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SEMI-INVARIANT SUBMANIFOLDS OF CODIMENSION 3 IN A COMPLEX SPACE FORM IN TERMS OF THE STRUCTURE JACOBI OPERATOR

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Abstract. Let M be a semi-invariant submanifold of codimension 3 with almost contact metric structure (ϕ, ξ, η, g) in a complex space form $M_{n+1}(c), c \neq 0$. We denote by A and R_ξ the shape operator in the direction of distinguished normal vector field and the structure Jacobi operator with respect to the structure vector ξ, respectively. Suppose that the third fundamental form t satisfies $dt(X, Y) = 2\theta g(\phi X, Y)$ for a scalar $\theta(< 2c)$ and any vector fields X and Y on M. In this paper, we prove that if it satisfies $R_\xi A = AR_\xi$ and at the same time $\nabla_\xi R_\xi = 0$ on M, then M is a Hopf hypersurface of type (A) provided that the scalar curvature s of M holds $s - 2(n-1)c \leq 0$.

1. Introduction

A submanifold M is called a *CR submanifold* of Kaehlerian manifold $\tilde{M}$ with complex structure J if there exists a differentiable distribution $\Delta : p \to \Delta_p \subset T_pM$ on M such that Δ is J-invariant and the complementary orthogonal distribution $\Delta^\perp$ is totally real, where T_pM denotes the tangent space at each point p in M ([1,25]). In particular, M said to be a *semi-invariant submanifold* provided that $\dim \Delta^\perp = 1$. The unit normal in $J\Delta^\perp$ is called the *distinguished normal* to the semi-invariant submanifold ([3, 23]). In this case, M admits an induced almost contact metric structure (ϕ, ξ, η, g). A typical example of semi-invariant submanifold is real hypersurfaces. And new examples of nontrivial semi-invariant submanifolds in a complex projective space $P_n\mathbb{C}$ are constructed in [14] and [20]. Therefore we may expect to generalize some results which are valid in a real hypersurface to a semi-invariant submanifold.

An n dimensional complex space form $M_n(c)$ is a Kaehlerian manifold of constant holomorphic sectional curvature $4c$. As is well known, complete and

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simply connected complex space forms are isometric to a complex projective space $P_n\mathbb{C}$, or a complex hyperbolic space $H_n\mathbb{C}$ according as $c > 0$ or $c < 0$.

For the real hypersurface of $M_n(c), c \neq 0$, many results are known. One of them, Takagi [21, 22] classified all the homogeneous real hypersurfaces in $P_n\mathbb{C}$ as six model spaces which are said to be A_1, A_2, B, C, D and E, and Cecil and Ryan [4] and Kimura [15] proved that they are realized as the tubes of constant radius over Kaehlerian submanifolds when the structure vector field ξ is principal.

On the other hand, real hypersurfaces in $H_n\mathbb{C}$ have been investigated by Berndt [2], Montiel and Romero [16] and so on. Berndt [2] classified all real hypersurfaces with constant principal curvature in $H_n\mathbb{C}$ and showed that they are realized as the tubes of constant radius over certain submanifolds when the structure vector field is principal. Also such kinds of tubes are said to be real hypersurfaces of type A_0, A_1, A_2 or type B.

Let M be a real hypersurface of type A_1 or type A_2 in a complex projective space $P_n\mathbb{C}$ or that of type A_0, A_1 or A_2 in a complex hyperbolic space $H_n\mathbb{C}$. Now, hereafter unless otherwise stated, such hypersurfaces are said to be of *type* (A) for our convenience sake.

Characterization problems for a real hypersurface of type (A) in a complex space form were studied by many authors ([6, 7, 10, 11, 16, 18] etc.).

Two of them, we introduce the following characterization theorems due to Okumura [18] for $c > 0$ and Montiel and Romero [16] for $c < 0$, respectively.

