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ASCENT AND DESCENT OF COMPOSITION OPERATORS ON LORENTZ SPACES

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Abstract. In this paper, we provide various characterizations for the composition operator on Lorentz spaces $L(p,q)$, $1 < p \leq \infty$, $1 \leq q \leq \infty$ to have finite ascent (descent) in terms of its inducing measurable transformation. At the end, in order to demonstrate our outcomes, some examples are given.

1. Introduction

Let $(X, \mathcal{A}, \mu)$ be a σ-finite complete measure space. Let h be a complex-valued measurable function defined on X. For $s \geq 0$, define μ_h, the *distribution function* of h, as

$$\mu_h(s) = \mu\{x \in X : |h(x)| > s\}.$$

The *non-increasing rearrangement* of h is represented by h^* and is given as

$$h^*(t) = \inf\{s > 0 : \mu_h(s) \leq t\},\ t \geq 0.$$

For $t > 0$, let

$$h^{**}(t) = \frac{1}{t}\int_0^t h^*(s)ds.$$

For a measurable function h on X, define $\|h\|_{pq},\ 1 < p \leq \infty,\ 1 \leq q \leq \infty$, as

$$\|h\|_{pq} = \begin{cases} \left\{ \frac{q}{p} \int_0^\infty \left(t^{1/p} h^{**}(t)\right)^q \frac{dt}{t} \right\}^{1/q}, & 1 < p < \infty,\ 1 \leq q < \infty \\ \sup\limits_{t>0}\ t^{1/p} h^{**}(t), & 1 < p \leq \infty,\ q = \infty \end{cases}$$

The *Lorentz space* denoted by $L(p,q)(X, \mathcal{A}, \mu)$ (or shortly written as $L(p,q)$) is defined to be the collection of all (equivalence classes of) measurable functions h on X such that $\|h\|_{pq} < \infty$. It is known that $L(p,q)$ is a Banach space with respect to the norm $\|\cdot\|_{pq}$. The Lebesgue spaces L^p, $1 < p \leq \infty$, are equivalent to the spaces $L(p,p)$. Whenever we write $L(p,q)$, $1 < p \leq \infty$, $1 \leq q \leq \infty$,

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we mean the Lorentz space $L(p,q)$ under the norm defined above for the cases $1 < p < \infty$, $1 \leq q < \infty$ and $1 < p \leq \infty$, $q = \infty$. For more on Lorentz spaces one can refer to [1, 9, 10] and the references therein.

On the measure space $(X, \mathcal{A}, \mu)$, let $T : X \to X$ be a measurable transformation, that is, preimage of measurable set is measurable. The transformation T is said to be non-singular if $\mu(A) = 0$, $A \in \mathcal{A}$, implies $\mu(T^{-1}(A)) = 0$. The non-singularity of a measurable transformation T ensures the well definedness of a linear transformation C_T given by $f \mapsto f \circ T$ (see [1, Corollary 2.2]) on the linear space of all complex-valued measurable functions on X. If C_T on the Lorentz space $L(p,q)$, $1 < p \leq \infty$, $1 \leq q \leq \infty$ is bounded with range in $L(p,q)$, then it is called a *composition operator* on $L(p,q)$ induced by T. An extensive literature is available for composition operators on measurable function spaces and their applications, one can refer to [1, 2, 4, 12] and references therein.

The non-singularity of the measurable transformation T also confirms that the measure $\mu \circ T^{-1}$ given by $\mu \circ T^{-1}(A) = \mu(T^{-1}(A))$, for $A \in \mathcal{A}$, is absolutely continuous with respect to the measure μ and it is symbolically written as $\mu \circ T^{-1} \ll \mu$. Now the Radon-Nikodym theorem shows the existence of a non-negative locally integrable function h_T $(= d\mu \circ T^{-1}/d\mu)$ on X, so that the measure $\mu \circ T^{-1}$ can be represented as

$$\mu \circ T^{-1}(A) = \int_A h_T(x) d\mu(x) \text{ for } A \in \mathcal{A}.$$

For a bounded linear operator L on a Banach space Y, we use the symbols $\mathcal{N}(L)$ and $\mathcal{R}(L)$ to denote the kernel and the range of L, respectively. The class of all bounded operators on Y is denoted by $\mathfrak{B}(Y)$. The kernel space $\mathcal{N}(L^k)$ and range space $\mathcal{R}(L^k)$ of L^k, $k \geq 0$, satisfy that

