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ON AN INTEGRAL INVOLVING Ī-FUNCTION

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Abstract. In this paper, an interesting integral involving the Ī-function of one variable introduced by Rathie has been derived. Since Ī-function is a very generalized function of one variable and includes as special cases many of the known functions appearing in the literature, a number of integrals can be obtained by reducing the Ī function of one variable to simpler special functions by suitably specializing the parameters. A few special cases of our main results are also discussed.

1. Introduction

The well known H-function of one variable is defined by Fox [3] and he proved the H-function as a symmetric Fourier kernel to Meijer G-function [2]. The H-function is often called Fox H-function. Later on many researchers studied and developed H-function. In 1997, Rathie [4] introduced a new function in the literature namely the I-function which is useful in Mathematics, Physics and other branches of applied mathematics. The I-function introduced by Rathie [4] is defined and represented by the following Mellin Barnes type contour integral:

$$I_{p,q}^{m,n}(z) \equiv I_{p,q}^{m,n}\left[z \;\middle|\; \begin{matrix} (a_1,e_1,A_1),\dots,(a_p,e_p,A_p) \\ (b_1,f_1,B_1),\dots,(b_q,f_q,B_q) \end{matrix} \right]$$

$$= \frac{1}{2\pi i}\int_{\mathcal{L}} \theta(s) z^s ds, \tag{1}$$

where

$$\theta(s) = \frac{\prod_{j=1}^{m} \Gamma^{B_j}(b_j - f_j s)\prod_{j=1}^{n}\Gamma^{A_j}(1-a_j+e_j s)}{\prod_{j=m+1}^{q}\Gamma^{B_j}(1-b_j+f_j s)\prod_{j=n+1}^{p}\Gamma^{A_j}(a_j-e_j s)}. \tag{2}$$

Also

(i) $i=\sqrt{-1}$;
(ii) $z \neq 0$;
(iii) m,n,p,q are integers satisfying $0 \leq m \leq q$, $0 \leq n \leq p$;
(iv) $\mathcal{L}$ is a suitable contour in the complex plane;

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(v) an empty product is to be interpreted as unity;
(vi) $e_j,\ j=1,\dots,p;\ f_j,\ j=1,\dots,q;\ A_j,\ j=1,\dots,p;$ and $B_j,\ j=1,\dots,q$ are positive numbers;
(vii) $a_j,\ j=1,\dots,p$ and $b_j,\ j=1,\dots,q$ are complex numbers such that no singularity of $\Gamma^{B_j}(b_j-f_js),\ j=1,\dots,m$, coincides with any singularity of $\Gamma^{A_j}(1-a_j+e_js),\ j=1,\dots,n$. In general these singularities are not poles.
(viii) The contour $\mathcal{L}$ goes from $\sigma-i\infty$ to $\sigma+i\infty$ (σ real) so that all the singularities of $\Gamma^{B_j}(b_j-f_js),\ j=1,\dots,m$, lie to the right of $\mathcal{L}$, and all the singularities of $\Gamma^{A_j}(1-a_j+e_js),\ j=1,\dots,n$, lie to the left of $\mathcal{L}$.

In short, (1) will be denoted by

$$I^{m,\,n}_{p,\,q}\left[\ z\ \middle|\ \begin{matrix} {}_1(a_j,e_j,A_j)_p \\ {}_1(b_j,f_j,B_j)_q \end{matrix}\ \right].$$

The function defined by (1) is convergent if

$$\Delta>0,\quad |\arg(z)|<\frac{1}{2}\Delta\pi, \tag{3}$$

where

$$\Delta=\sum_{j=1}^{m}B_jf_j-\sum_{j=m+1}^{q}B_jf_j+\sum_{j=1}^{n}A_je_j-\sum_{j=n+1}^{p}A_je_j. \tag{4}$$

When $A_1=A_2=\cdots=A_p=1=B_1=B_2=\cdots=B_q$, (1) reduces to the H-function introduced by Fox [3] and studied by Braaksma [1].