Theorem O ([18])**.** *Let M be a real hypersurface in $P_n\mathbb{C}$, $n \geq 2$. If it satisfies*

$$g((A\phi - \phi A)X, Y) = 0 \tag{1.1}$$

for any vector fields X and Y, then M is locally congruent to a tube of radius r over one of the following Kaehlerian submanifolds:

- (A_1) *a geodesic hyperplane of radius r, where $0 < r < \pi/2$ and $r \neq \pi/4$,*
- (A_2) *a tube of radius r over a totally geodesic $P_k\mathbb{C}$ for some $k \in \{1, \ldots, n-2\}$, where $0 < r < \pi/2$ and $r \neq \pi/4$.*

Theorem MR ([16])**.** *Let M be a real hypersurface in $H_n\mathbb{C}$, $n \geq 2$. If it satisfies* (1.1)*, then M is locally congruent to one of the following hypersurface*:

- (A_0) *a horosphere in $H_n\mathbb{C}$,*
- (A_1) *a geodesic hypersphere or a tube over a complex hyperplane $H_{n-1}\mathbb{C}$,*
- (A_2) *a tube over a totally geodesic $H_k\mathbb{C}$ for some $k \in \{1, \ldots, n-2\}$.*

Denoting by R the curvature tensor of the submanifold, we define the Jacobi operator $R_\xi = R(\cdot, \xi)\xi$ with respect to the structure vector ξ. Then R_ξ is a self adjoint endomorphism on the tangent space of a CR submanifold.

Using several conditions on the structure Jacobi operator R_ξ, characterization problems for real hypersurfaces of type (A) have recently studied (cf. [7,10,11]). In the previous paper [6,7], Cho and one of the present authors gave another characterization of real hypersurface of type (A) in a complex space form $M_n(c)$. Namely they prove following:

Theorem CK ([7])**.** *Let M be a connected real hypersurface in $M_n(c)$ if it satisfies $R_\xi A = AR_\xi$ and $\nabla_\xi R_\xi = 0$, then M is of type (A), where A denotes the shape operator of M.*

On the other hand, semi-invariant submanifolds of codimension 3 in a complex projective space $P_{n+1}\mathbb{C}$ have been studied in [5, 9, 12, 14] and so on by using properties of induced almost contact metric structure and those of third fundamental form of the submanifold.

Now, let M be a semi-invariant submanifold of codimension 3 in a complex space form $M_{n+1}(c), c \neq 0$ such that the third fundamental form t satisfies $dt = 2\theta\omega$ for a scalar $\theta(\neq 2c)$, where $\omega(X,Y) = g(\phi X, Y)$ for any vector fields X and Y on M. We denote by A and S the shape operator in the direction of the distinguished normal and the Ricci tensor of M, respectively.

In the preceding work [10], it is proved that the submanifold M above is a Hopf hypersurface in $P_n\mathbb{C}$ provided that $A\xi = \alpha\xi$ and $\theta - 2c < 0$ for $c > 0$.

Further, one of present authors and Song ([13]) proved that if it satisfies $R_\xi\phi = \phi R_\xi$ and at the same time $S\xi = g(S\xi,\xi)\xi$, then M is a Hopf hypersurface of type (A) in $M_n(c)$ provided that the scalar curvature s of M holds $s - 2(n-1)c \leq 0$. This is a semi-invariant version of the main theorem stated in [11].

In this paper, we consider a semi-invariant submanifold of codimension 3 in $M_{n+1}(c)$ satisfying $R_\xi A = AR_\xi$ and at the same time $\nabla_\xi R_\xi = 0$, that is, the semi-invariant version of Theorem CK. In this case, M is a Hopf hypersurface of type (A) in $M_n(c)$ provided that $\theta - 2c < 0$ for $c > 0$ or $s - 2(n-1)c \leq 0$ for $c < 0$.

All manifolds in the present paper are assumed to be connected and of class C^∞ and the semi-invariant are supposed to be orientable.

2. Preliminaries

Let $\tilde{M}$ be a real $2(n+1)$-dimensional Kaehlerian manifold with parallel complex structure J and Riemannian metric tensor G. Let M be a real $(2n-1)$-dimensional Riemannian manifold immersed isometrically in $\tilde{M}$ by the immersion $i : M \to \tilde{M}$. In the sequel we identify $i(M)$ with M itself. We denote by g the Riemannian metric tensor on M from that of $\tilde{M}$.