$$\{0\} = \mathcal{N}(I) \subseteq \mathcal{N}(L) \subseteq \mathcal{N}(L^2) \subseteq \cdots \subseteq \mathcal{N}(L^k) \subseteq \mathcal{N}(L^{k+1}) \subseteq \cdots$$

and

$$X = \mathcal{R}(I) \supseteq \mathcal{R}(L) \supseteq \mathcal{R}(L^2) \supseteq \cdots \supseteq \mathcal{R}(L^k) \supseteq \mathcal{R}(L^{k+1}) \supseteq \cdots .$$

We say that L is of finite ascent (descent) if $\mathcal{N}(L^k) = \mathcal{N}(L^{k+1})$ $\big(\mathcal{R}(L^k) = \mathcal{R}(L^{k+1})\big)$ for some non-negative integer k and in this case we define such smallest number as the ascent (descent) of L. If no such k is there, then we say that the ascent (descent) of L is infinite. We denote the ascent and descent of L by $\alpha(L)$ and $\beta(L)$, respectively. These quantities were introduced by F. Riesz [11] in his original investigation of compact linear operators. Furthermore, the properties and relationships of these quantities can be found in [13, 14].

Lorentz spaces first appeared in 1950 in the work of Lorentz [9], whereas in 1966, Hunt [8] discussed the conjugate of these spaces. But in the later years various function spaces have been introduced along with various measures, like Orlicz-Lorentz spaces, Lorentz Karamata spaces, Lorentz-Zygmund spaces etc. Simultaneously, multiplication, composition and weighted composition operators are discussed on all these spaces. The properties of ascent and descent, which were introduced with the aim of studying compactness of the operators, were discussed for the composition operators on l^p spaces [5] and recently in Orlicz spaces as well in [6, 7]. With the existence of so many function spaces involving Lorentz spaces, there seems to be a need to discuss these properties for composition operators on Lorentz spaces, which can then be a mile stone to extend the study over various function spaces.

Our aim in this paper is to look the ascent and descent of composition operators in terms of the inducing function and the measure under consideration.

2. Ascent of composition operators

Recall that, if $T : X \to X$ is a non-singular measurable transformation, then T^k is also a non-singular measurable transformation for every non-negative integer k with respect to the measure μ. Thus, we can define the composition operator C_{T^k} on Lorentz space $L(p, q)$, $1 < p \leq \infty$, $1 \leq q \leq \infty$ such that $C_T^k(h) = h \circ T^k = C_{T^k}(h)$ for every measurable function h of the Lorentz space. Also, define the measure $\mu \circ T^{-k}$ on the measure space $(X, \mathcal{A}, \mu)$ as

$$\mu \circ T^{-k}(A) = \mu \circ T^{-(k-1)}(T^{-1}(A)) = \mu \circ T^{-1}(T^{-(k-1)}(A)) \text{ for } A \in \mathcal{A}.$$

Then

$$\cdots \ll \mu \circ T^{-(k+1)} \ll \mu \circ T^{-k} \ll \mu T^{-(k-1)} \ll \cdots \ll \mu \circ T^{-1} \ll \mu. \tag{1}$$

If we put $\mu_k := \mu \circ T^{-k}$, then, by Radon-Nikodym theorem, there exists a non negative locally integrable function h_{T^k} on X satisfying

$$\mu_k(A) = \int_A h_{T^k}(x)\, d\mu(x) \text{ for all } A \in \mathcal{A}. \tag{2}$$

Here, $h_{T^k} \left(= \frac{d\mu_k}{d\mu}\right)$ is called the Radon-Nikodym derivative of μ_k with respect to μ. We claim that for each $k \geq 0$, the kernel $\mathcal{N}(C_T^k)$ of the operator C_T^k is $L(p, q)(X_k)$, the collection of all the measurable functions h in $L(p, q)$ satisfying $h(x) = 0$ for $x \in X \setminus X_k$, where $X_k = \{x \in X : h_{T^k}(x) = 0\}$. In order to prove this, we first assume that h is an element of $L(p, q)(X_k)$. Then

$$\mu \left\{x \in X : h \circ T^k(x) \neq 0\right\} \leq \mu \circ T^{-k} (X_k) = \int_{X_k} h_{T^k}(x)\, d\mu(x) = 0.$$