In the present paper we establish an integral formulae of $\bar{I}$-function of one variable which is defined and represented by the following Mellin Barnes type contour integral:

$$\begin{aligned}\bar{I}(z)&=\bar{I}^{m,\,n}_{p,\,q}\left[z\ \middle|\ \begin{matrix} {}_1(a_j,e_j,A_j)_n,\ {}_{n+1}(a_j,e_j,A_j)_p \\ {}_1(b_j,f_j,1)_m,\ {}_{m+1}(b_j,f_j,B_j)_q \end{matrix}\ \right]\\ &=\frac{1}{2\pi i}\int_{\mathcal{L}}\phi(s)z^s ds,\end{aligned} \tag{5}$$

where

$$\phi(s)=\frac{\prod_{j=1}^{m}\Gamma\left(b_j-f_js\right)\prod_{j=1}^{n}\Gamma^{A_j}\left(1-a_j+e_js\right)}{\prod_{j=m+1}^{q}\Gamma^{B_j}\left(1-b_j+f_js\right)\prod_{j=n+1}^{p}\Gamma^{A_j}\left(a_j-e_js\right)}. \tag{6}$$

The function defined by (2) is convergent if

$$\Delta>0,\quad |\arg(z)|<\frac{1}{2}\Delta\pi, \tag{7}$$

where

$$\Delta=\sum_{j=1}^{m}f_j-\sum_{j=m+1}^{q}f_j+\sum_{j=1}^{n}A_je_j-\sum_{j=n+1}^{p}A_je_j. \tag{8}$$

Also, in the same paper, Rathie [4] showed that

$$\bar{I}(z) \sim z^{c}, \text{ where } c = \min_{1 \leq j \leq m} \left(Re \left[\frac{b_j}{f_j} \right] \right). \tag{9}$$

In our present investigation, we shall require the following result [2].

$$\int_0^{\pi} (\sin\phi)^{c-1} e^{i\delta\phi} d\phi = \frac{\pi\, \Gamma(c)\, e^{\frac{1}{2} i\delta\pi}}{2^{c-1}\, \Gamma(\frac{c+\delta+1}{2})\, \Gamma(\frac{c-\delta+1}{2})} \tag{10}$$

provided $Re(c) > 0$.

2. Main result

The integral involving $\overline{I}$-function to be established in this note is the following.

Consider the integral

(11)

$$S = \int_0^{\pi} (\sin\phi)^{c-1} e^{-i\delta\phi}\, \bar{I}^{\,m,\,n}_{\,p,\,q} \left[z\,(\sin\phi)^{\sigma}\, e^{i\sigma\phi} \;\middle|\; \begin{matrix} {}_1(a_j, e_j, A_j)_n,\ {}_{n+1}(a_j, e_j, A_j)_p \\ {}_1(b_j, f_j, 1)_m,\ {}_{m+1}(b_j, f_j, B_j)_q \end{matrix} \right].$$

Expressing the $\overline{I}$-function with the help of its definition, we get

$$S = \int_0^{\pi} (\sin\phi)^{c-1}\, e^{-i\delta\phi} \frac{1}{2\pi i} \int_L \theta(s)\, [z\,(\sin\phi)^{\sigma} e^{i\alpha\phi}]^{s}\, ds d\phi. \tag{12}$$

On changing the order of integration, we get

$$S = \frac{1}{2\pi i} \int_L \theta(s) z^{s} ds \int_0^{\pi} (\sin\phi)^{\sigma s + c - 1} e^{-i\delta\phi + i\alpha\phi s} d\phi. \tag{13}$$