We denote by $\tilde{\nabla}$ the operator of covariant differentiation with respect to the metric tensor G on $\tilde{M}$ and by ∇ the one on M. Then the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + \sum_{i=1}^{3} g(A_{N_i}X, Y)N_i, \tag{2.1}$$

$$\tilde{\nabla}_X N_i = -A_{N_i}X + \sum_{j=1}^{3} l_j^{(i)}(X)N_j \tag{2.2}$$

for any vector fields X and Y tangent to M and any vector fields N_1, N_2 and N_3 normal to M, where A_{N_i} is called a *second fundamental form* with respect to the normal vector N_i. Let $\nabla^\perp$ be the induced normal connection from $\tilde\nabla$ to $T^\perp M$. Then above equation (2.2) implies that

$$\nabla^\perp_X N_i = \sum_{j=1}^{3} l_j^{(i)}(X) N_j. \tag{2.3}$$

As is well-known, a submanifold M of a Kaehlerian manifold $\tilde M$ is said to be a *CR submanifold* ([1,3]) if it is endowed with a pair of mutually orthogonal and complementally differentiable distribution $(\Delta, \Delta^\perp)$ such that for any point $p \in M$ we have $J\Delta_p = T_pM$, $J\Delta_p^\perp \subset T_p^\perp M$, where T_pM denotes the tangent space of M at each point p on M. In particular, M is said to be a *semi-invariant submanifold* ([3,23]) provided that $\dim \Delta^\perp = 1$ or to be a CR submanifold with *CR dimension* $n-1$ ([19]). In this case the unit normal vector field in $J\Delta^\perp$ is called a *distinguished normal* to the semi-invariant submanifold and denote this by C ([3,23]).

In what follows we consider that M is a real $(2n-1)$-dimensional semi-invariant submanifold of codimension 3 in a Kaehlerian manifold $\tilde M$ of real dimension $2(n+1)$. Then we can choose a local orthonormal frame field $\{e_1, \ldots, e_{n-1}, Je_1, \ldots, Je_{n-1}, e_0 = \xi, J\xi = C, C_1 = JC_2, C_2\}$ on the tangent bundle $T\tilde M$ such that $e_1, \ldots, e_{n-1},\ Je_1, \ldots, Je_{n-1}, \xi \in TM$ and C, C_1 and $C_2 \in T^\perp M$ where $T^\perp M$ is the normal bundle. So, (2.1) can be written as

$$\tilde\nabla_X Y = \nabla_X Y + g(AX, Y)C + g(KX, Y)C_1 + g(LX, Y)C_2$$

for any vector fields X and Y on M, where we put $A_C = A$, $A_{C_1} = K$, $A_{C_2} = L$. If we put $l_2^{(1)} = l$, $l_3^{(1)} = m$, $l_3^{(2)} = t$, then the equations of Weingarten are also given by

$$\begin{aligned} \tilde\nabla_X C &= -AX + l(X)C_1 + m(X)C_2, \\ \tilde\nabla_X C_1 &= -KX - l(X)C + t(X)C_2, \\ \tilde\nabla_X C_2 &= -LX - m(X)C - t(X)C_1 \end{aligned}$$

because C, C_1 and C_2 are mutually orthogonal.

Now, let ϕ be the restriction of J on M, then we have (cf. [23,24])

$$JX = \phi X + \eta(X)C, \quad \eta(X) = g(\xi, X), \quad JC = -\xi \tag{2.4}$$

for any vector field X on M. From this, we see, using Hermitian property of J, that the aggregate (ϕ, ξ, η, g) is an *almost contact metric structure* on M, that is, we have

$$\begin{aligned} &\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad g(\xi, X) = \eta(X), \\ &\phi\xi = 0, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \end{aligned}$$

for any vector fields X and Y on M.