This infers that $h \circ T^k = 0$ a.e., that is, $h \in \mathcal{N}\left(C_T^k\right)$, which proves that $L(p, q)(X_k) \subseteq \mathcal{N}\left(C_T^k\right)$. On the other hand, if $h \in \mathcal{N}\left(C_T^k\right)$, then $\mu \circ T^{-k} \{x \in X : h(x) \neq 0\} = 0$. Take $E = \{x \in X \setminus X_k : h(x) \neq 0\}$ and $F = \{x \in X_k : h(x) \neq 0\}$. From (2), we have

$$\begin{aligned} 0 &= \int_E h_{T^k}(x)\, d\mu(x) + \int_F h_{T^k}(x)\, d\mu(x) \\ &= \int_E h_{T^k}(x)\, d\mu(x) \geq \frac{1}{n} \int_{E_n \cap E} d\mu = \frac{1}{n} \mu\left(E_n \cap E\right), \end{aligned}$$

for each n, where $E_n = \{x \in X \setminus X_k : h_{T^k}(x) > 1/n\}$. This means that $\mu(E_n \cap E) = 0$ for each n. Since $E = \cup_{n=1}^{\infty}(E_n \cap E)$, we have $\mu(E) = 0$, that is, $h = 0$ a.e. on $X \setminus X_k$. Hence, $h \in L(p,q)(X_k)$. This justifies our claim.

Now we prove a result which, along with the preceding observation, will be useful for yielding information about the ascent of composition operators in Lorentz spaces.

Lemma 2.1. *Let T be a non-singular measurable transformation on the measure space X that induces the composition operator C_T on the Lorentz space $L(p,q)$, $1 < p \leq \infty$, $1 \leq q \leq \infty$. Then $\mathcal{N}(C_T^k) = \mathcal{N}(C_T^{k+1})$ if and only if measures μ_k and μ_{k+1} are equivalent.*

Proof. Let $C_T \in \mathfrak{B}(L(p,q))$, $1 < p \leq \infty$, $1 \leq q \leq \infty$. Suppose that μ_k and μ_{k+1} are equivalent. Then $\mu_{k+1} \ll \mu_k \ll \mu_{k+1}$. So, from (1), we have $\mu_k \ll \mu_{k+1} \ll \mu$ and $\mu_{k+1} \ll \mu_k \ll \mu$, and hence, by the chain rule,

$$h_{T^k}(x) = \frac{d\mu_k}{d\mu_{k+1}}(x) \cdot h_{T^{k+1}}(x),$$
$$h_{T^{k+1}}(x) = \frac{d\mu_{k+1}}{d\mu_k}(x) \cdot h_{T^k}(x).$$

Consequently, we have $X_k = X_{k+1}$. Since $\mathcal{N}(C_T^k) = L(p,q)(X_k)$ for each $k \geq 0$, we have

$$\mathcal{N}(C_T^k) = L(p,q)(X_k) = L(p,q)(X_{k+1}) = \mathcal{N}(C_T^{k+1}).$$

Conversely, suppose that $\mathcal{N}(C_T^k) = \mathcal{N}(C_T^{k+1})$. This implies $L(p,q)(X_k) = L(p,q)(X_{k+1})$. We first claim that $\mu(X_k \setminus X_{k+1}) = 0$. The assumption of $\mu(X_k \setminus X_{k+1}) > 0$ provides a set $Y_n (= \{x \in X_k : h_{T^{k+1}}(x) > 1/n\})$, $n \in \mathbb{N}$, of non zero finite measure. Now,

$$\|\chi_{Y_n}\|_{pq} = \begin{cases} (p')^{1/q}\ (\mu(Y_n))^{1/p}, & 1 < p < \infty,\ 1 \leq q < \infty, \\ (\mu(Y_n))^{1/p}, & 1 < p \leq \infty,\ q = \infty, \end{cases} \tag{3}$$
$$< \infty,$$

where $1/p + 1/p' = 1$. Thus, $\chi_{Y_n} \in L(p,q)(X_k) = L(p,q)(X_{k+1})$. As a result, χ_{Y_n} vanishes outside X_{k+1}, which implies that $Y_n \subseteq X_{k+1}$. Therefore

$$0 \leq \frac{1}{n}\int_{Y_n} \chi_{Y_n}\, d\mu \leq \int_{Y_n} h_{T^{k+1}}\, d\mu = 0.$$

