 2.2. For $M = 1$, we conclude Theorem 1.13.

Acknowledgements. The authors are greatly indebted to the editor and the referee for many valuable comments and suggestions improving the first version of this paper.

References

[1] T. Diagana, *Non-Archimedean Linear Operators and Applications*, Nova Science Publishers, Inc., Huntington, NY, 2007.

[2] A. El Amrani, A. Blali, J. Ettayb, and M. Babahmed, *A note on C_0-groups and C-groups on non-archimedean Banach spaces*, Asian-European Journal of Mathematics, 2020.

[3] A. El Amrani, J. Ettayb, and A. Blali, *p-adic discrete semigroup of contractions*, Proyecciones, accepted.

[4] A. G. Gibson, *A discrete Hille-Yosida-Phillips theorem*, J. Math. Anal. Appl. **39** (1972), 761–770. `https://doi.org/10.1016/0022-247X(72)90196-5`

[5] N. Koblitz, *p-adic analysis: a short course on recent work*, London Mathematical Society Lecture Note Series, 46, Cambridge University Press, Cambridge, 1980.

[6] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences, 44, Springer-Verlag, New York, 1983. `https://doi.org/10.1007/978-1-4612-5561-1`

[7] A. C. M. van Rooij, *Non-Archimedean functional analysis*, Monographs and Textbooks in Pure and Applied Mathematics, 51, Marcel Dekker, Inc., New York, 1978.
[8] W. H. Schikhof, *Ultrametric Calculus*, Cambridge Studies in Advanced Mathematics, 4, Cambridge University Press, Cambridge, 1984.
[9] W. H. Schikhof, *On p-adic compact operators*, Tech. Report 8911, Departement of Mathematics, Catholic University, Nijmengen, The Netherlands (1989), 1–28.

Aziz Blali
Department of Mathematics and Computer Science
Sidi Mohamed Ben Abdellah University
ENS B. P. 5206 Bensouda, Fez, Morocco
Email address: aziz.blali@usmba.ac.ma

Abdelkhalek EL Amrani
Department of Mathematics and Computer Science
Sidi Mohamed Ben Abdellah University
Faculty of Sciences Dhar El Mahraz, Fez, Morocco
Email address: abdelkhalek.elamrani@usmba.ac.ma

Jawad Ettayb
Department of Mathematics and Computer Science
Sidi Mohamed Ben Abdellah University
Faculty of Sciences Dhar El Mahraz, Fez, Morocco
Email address: jawad.ettayb@usmba.ac.ma