Using (10) we obtain,

$$S = \frac{1}{2\pi i} \int_L z^{s}\theta(s) \frac{\pi\, \Gamma(\sigma s + c)\, e^{\frac{1}{2} i(\alpha s - \delta)\pi}}{2^{\sigma s + c - 1}\, \Gamma(\frac{\sigma s + c + \alpha s - \delta + 1}{2})\, \Gamma(\frac{\sigma s + c - \alpha s + \delta + 1}{2})}\, ds \tag{14}$$

which can also be written as

$$S = \frac{1}{2\pi i} \int_L z^{s}\theta(s) \frac{\Gamma(c + \sigma s)\; 2^{-\sigma s}\; e^{\frac{1}{2} i\pi\alpha s}}{\Gamma(\frac{c+\delta+1}{2} + \frac{(\sigma-\alpha)s}{2})\, \Gamma(\frac{c-\delta+1}{2} + \frac{(\sigma+\alpha)s}{2})}\, ds\, \frac{\pi\; 2^{1-c}}{e^{\frac{1}{2} i\pi\delta}}. \tag{15}$$

This represents the constant $\frac{\pi\, 2^{1-c}}{e^{\frac{1}{2} i\pi\delta}}$ multiplied by an Ī-function with the argument $z\; 2^{-\sigma}\; e^{\frac{1}{2} i\pi\alpha}$.

Let $P = \frac{\pi 2^{1-c}}{e^{\frac{i\pi\delta}{2}}}$ and $Q = z 2^{-\sigma} e^{\frac{1}{2} i\pi\alpha}$.

If σ, $\sigma - \alpha$ or $\sigma + \alpha$ is zero, the corresponding gamma function inside the integral is free from the variable of integration s. Then the gamma function is a constant and can be taken outside the integral sign. For example when $\sigma = 0$, the integral is $P\, \Gamma(c)$ multiplied by an Ī-function with argument $z\; e^{\frac{1}{2} i\pi\alpha}$.

If $\sigma > 0$, $\Gamma(c + \sigma s)$ will be taken with $\prod_{j=1}^{n} \Gamma^{A_j}(1 - a_j + \alpha_j s)$ of $\theta(s)$ and hence the value of n (and of p) will be increased by 1. The corresponding pair of parameters in the Ī-function is obtained by comparing $c + \sigma s$ with $1 - a_j + \alpha_j s$.

If $\sigma < 0$, $\Gamma(c+\sigma s)$ will be taken with $\prod_{j=1}^{m}\Gamma^{B_j}(b_j-\beta_j s)$ of $\theta(s)$ and hence the value of m (and of q) will be increased by 1. The corresponding pair of parameters in the $\bar{\text{I}}$-function is obtained by comparing $c+\sigma s$ with $b_j-\beta_j s$.

If $\frac{\sigma-\alpha}{2}$ or $\frac{\sigma+\alpha}{2}$ is positive, the gamma function in the denominator is taken with $\prod_{j=m+1}^{q}\Gamma^{B_j}(1-b_j+\beta_j s)$ and hence q increases by 1.

If $\frac{\sigma-\alpha}{2}$ or $\frac{\sigma+\alpha}{2}$ is negative, the gamma function in the denominator is taken with $\prod_{j=n+1}^{p}\Gamma^{A_j}(a_j-\alpha_j s)$ and hence p increases by 1.

The following cases can be considered.

1. $\sigma = 0,\ \alpha > 0$.
2. $\sigma = 0,\ \alpha < 0$.
3. $\alpha = \sigma,\ \sigma > 0$.
4. $\alpha = \sigma,\ \sigma < 0$.
5. $\alpha = -\sigma,\ \sigma > 0$.
6. $\alpha = -\sigma,\ \sigma < 0$.
7. $\sigma > 0,\ \sigma+\alpha > 0,\ \sigma-\alpha > 0$.
8. $\alpha < 0 < \sigma,\ \sigma+\alpha < 0$.
9. $\alpha < \sigma < 0$.
10. $0 < \sigma < \alpha$.
11. $\sigma < 0 < \alpha,\ \sigma-\alpha > 0$.
12. $\sigma < 0,\ \sigma+\alpha < 0,\ \sigma-\alpha < 0$.