This infers that $\mu(Y_n) = 0$, which contradicts our assumption and proves our claim. Similarly, $\mu(X_{k+1} \setminus X_k) = 0$. Keeping (1) in mind, we only need to show $\mu_k \ll \mu_{k+1}$ to prove that μ_k and μ_{k+1} are equivalent. For, let us suppose that $\mu_{k+1}(A) = 0$, $A \in \mathcal{A}$. This yields that for each subset B of A

$$\int_B h_{T^{k+1}}\, d\mu \leq \int_A h_{T^{k+1}}\, d\mu = \mu_{k+1}(A) = 0,$$

which on following the same steps as we have used in setting the claim that $\mu\left(X_k \setminus X_{k+1}\right) = 0$, provides that

$$\mu\left\{x \in A : h_{T^k}(x) \neq 0, h_{T^{k+1}}(x) \neq 0\right\} = 0.$$

Also, the measure of the set $\{x \in A : h_{T^k}(x) \neq 0, h_{T^{k+1}}(x) = 0\}$ is zero as it is a subset of $X_{k+1} \setminus X_k$. Thus

$$\begin{aligned}\mu_k(A) &= \int_{\left\{x \in A \,:\, h_{T^k}(x)=0\right\}} h_{T^k}\, d\mu + \int_{\left\{x \in A \,:\, h_{T^k}(x)\neq 0\right\}} h_{T^k}\, d\mu \\ &= 0.\end{aligned}$$

This completes the proof. □

Using Lemma 2.1, it is easy to attain the following.

Theorem 2.2. *Let T be a non-singular measurable transformation on the measure space X inducing the composition operator C_T on the Lorentz space $L(p, q)$, $1 < p \leq \infty$, $1 \leq q \leq \infty$. A necessary and sufficient condition for k to be the ascent of C_T is that k is the least non negative integer such that the measures μ_k and μ_{k+1} are equivalent.*

A measurable transformation T is said to be measure preserving if it preserves the measure in the sense that $\mu\left(T^{-1}(A)\right) = \mu(A)$ for all $A \in \mathcal{A}$. The findings of [1] show that a measure preserving transformation T always induces the composition operator C_T on Lorentz space $L(p, q)$, $1 < p \leq \infty$, $1 \leq q \leq \infty$. However, a measurable transformation T inducing the composition operator C_T on Lorentz space is surjective if and only if C_T is injective. One can easily attain the following as a consequence of the last theorem.

Corollary 2.3. *Let $C_T \in \mathfrak{B}(L(p, q))$, $1 < p \leq \infty$, $1 \leq q \leq \infty$. Then the ascent of C_T is 0 in each of the following situations.*

(i) *T is a measure preserving.*
(ii) *T is a surjective.*

Theorem 2.2 can be restated as follows.

Theorem 2.4. *A necessary and sufficient condition for a composition operator C_T on Lorentz space $L(p, q)$, $1 < p \leq \infty$, $1 \leq q \leq \infty$, to have the ascent infinite is that the measures μ_k and μ_{k-1} can not be equivalent for any natural number k.*

Proposition 2.5. *Let $C_T \in \mathfrak{B}(L(p, q))$, $1 < p \leq \infty$, $1 \leq q \leq \infty$. If k is the least non-negative integer such that $X_k^c \subset T^{k+1}\left(X_k^c\right)$, then $\alpha(C_T) = k$, where X_k^c denotes the complement of X_k in X.*