In the sequel, we denote the normal components of $\tilde\nabla_X C$ by $\nabla^\perp_X C$. The distinguished normal C is said to be *parallel* in the normal bundle if $\nabla^\perp_X C = 0$ for any vector fields X on M, that is, from (2.3) l and m vanish identically.

From the Kaehler condition $\tilde{\nabla}J = 0$ and using the Gauss and Weingarten formulas, we obtain from (2.4)

$$\nabla_X \xi = \phi AX, \tag{2.5}$$

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{2.6}$$

$$KX = \phi LX - m(X)\xi, \tag{2.7}$$

$$LX = -\phi KX + l(X)\xi \tag{2.8}$$

for any vector fields X and Y on M. From the last two relationships, we have

$$g(K\xi, X) = -m(X), \tag{2.9}$$

$$g(L\xi, X) = l(X). \tag{2.10}$$

Using the frame field $\{e_0 = \xi, e_1, \ldots, e_{n-1}, e_n = \phi e_1, \ldots, e_{2n-2} = \phi e_{n-1}\}$ on M, it follows from (2.7)–(2.10) that

$$\mathrm{Tr}K = \eta(K\xi) = -m(\xi), \quad \mathrm{Tr}L = \eta(L\xi) = l(\xi). \tag{2.11}$$

Now we retake C_1 and C_2, there is no loss of generality such that we may assume $\mathrm{Tr}L = 0$, that is, for example if $\mathrm{Tr}L \neq 0$, then there exists $a \in \mathbb{R}$ such that $\mathrm{Tr}K + a\mathrm{Tr}L = 0$. We may retake C_2 by $C_1 + aC_2$.

So we have

$$l(\xi) = 0. \tag{2.12}$$

Notation. To write our formulas in a convention from, in the sequel we denote by $\alpha = \eta(A\xi)$, $\beta = \eta(A^2\xi)$, $\gamma = \eta(A^3\xi)$, $h = \mathrm{Tr}A$, $k = \mathrm{Tr}K$, $h_{(2)} = \mathrm{Tr}({}^tAA)$, and for a function f we denote by ∇f the gradient vector field of f.

From (2.11) we have

$$m(\xi) = -k. \tag{2.13}$$

Using (2.7) and (2.8) we have

$$-m(X)\eta(Y) + m(Y)\eta(X) = \eta(X)l(\phi Y) - \eta(Y)l(\phi X).$$

If we put $Y = \xi$ in this, and take account of (2.13), then we find

$$l(\phi X) = m(X) + k\eta(X), \tag{2.14}$$

which tells us, using (2.12), that

$$m(\phi X) = -l(X). \tag{2.15}$$

Taking the inner product with LY to (2.7) and using (2.10), we get

$$g(KLX, Y) + g(LKX, Y) = -(l(X)m(Y) + l(Y)m(X)). \tag{2.16}$$

In the rest of this paper we shall suppose that $\tilde{M}$ is a Kaehlerian manifold of constant holomorphic sectional curvature $4c$, which is called a *complex space form* and denote by $M_{n+1}(c)$, that is,

$$\begin{aligned}\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z} = c(&G(\tilde{Y}, \tilde{Z})\tilde{X} - G(\tilde{X}, \tilde{Z})\tilde{Y} + G(J\tilde{Y}, \tilde{Z})J\tilde{X} \\ &- G(J\tilde{X}, \tilde{Z})J\tilde{Y} - 2G(J\tilde{X}, \tilde{Y})J\tilde{Z}),\end{aligned}$$

where $\tilde{R}$ is the curvature tensor of $M_{n+1}(c)$. Then the codimension reduction theorem is given by