Special Cases:

(i) When $\sigma = 0$, $\alpha > 0$, (15) takes the following form.

$$\int_0^{\pi}(\sin\phi)^{c-1}e^{-i\delta\phi}\ \bar{I}^{\,m,\,n}_{\,p,\,q}\left[\ z\,e^{i\sigma\phi}\ \middle|\ \begin{matrix} {}_1(a_j,e_j,A_j)_n,\ {}_{n+1}(a_j,e_j,A_j)_p \\ {}_1(b_j,f_j,1)_m,\ {}_{m+1}(b_j,f_j,B_j)_q \end{matrix}\right]$$

$$= P\,\Gamma(c)\ \bar{I}^{\,m,n}_{\,p+1,q+1}\left[\ z\,e^{\frac{i\pi\alpha}{2}}\ \middle|\ \begin{matrix} {}_1(a_j,e_j,A_j)_p,\ (\frac{c+\delta+1}{2},\frac{\alpha}{2},1) \\ {}_1(b_j,f_j,1_j)_m,\ {}_{m+1}(b_j,f_j,B_j)_q\ (\frac{1-c+\delta}{2},\frac{\alpha}{2},1) \end{matrix}\right].$$

(ii) When $\sigma = 0$, $\alpha < 0$, (15) takes the following form.

$$\int_0^{\pi}(\sin\phi)^{c-1}e^{-i\delta\phi}\ \bar{I}^{\,m,\,n}_{\,p,\,q}\left[z\,e^{i\sigma\phi}\middle| \begin{matrix} {}_1(a_j,e_j,A_j)_n,\ {}_{n+1}(a_j,e_j,A_j)_p \\ {}_1(b_j,f_j,1)_m,\ {}_{m+1}(b_j,f_j,B_j)_q \end{matrix}\right]$$

$$= P\,\Gamma(c)\ \bar{I}^{\,m,n}_{\,p+1,q+1}\left[z\,e^{\frac{i\pi\alpha}{2}}\ \middle|\ \begin{matrix} {}_1(a_j,e_j,A_j)_p,\ (\frac{c-\delta+1}{2},\frac{-\alpha}{2},1) \\ {}_1(b_j,f_j,1_j)_m,\ {}_{m+1}(b_j,f_j,B_j)_q\ (\frac{1-c-\delta}{2},\frac{-\alpha}{2},1) \end{matrix}\right].$$

(iii) When $\alpha = \sigma > 0$, (15) takes the following form.

$$\int_0^{\pi}(\sin\phi)^{c-1}e^{-i\delta\phi}\ \bar{I}^{\,m,\,n}_{\,p,\,q}\left[z\,(\sin\phi)^{\sigma}\ e^{i\sigma\phi}\ \middle|\ \begin{matrix} {}_1(a_j,e_j,A_j)_n,\ {}_{n+1}(a_j,e_j,A_j)_p \\ {}_1(b_j,f_j,1)_m,\ {}_{m+1}(b_j,f_j,B_j)_q \end{matrix}\right]$$

$$= \frac{P}{\Gamma(\frac{c+\delta+1}{2})}\ \bar{I}^{\,m,n+1}_{\,p+1,q+1}\left[z\,2^{-\sigma}\ e^{\frac{i\pi\sigma}{2}}\ \middle| \begin{matrix} (1-c,\sigma,1),\ {}_1(a_j,e_j,A_j)_p \\ {}_1(b_j,f_j,1_j)_m,\ {}_{m+1}(b_j,f_j,B_j)_q,\ (\frac{1-c+\delta}{2},\sigma,1) \end{matrix}\right].$$

(iv) When $\alpha = \sigma < 0$, (15) takes the following form.