Proof. Let k be the integer such that $X_k^c \subset T^{k+1}\left(X_k^c\right)$. It is easy to verify that $L(p, q)(X) = L(p, q)(X_k^c) \bigoplus L(p, q)(X_k)$. Thus, if $h \in \mathcal{N}(C_T^{k+1})$, then there exist $h_1 \in L(p, q)(X_k^c)$ and $h_2 \in L(p, q)(X_k) = \mathcal{N}(C_T^k) \subseteq \mathcal{N}(C_T^{k+1})$, with $h = h_1 + h_2$. This indicates that $0 = C_T^{k+1} h = C_T^{k+1} h_1 = h_1 \circ T^{k+1}$. This along with the fact that $X_k^c \subset T^{k+1}(X_k^c)$ provides that $h_1(x) = 0$ for $x \in X_k^c$. However, $h_1(x) = 0$ for $x \in X_k$ being $h_1 \in L(p,q)(X_k^c)$. Hence, $h_1 = 0$, which further implies that, $h = h_2 \in \mathcal{N}(C_T^k)$. Consequently, $\mathcal{N}(C_T^k) = \mathcal{N}(C_T^{k+1})$ and $\alpha(C_T) = k$. □

We now introduce a notion which provides a sufficient condition for the composition operators C_T on $L(p,q)$, $1 < p \leq \infty,\ 1 \leq q \leq \infty$, to have finite ascent.

Definition. Let $(X, \mathcal{A}, \mu)$ be a measure space. A measurable transformation $T : X \to X$ is said to be *pre-positive* if it satisfies the condition $\mu\left(T^{-1}(A)\right) > 0$ whenever $\mu(A) > 0$.

Example 2.6. (1) Let $(\mathbb{R}, \mathcal{A}, \mu)$ be the Lebesgue measure space with Borel σ-algebra. Then, the transformation $T : \mathbb{R} \to \mathbb{R}$ defined as $T(x) = \frac{x}{2}$ is a pre-positive measurable transformation.

(2) If $\left(\mathbb{N}, 2^{\mathbb{N}}, \mu\right)$ is the measure space, where $2^{\mathbb{N}}$ and μ denotes the power set and counting measure respectively, then every surjective measurable transformation T on $\mathbb{N}$ becomes pre-positive measurable transformation. In fact, a transformation $T : \mathbb{N} \to \mathbb{N}$ is pre-positive if and only if it is surjective.

Theorem 2.7. *A sufficient condition for an operator $C_T \in \mathfrak{B}(L(p,q))$, $1 < p \leq \infty,\ 1 \leq q \leq \infty$, induced by a measurable transformation T, to have the ascent finite is that T is a pre-positive measurable transformation.*

Proof. Let T be a pre-positive measurable transformation inducing C_T on $L(p,q)$, $1 < p \leq \infty,\ 1 \leq q \leq \infty$. If we assume that the ascent of C_T is not finite, then for each natural number k, there exists $h_k \in \mathcal{N}(C_T^k)$ such that $\mu(X_k') > 0$, where $X_k' := \{x \in X : h_k \circ T^{k-1}(x) \neq 0\}$. Since T is pre-positive, $\mu\left(T^{-1}(X_k')\right) = \mu\left(\left\{x \in X : \left(h_k \circ T^k\right)(x) \neq 0\right\}\right) > 0$, which contradicts the entity of h_k in $\mathcal{N}(C_T^k)$. Therefore, the presumed condition on C_T is false, and it follows that the ascent of C_T is finite. □

The following is an immediate consequence of the preceding theorem.

Corollary 2.8. *If the measurable transformation T inducing $C_T \in \mathfrak{B}(L(p,q))$, $1 < p \leq \infty$, $1 \leq q \leq \infty$, is pre-positive, then the ascent of C_T is zero, i.e., $\alpha(C_T) = 0$.*

Theorem 2.9. *Let $C_T \in \mathfrak{B}(L(p,q))$, $1 < p \leq \infty,\ 1 \leq q \leq \infty$. If there exists a sequence $\{A_k\}_{k \geq 1}$ of measurable sets such that for each k, $0 < \mu(A_k) < \infty$, $\mu\left(T^{-k}(A_k)\right) = 0$ and $\mu\left(T^{-(k-1)}(A_k)\right) \neq 0$, then the ascent of C_T can not be finite.*