Theorem 2.1 ([8, 19])**.** *Let $N_0(p)$ the orthogonal complement of first normal space in $T_p^{\perp}M$, that is, $N_0(p) = \{v \in T_p^{\perp}M; A_v = 0\}$ and $H_0(p)$ be the maximal J-invariant subspace of $N_0(p)$. If the orthogonal complement $H_2(p)$ of J-invariant subspace of $H_0(p)$ in $T_p^{\perp}M$ is invariant under parallel translation with respect to the normal connection and if q is the dimension of $H_2(p)$, then there exists a real $(2n-1+q)$ dimensional totally geodesic complex space form $M_{(2n-a+q)/2}(c)$ in $M_{n+1}(c)$ such that $M \subset M_{(2n-a+q)/2}(c)$.*

Moreover equations of Gauss and Codazzi are given by

$$\begin{aligned} R(X,Y)Z = {} & c(g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X \\ & - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z) \\ & + g(AY,Z)AX - g(AX,Z)AY + g(KY,Z)KX \\ & - g(KX,Z)KY + g(LY,Z)LX - g(LX,Z)LY, \end{aligned} \tag{2.17}$$

$$\begin{aligned} & (\nabla_X A)Y - (\nabla_Y A)X - l(X)KY + l(Y)AX - m(X)LY + m(Y)LX \\ & = c(\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi), \end{aligned} \tag{2.18}$$

$$(\nabla_X K)Y - (\nabla_Y K)X = -l(X)AY + l(Y)AX + t(X)LY - t(Y)LX, \tag{2.19}$$

$$\begin{aligned} & (\nabla_X L)Y - (\nabla_Y L)X \\ & = -m(X)AY + m(Y)AX - t(X)KY + t(Y)KX, \end{aligned} \tag{2.20}$$

where R is the Riemannian curvature tensor of M, and those of the Ricci tensor by

$$\begin{aligned} & (\nabla_X l)(Y) - (\nabla_Y l)(X) \\ & = g((AK - KA)X,Y) - m(X)t(Y) + m(Y)t(X), \end{aligned} \tag{2.21}$$

$$\begin{aligned} & (\nabla_X m)(Y) - (\nabla_Y m)(X) \\ & = g((AL - LA)X,Y) - t(X)l(Y) + t(Y)l(X), \end{aligned} \tag{2.22}$$

$$\begin{aligned} & (\nabla_X t)(Y) - (\nabla_Y t)(X) \\ & = g((KL - LK)X,Y) + l(Y)m(X) - l(X)m(Y) + 2cg(\phi X,Y). \end{aligned} \tag{2.23}$$

Now, we put $\nabla_\xi \xi = U$ in the sequel. Then U is orthogonal to ξ because of (2.5). We put

$$A\xi = \alpha\xi + \mu W, \tag{2.24}$$

where W is a unit vector field orthogonal to ξ. Then we have

$$U = \mu\phi W \tag{2.25}$$

by virtue of (2.5). So, W is also orthogonal to U. Further, we have

$$\mu^2 = \beta - \alpha^2. \tag{2.26}$$

From (2.24) and (2.25) we have

$$\phi U = -A\xi + \alpha\xi. \tag{2.27}$$

If we take account of (2.5), (2.24) and the last equation, then we find

$$g(\nabla_X \xi, U) = \mu g(AW, X). \tag{2.28}$$

Since W is orthogonal to ξ, we see, using (2.5) and (2.25), that

$$\mu g(\nabla_X W, \xi) = g(AU, X). \tag{2.29}$$

Differentiating (2.27) covariantly along M and using (2.5) and (2.6), we find

$$(\nabla_X A)\xi = -\phi\nabla_X U + g(AU + \nabla\alpha, X)\xi - A\phi AX + \alpha\phi AX. \tag{2.30}$$

Taking the inner product with ξ to this and using (2.9), (2.10), (2.12), (2.18) and (2.27), we have

$$(\nabla_\xi A)\xi = 2AU + \nabla\alpha + \eta(L\xi)K\xi - 2\eta(K\xi)L\xi. \tag{2.31}$$