$$\int_0^{\pi}(\sin\phi)^{c-1}e^{-i\delta\phi}\ \bar{I}^{\,m,\,n}_{\,p,\,q}\left[z(\sin\phi)^{\sigma}\ e^{i\sigma\phi}\middle| \begin{matrix} {}_1(a_j,e_j,A_j)_n,\ {}_{n+1}(a_j,e_j,A_j)_p \\ {}_1(b_j,f_j,1)_m,\ {}_{m+1}(b_j,f_j,B_j)_q \end{matrix}\right]$$

$$= \frac{P}{\Gamma(\frac{c+\delta+1}{2})}\, \bar{I}^{\,m+1,n}_{\,p+1,q+1}\left[z\, 2^{-\sigma}\, e^{\frac{i\pi\sigma}{2}} \,\middle|\, \begin{matrix} {}_1(a_j,e_j,A_j)_p,\ (\frac{1+c-\delta}{2},-\sigma,1) \\ (c,-\sigma,1),\ {}_1(b_j,f_j,1)_m,\ {}_{m+1}(b_j,f_j,B_j)_q \end{matrix} \right].$$

(v) When $\alpha=-\sigma$, $\sigma>0$, (15) takes the following form.

$$\int_0^\pi (\sin\phi)^{c-1} e^{-i\delta\phi}\, \bar{I}^{\,m,\,n}_{\,p,\,q}\left[z\,(\sin\phi)^\sigma\, e^{i\sigma\phi} \,\middle|\, \begin{matrix} {}_1(a_j,e_j,A_j)_n,\ {}_{n+1}(a_j,e_j,A_j)_p \\ {}_1(b_j,f_j,1)_m,\ {}_{m+1}(b_j,f_j,B_j)_q \end{matrix} \right]$$

$$= \frac{P}{\Gamma(\frac{c-\delta+1}{2})}\, \bar{I}^{\,m,n+1}_{\,p+1,q+1}\left[z\, 2^{-\sigma}\, e^{\frac{-i\pi\sigma}{2}} \,\middle|\, \begin{matrix} (1-c,\sigma,1),\ {}_1(a_j,e_j,A_j)_p \\ {}_1(b_j,f_j,1)_m,\ {}_{m+1}(b_j,f_j,B_j)_q\ ,\ (\frac{1-c-\delta}{2},\sigma,1) \end{matrix} \right].$$

(vi) When $\alpha=-\sigma$, $\sigma<0$, (15) takes the following form.

$$\int_0^\pi (\sin\phi)^{c-1} e^{-i\delta\phi}\, \bar{I}^{\,m,\,n}_{\,p,\,q}\left[z\,(\sin\phi)^\sigma\, e^{i\sigma\phi} \,\middle|\, \begin{matrix} {}_1(a_j,e_j,A_j)_n,\ {}_{n+1}(a_j,e_j,A_j)_p \\ {}_1(b_j,f_j,1)_m,\ {}_{m+1}(b_j,f_j,B_j)_q \end{matrix} \right]$$

$$= \frac{P}{\Gamma(\frac{c-\delta+1}{2})}\, \bar{I}^{\,m+1,n}_{\,p+1,q+1}\left[z\, 2^{-\sigma}\, e^{\frac{-i\pi\sigma}{2}} \,\middle|\, \begin{matrix} 1(a_j,e_j,A_j)_p,\ (\frac{1+c+\delta}{2},-\sigma,1) \\ (c,-\sigma,1),_1(b_j,f_j,1)_m,\ {}_{m+1}(b_j,f_j,B_j)_q \end{matrix} \right].$$

(vii) When $\alpha>0$, $\alpha+\sigma>0$, $\sigma-\alpha>0$, (15) takes the following form.