Proof. Let $\{A_k\}_{k \geq 1}$ be a sequence of measurable sets satisfying $0 < \mu(A_k) < \infty$, $\mu(T^{-k}(A_k)) = 0$ and $\mu\left(T^{-(k-1)}(A_k)\right) \neq 0$ for each k. For each measurable set B, we know that

$$\|\chi_B\|_{pq} = \begin{cases} (p')^{1/q}(\mu(B))^{1/p}, & 1 < p < \infty,\ 1 \leq q < \infty, \\ (\mu(B))^{1/p}, & 1 < p \leq \infty,\ q = \infty, \end{cases}$$

where $1/p + 1/p' = 1$. Hence, the given hypothesis provides that for each natural number k, the characteristics function $\chi_{A_k} \in L(p,q)$ and $\chi_{A_k} \in \big(\mathcal{N}(C_T^k) \setminus \mathcal{N}(C_T^{k-1})\big)$. As a consequence, the ascent of the composition operator C_T is infinite. □

Now, we examine results yielding composition operators on the Lorentz spaces $L(p,q),\ 1 < p \leq \infty,\ 1 \leq q \leq \infty$ whose ascent are not finite.

Theorem 2.10. *Let the measurable transformation T on X inducing C_T on $L(p,q),\ 1 < p \leq \infty,\ 1 \leq q \leq \infty$ be such that the image of each measurable set is measurable. If the ascent of the composition operator C_T on $L(p,q)$ space is not finite, then there exists a sequence of subsets $\{A_k\}$ of X such that for all $k \geq 1$*

(i) $0 < \mu(A_k) < \infty$,
(ii) $A_k \subseteq T^{k-1}(B)$ *for some* $B \in \mathcal{A}$,
(iii) $A_k \notin \big\{T^k(D) : D \in \mathcal{A}$ *and* $\mu(D) > 0\big\}$.

Proof. Let the ascent of C_T be infinite. Then, for each positive integer k, we get $\mathcal{N}(C_T^{k-1}) \subsetneq \mathcal{N}(C_T^k)$. So, we can extract a measurable function $h_k \in \mathcal{N}(C_T^k)$ such that $h_k \notin \mathcal{N}(C_T^{k-1})$. Take $X_k' := \{x \in X : h_k \circ T^{k-1}(x) \neq 0\}$. Clearly, $\mu\left(X_k'\right) > 0$ and, by given condition on T, $T^{k-1}(X_k')$ is measurable. Now, we claim that $\mu(T^{k-1}(X_k')) > 0$. To demonstrate this, suppose that $\mu(T^{k-1}(X_k')) = 0$. Since T is non-singular and $X_k' \subseteq T^{-(k-1)}\left(T^{k-1}\left(X_k'\right)\right)$,

$$\mu\left(X_k'\right) \leq \mu\left(T^{-(k-1)}\left(T^{k-1}\left(X_k'\right)\right)\right) = 0,$$

which is a contradiction. As a consequence, we achieve our claim.

Now, since the measure μ is σ-finite, we can choose a measurable subset A_k of $T^{k-1}(X_k')$ such that $0 < \mu\left(A_k\right) < \infty$. With little efforts, we can deduce that $A_k = \{x \in A_k : h_k(x) \neq 0\}$. To settle the final assertion, if we assume on contrary that $A_k = T^k\left(D\right)$ for some measurable set D having positive measure, then

$$\mu\left(\left\{x \in D : h_k \circ T^k(x) = 0\right\}\right) = 0.$$

Since $h_k \in \mathcal{N}(C_T^k)$, $\mu\left(\left\{x \in D : \left(h_k \circ T^k\right)(x) \neq 0\right\}\right) = 0$. The above two sets each having measure zero mean that $\mu(D) = 0$, this contradicts the fact that $\mu(D) > 0$. This completes the proof. □

If we put $X = \mathbb{N}, \mathcal{A} = 2^{\mathbb{N}}$ and μ is the counting measure, then the corresponding Lorentz space is denoted by $l(p,q)$ and is known as Lorentz sequence space (see [3]). The forthcoming theorem is the consequence of the previous two theorems on the Lorentz sequence space $l(p,q)$.