Applying (2.30) by ϕ and making use of (2.28), we obtain

$$\phi(\nabla_X A)\xi = \nabla_X U + \mu g(AW, X)\xi - \phi A\phi AX - \alpha AX + \alpha g(A\xi, X)\xi, \tag{2.32}$$

which enables us to obtain

$$\nabla_U U = \phi(\nabla_U A)\xi + \phi A\phi AU + \alpha AU.$$

Finally, we introduce the structure Jacobi operator R_ξ with respect to the structure vector field ξ which defined by $R_\xi X = R(X, \xi)\xi$ for any vector field X. Then we have from (2.17)

$$\begin{aligned} R_\xi X = {} & c(X - \eta(X)\xi) + \alpha AX - \eta(AX)A\xi \\ & + \eta(K\xi)KX - \eta(KX)K\xi + \eta(L\xi)LX - \eta(LX)L\xi. \end{aligned}$$

Since l and m are dual 1-forms of $L\xi$ and $-K\xi$ respectively because of (2.9) and (2.10), the last equation can be written as

$$R_\xi X = c(X - \eta(X)\xi) + \alpha AX - \eta(AX)A\xi + kKX + m(X)K\xi - l(X)L\xi, \tag{2.33}$$

where we have used (2.9)–(2.13).

3. The third fundamental form of semi-invariant submanifolds

In this section we will suppose that M is a semi-invariant submanifold of codimension 3 in a complex space form $M_{n+1}(c), c \neq 0$ and that the third fundamental form t satisfies

$$dt = 2\theta\omega, \quad \omega(X, Y) = g(\phi X, Y) \tag{3.1}$$

for any vector fields X and Y on M and a certain scalar θ, where d denotes the exterior differential operator. Then (2.23) reformed as

$$g((LK - KL)X, Y) + l(X)m(Y) - l(Y)m(X) = -2(\theta - c)g(\phi X, Y),$$

or, using (2.16)

$$g(LKX, Y) + l(X)m(Y) = -(\theta - c)g(\phi X, Y), \tag{3.2}$$

which together with (2.9), (2.10) and (2.12) gives

$$l(KX) = kl(X), \quad m(LX) = 0 \tag{3.3}$$

for any vector X on M, that is

$$KL\xi = kL\xi, \quad LK\xi = 0.$$

Differentiating (3.1) covariantly along M and using (2.6) and the first Bianchi identity, we find

$$(X\theta)\omega(Y,Z) + (Y\theta)\omega(Z,X) + (Z\theta)\omega(X,Y) = 0,$$

which implies $(n-2)X\theta = 0$. Thus, θ is constant if $n > 2$.

For the case where $\theta = c$ in (3.1) we have $dt = 2c\omega$. In this case, the normal connection of M is said to be *L-flat* ([19]).

Replacing Y by ϕY in (3.2) and using (2.7) and (2.15), we have

$$g(K^2X,Y) + m(KX)\eta(Y) + l(X)l(Y) = (\theta - c)(g(X,Y) - \eta(X)\eta(Y)).$$

Putting $X = Y = e_i$ in this and summing up to $i = 0, 1, \dots, 2n-2$, we have

$$\mathrm{Tr}({}^tKK) - \|K\xi\|^2 + \|L\xi\|^2 = 2(n-1)(\theta - c),$$

which together with (2.9) implies that

$$\|K - k\eta \otimes \xi\|^2 + \|L\xi\|^2 = 2(n-1)(\theta - c), \tag{3.4}$$

where $\|F\|^2 = g(F,F)$ for any tensor field F on M. Thus, $\theta - c$ is nonnegative.

In the same way, we have, using (2.8), (2.12), (2.15) and (3.2), that

$$-\mathrm{Tr}({}^tLL) - \|L\xi\|^2 + \|K\xi - k\xi\|^2 = 2(n-1)(\theta - c), \tag{3.5}$$

Lemma 3.1. *Let M be a semi-invariant submanifold with L-flat normal connection in $M_{n+1}(c), c \neq 0$. If $A\xi = \alpha\xi$, then the distinguished normal C is parallel in the normal bundle.*