$$\int_0^\pi (\sin\phi)^{c-1} e^{-i\delta\phi}\, \bar{I}^{\,m,\,n}_{\,p,\,q}\left[z\,(\sin\phi)^\sigma\, e^{i\sigma\phi} \,\middle|\, \begin{matrix} {}_1(a_j,e_j,A_j)_n,\ {}_{n+1}(a_j,e_j,A_j)_p \\ {}_1(b_j,f_j,1)_m,\ {}_{m+1}(b_j,f_j,B_j)_q \end{matrix} \right]$$

$$= P\, \bar{I}^{\,m,n+1}_{\,p+1,q+2}\left[z\, 2^{-\sigma}\, e^{\frac{i\pi\sigma}{2}} \,\middle|\, \begin{matrix} (1-c,\sigma,1),\ {}_1(a_j,e_j,A_j)_p \\ {}_1(b_j,f_j,1)_m,\ {}_{m+1}(b_j,f_j,B_j)_q,\ (\frac{1-c+\delta}{2},\frac{\sigma+\alpha}{2},1),\ (\frac{1-c-\delta}{2},\frac{\sigma-\alpha}{2},1) \end{matrix} \right].$$

(viii) When $\alpha<0<\sigma$, $\alpha+\sigma<0$, (15) takes the following form.

$$\int_0^\pi (\sin\phi)^{c-1} e^{-i\delta\phi}\, \bar{I}^{\,m,\,n}_{\,p,\,q}\left[z\,(\sin\phi)^\sigma\, e^{i\sigma\phi} \,\middle|\, \begin{matrix} {}_1(a_j,e_j,A_j)_n,\ {}_{n+1}(a_j,e_j,A_j)_p \\ {}_1(b_j,f_j,1)_m,\ {}_{m+1}(b_j,f_j,B_j)_q \end{matrix} \right]$$

$$= P\, \bar{I}^{\,m,n+1}_{\,p+2,q+1}\left[z\, 2^{-\sigma}\, e^{\frac{i\pi\sigma}{2}} \,\middle|\, \begin{matrix} (1-c,\sigma,1),\ {}_1(a_j,e_j,A_j)_p,\ (\frac{1+c-\delta}{2},\frac{-(\sigma+\alpha)}{2},1) \\ {}_1(b_j,f_j,1)_m,\ {}_{m+1}(b_j,f_j,B_j)_q,\ (\frac{1-c-\delta}{2},\frac{\sigma-\alpha}{2},1) \end{matrix} \right].$$

(ix) When $\alpha<\sigma<0$, (15) takes the following form.

$$\int_0^\pi (\sin\phi)^{c-1} e^{-i\delta\phi}\, \bar{I}^{\,m,\,n}_{\,p,\,q}\left[z\,(\sin\phi)^\sigma\, e^{i\sigma\phi} \,\middle|\, \begin{matrix} {}_1(a_j,e_j,A_j)_n,\ {}_{n+1}(a_j,e_j,A_j)_p \\ {}_1(b_j,f_j,1)_m,\ {}_{m+1}(b_j,f_j,B_j)_q \end{matrix} \right]$$

$$= P\, \bar{I}^{\,m+1,n}_{\,p+1,q+2}\left[z\, 2^{-\sigma}\, e^{\frac{i\pi\sigma}{2}} \,\middle|\, \begin{matrix} {}_1(a_j,e_j,A_j)_p,\ (\frac{1+c-\delta}{2},\frac{-(\sigma+\alpha)}{2},1) \\ (c,-\sigma,1),\ {}_1(b_j,f_j,1)_m,\ {}_{m+1}(b_j,f_j,B_j)_q,\ (\frac{1-c-\delta}{2},\frac{\sigma-\alpha}{2},1) \end{matrix} \right].$$

(x) When $0<\sigma<\alpha$, (15) takes the following form.