Theorem 2.11. *A necessary and sufficient condition for the composition operator C_T on the Lorentz sequence space $l(p,q),\ 1 < p \leq \infty,\ 1 \leq q \leq \infty$ induced by $T : \mathbb{N} \to \mathbb{N}$ to have the ascent infinite is that there exists a sequence* *of distinct natural numbers $\langle n_k \rangle$ such that $n_k \notin T^k(\mathbb{N})$ but $n_k \in T^{k-1}(\mathbb{N})$ for each $k \geq 1$.*

Proof. The sufficient part is a direct consequence of Theorem 2.9. For the necessary part, we apply Theorem 2.10 to get a sequence of non empty measurable subsets $\{A_k\}_{k\geq 1}$ of $\mathbb{N}$ satisfying $A_k \subseteq T^{k-1}(\mathbb{N})$ and $A_k \nsubseteq T^k(\mathbb{N}) \subseteq T^{k-1}(\mathbb{N})$. Hence we can take a sequence of distinct natural number n_k satisfying $n_k \in T^{k-1}(\mathbb{N})$ and $n_k \notin T^k(\mathbb{N})$. □

3. Descent of composition operators

Next, we discuss the results on descent of the composition operator $C_T \in \mathfrak{B}(L(p,q))$ on $L(p,q)$, $1 < p \leq \infty$, $1 \leq q \leq \infty$. For a σ-finite measure space $(X, \mathcal{A}, \mu)$, Theorem 3.3 of [1] states that a composition operator C_T on $L(p,q)$ is surjective if and only if h_T, the Radon Nikodym derivative of $\mu \circ T^{-1}$ with respect to μ, is bounded away from zero on its support and $T^{-1}(\mathcal{A}) = \mathcal{A}$. This fact suggests the following.

Theorem 3.1. *If T inducing C_T on the Lorentz space $L(p,q)$, $1 < p \leq \infty$, $1 \leq q \leq \infty$, is such that h_T is bounded away from zero on its support and $T^{-1}(\mathcal{A}) = \mathcal{A}$, then the descent of C_T is* 0.

Now onward, we assume the measure space $(X, \mathcal{A}, \mu)$ under consideration is a separable σ-finite measure space unless stated otherwise. We recall the definition of separable measure space from [6].

Definition. A measure space $(X, \mathcal{A}, \mu)$ is said to be separable if for every distinct pair of points x_1 and x_2 in X, we can find disjoint positive measurable sets X_1 and X_2 such that $x_1 \in X_1$ and $x_2 \in X_2$.

Theorem 3.2. *Suppose that $(X, \mathcal{A}, \mu)$ is a separable σ-finite measure space. A necessary condition for the composition operator C_T on $L(p,q)$, $1 < p \leq \infty$, $1 \leq q \leq \infty$ induced by a measurable transformation $T : X \to X$ to have the descent finite is that $\hat{T}_k$ is injective for some non negative integer k, where $\hat{T}_k = T\big|_{\mathcal{R}(T^k)} : \mathcal{R}(T^k) \to \mathcal{R}(T^k)$.*

Proof. Suppose that the descent of the composition operator C_T on $L(p,q)$, $1 < p \leq \infty$, $1 \leq q \leq \infty$ is finite. On contrary, if we assume that for each k, the mapping $\hat{T}_k$ is not injective, where $\hat{T}_k$ is the same as stated in the statement of the theorem, then there exist x_1', x_2' in X such that $T^k(x_1') \neq T^k(x_2')$ and $T^{k+1}(x_1') = T^{k+1}(x_2')$. Furthermore, the separability and σ-finite conditions of the measure space give disjoint measurable sets X_1 and X_2 with non-zero finite measures containing $x_1 (= T^k(x_1'))$ and $x_2 (= T^k(x_2'))$, respectively. Hence $\chi_{X_1}, \chi_{X_2} \in L(p,q)$. Now consider the element f of $L(p,q)$ given by $f := \chi_{X_1} - \chi_{X_2}$. Then $g = C_T^k f \in \mathcal{R}(C_T^k)$. But $g \notin \mathcal{R}(C_T^{k+1})$ because $g = C_T^{k+1} h$ for some $h \in L(p,q)(X)$ gives an absurd like $1 = f(x_1) = g(x_1') = C_T^{k+1} h(x_1') = C_T^{k+1} h(x_2') = g(x_2') = f(x_2) = -1$.