$$\int_0^\pi (\sin\phi)^{c-1} e^{-i\delta\phi}\, \bar{I}^{\,m,\,n}_{\,p,\,q}\left[z\,(\sin\phi)^\sigma\, e^{i\sigma\phi} \,\middle|\, \begin{matrix} {}_1(a_j,e_j,A_j)_n,\ {}_{n+1}(a_j,e_j,A_j)_p \\ {}_1(b_j,f_j,1)_m,\ {}_{m+1}(b_j,f_j,B_j)_q \end{matrix} \right]$$

$$= P\, \bar{I}^{\,m,n+1}_{\,p+2,q+1}\left[z\, 2^{-\sigma}\, e^{\frac{i\pi\sigma}{2}} \,\middle|\, \begin{matrix} (1-c,\sigma,1)\ {}_1(a_j,e_j,A_j)_p,\ (\frac{1+c+\delta}{2},\frac{-(\sigma-\alpha)}{2},1) \\ {}_1(b_j,f_j,1)_m,\ {}_{m+1}(b_j,f_j,B_j)_q,\ (\frac{1-c+\delta}{2},\frac{\sigma+\alpha}{2},1) \end{matrix} \right].$$

(xi) When $\sigma<0<\alpha$, $\alpha+\sigma>0$, (15) takes the following form.

$$\int_0^\pi (\sin\phi)^{c-1} e^{-i\delta\phi}\, \bar{I}^{\,m,\,n}_{\,p,\,q}\left[z\,(\sin\phi)^\sigma\, e^{i\sigma\phi} \,\middle|\, \begin{matrix} {}_1(a_j,e_j,A_j)_n,\ {}_{n+1}(a_j,e_j,A_j)_p \\ {}_1(b_j,f_j,1)_m,\ {}_{m+1}(b_j,f_j,B_j)_q \end{matrix} \right]$$

$$= P\, \bar{I}^{\,m+1,n}_{\,p+1,q+2}\left[z\, 2^{-\sigma}\, e^{\frac{i\pi\sigma}{2}} \,\middle|\, \begin{matrix} {}_1(a_j,e_j,A_j)_p,\ (\frac{1+c+\delta}{2},\frac{\sigma-\alpha}{2},1) \\ (c,-\sigma,1),\ {}_1(b_j,f_j,1)_m,\ {}_{m+1}(b_j,f_j,B_j)_q,\ (\frac{1-c+\delta}{2},\frac{\sigma+\alpha}{2},1) \end{matrix} \right].$$

(xii) When $\sigma < 0$, $\sigma + \alpha < 0$, $\sigma - \alpha < 0$, (15) takes the following form.

$$\int_0^{\pi} (\sin\phi)^{c-1} e^{-i\delta\phi}\, \bar{I}^{\,m,\,n}_{p,\,q}\left[z\,(\sin\phi)^{\sigma}\, e^{i\sigma\phi} \;\middle|\; \begin{matrix} {}_1(a_j, e_j, A_j)_n,\; {}_{n+1}(a_j, e_j, A_j)_p \\ {}_1(b_j, f_j, 1)_m,\; {}_{m+1}(b_j, f_j, B_j)_q \end{matrix} \right]$$

$$= P\, \bar{I}^{\,m+1,n}_{p+2,q+1}\left[z\, 2^{-\sigma}\, e^{\frac{i\pi\sigma}{2}} \;\middle|\; \begin{matrix} {}_1(a_j, e_j, A_j)_p,\; (\frac{1+c-\delta}{2}, \frac{-(\sigma+\alpha)}{2}, 1),\; (\frac{1+c+\delta}{2}, \frac{-(\sigma+\alpha)}{2}, 1) \\ (c, -\sigma, 1),\; {}_1(b_j, f_j, 1)_m,\; {}_{m+1}(b_j, f_j, B_j)_q \end{matrix} \right].$$

3. Conclusion

Since Ī-function of one variable is of very general character and includes H-function, G-function and a large number of other elementary functions as special cases, so by specializing the parameters therein, we can obtain a large number of results involving elementary functions. But we shall not include due to lack of space.

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