This justifies the strict inclusion of $\mathcal{R}\left(C_T^{k+1}\right)$ in $\mathcal{R}\left(C_T^{k}\right)$ for each non negative integer k. As a result, we get a contradiction to the fact that the descent of C_T is finite. This completes the proof. □

The lines of proof of the above theorem suggest the following.

Theorem 3.3. *If the measure space under consideration is a separable σ-finite measure space, then a necessary condition for the composition operator C_T on $L(p,q)$, $1 < p \leq \infty$, $1 \leq q \leq \infty$, to have the descent at most k is that the mapping $\hat{T}_k$ is injective.*

If the measure space under consideration has a positive measure for every singleton set, it is always separable. The following can be deduced immediately from the above theorem.

Corollary 3.4. *Let the measure space X be such that every singleton set has positive measure and $T : X \to X$ be such that $C_T \in \mathfrak{B}(L(p,q))$, $1 < p \leq \infty$, $1 \leq q \leq \infty$. Then the descent of C_T is greater than k if the mapping $T\big|_{\mathcal{R}(T^k)} : \mathcal{R}(T^k) \to \mathcal{R}(T^k)$ is not injective.*

Proof. If $T\big|_{\mathcal{R}(T^k)}$ is not injective, then on taking the measurable sets X_1 and X_2 (used in the proof of Theorem 3.2) containing $x_1(= T^k(x_1'))$ and $x_2(= T^k(x_2'))$ as the singleton sets $\{x_1\}$ and $\{x_2\}$ respectively, we can attain the desired result. □

Now we present some examples in support of our study and justify the relevance of our findings.

Example 3.5. Consider the measure space $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$, where $2^{\mathbb{N}}$ and μ denotes the power set and counting measure, respectively.

(a) Define $T : \mathbb{N} \to \mathbb{N}$ as $T(2n-1) = T(2n) = n$. It is easy to check $\mathcal{R}(T^k) = \mathbb{N}$. Hence, the mapping $T : \mathcal{R}(T^n) \to \mathcal{R}(T^n)$ is not one-one for all n. By Corollary 2.8 and Corollary 3.4, the ascent of $C_T : l(p,q) \to l(p,q)$ is zero and descent is infinite.

(b) Now, take $T : \mathbb{N} \to \mathbb{N}$ as $T(2n-1) = 2n-1$ and $T(2n) = 2n+2$. We can extract a sequence $\langle 2k \rangle_{k\in\mathbb{N}}$ from $\mathbb{N}$ such that $2k \in T^{k-1}(\mathbb{N})$ and $2k \notin T^k(\mathbb{N})$. Therefore, by Theorem 2.11, $\alpha(C_T) = \infty$.

Example 3.6. Let $([0,1], \mathcal{A}, \mu)$ be the measure space, where μ is the Lebesgue measure.

- Let $\mathcal{A}$ be the σ-algebra of all Lebesgue measurable subsets of $[0,1]$. For a fixed $a \in (0,1)$, define $T : [0,1] \to [0,1]$ as $T(x) = ax$. Then, it is a non-singular measurable transformation inducing the composition operator C_T on Lorentz space $L(p,q)$, $1 < p \leq \infty$, $1 \leq q \leq \infty$. T is not pre-positive. In fact, $\mu((a,1]) = 1-a$, but $\mu\left(T^{-1}(a,1]\right) = 0$. Now, we use Theorem 2.9 to confirm that the ascent of C_T is infinite.

Take $A_k = \left(a^{k+1}, a^k\right)$. Then $\mu\left(T^{-k}\left(A_k\right)\right) = \mu\left(a, 1\right) = 1 - a \neq 0$ and $\mu\left(T^{-(k+1)}\left(A_k\right)\right) = \mu\left(\emptyset\right) = 0$. As a consequence, $\alpha\left(C_T\right) = \infty$.

- If we take $\mathcal{A} = \{\emptyset, [0,1]\}$ and $T : [0,1] \to [0,1]$ as
$$T(x) = \begin{cases} 2x & 0 \leq x \leq \frac{1}{2}, \\ 1 & \frac{1}{2} \leq x \leq 1, \end{cases}$$
then T is a non-singular and pre-positive measurable transformation. Therefore, by Corollary 2.8, $\alpha\left(C_T\right) = 0$.

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