Proof. Since $\theta - c = 0$ was assumed, we have $L = 0$ and $KX = k\eta(X)\xi$ because of (3.4) and (3.5). By virtue of (2.11), it follows that $m(X) = -k\eta(X)$. We also have $l = 0$ because of (2.10). Thus, it suffices to show that $k = 0$. Using these facts, (2.21) reformed as

$$k(\eta(AX)\eta(Y) - \eta(X)\eta(AY)) = k(\eta(X)t(Y) - t(X)\eta(Y)),$$

which together with $A\xi = \alpha\xi$ gives $k(t(X) - t(\xi)\eta(X)) = 0$. If we suppose that $k \neq 0$ on M, then we have

$$t(X) = t(\xi)\eta(X) \tag{3.6}$$

on this open subset. Differentiating this covariantly and using (2.5) and (3.1) with $\theta = c$, we find

$$2cg(\phi X, Y) = t(\xi)g((A\phi - \phi A)X, Y)$$

by virtue of $A\xi = \alpha\xi$, which implies

$$2c(n-1) = t(\xi)(h - \alpha).$$

On the other side, since (2.20), (3.6) and $k \neq 0$, we have

$$\eta(X)AY - \eta(Y)AX = 0,$$

and hence $AY - \eta(Y)A\xi = 0$, which implies $h - \alpha = 0$, a contradiction. Hence $k = 0$ on M. □

From (3.3) we have $LK\xi = 0$ which together with (2.8) gives

$$K^2\xi = \|K\xi\|^2\xi. \tag{3.7}$$

Replacing Y by ϕY in (3.2) and using (2.7), (2.9) and (3.7), we have

$$K^2X + l(X)L\xi - \|K\xi\|^2\eta(X)\xi = (\theta - c)(X - \eta(X)\xi). \tag{3.8}$$

In the same way replacing X by ϕX in (3.2) and using (2.8) we have

$$\begin{aligned} g(L^2X, Y) + \eta(X)l(LY) + m(X)m(Y) + k\eta(X)m(Y) \\ = (\theta - c)(g(X, Y) - \eta(X)\eta(Y)). \end{aligned}$$

If we put $Y = \xi$ in this, we have

$$g(L^2\xi, X) = km(X) + (\|L\xi\|^2 + k^2)\eta(X). \tag{3.9}$$

Putting $Y = K\xi$ in (3.2), from (2.15) and (3.3), we have

$$(\theta - c - \|K\xi\|^2)L\xi = 0. \tag{3.10}$$

On the other hand, taking an inner product $L\xi$ to (3.8) and using (3.3), we have

$$(\theta - c - \|L\xi\|^2 - k^2)L\xi = 0. \tag{3.11}$$

Lemma 3.2. *Let M be a semi-invariant submanifold of codimension 3 in $M_{n+1}(c)$, $c \neq 0$ satisfying $dt = 2\theta\omega$ for a scalar $\theta(\neq 2c)$. Then $l = 0$ on M.*

Proof. Let Ω_0 be a set of points such that $\|L\xi\| \neq 0$ on M and suppose that Ω_0 be nonvoid. From now on we discuss our arguments on the open set Ω_0 of M. Then, by (3.10) and (3.11) we have

$$\|K\xi\|^2 = \theta - c, \quad \|L\xi\|^2 + k^2 = \theta - c. \tag{3.12}$$

Thus (3.8) turns out to be

$$K^2X = (\theta - c)X - l(X)L\xi. \tag{3.13}$$

Differentiating (3.13) covariantly and using (2.19), (2.20) and other equations already obtained, we find (for detail, see (2.19) and (2.24) of [14])

$$(\nabla_X K)Y = t(X)LY + l(Y)AX + g(AX, Y)L\xi + \sigma l(X)l(Y)L\xi, \tag{3.14}$$

$$(\nabla_X l)(Y) = t(X)m(Y) - g(AX, KY) - kg(AX, Y) \tag{3.15}$$

for some smooth function σ.

By (2.11) we have $\eta(K\xi) = k$. Differentiating this covariantly and using (3.14), we have

$$Yk = 2l(AY), \tag{3.16}$$

which implies that $\sigma = 0$ (for detail, see [14]). Thus (3.14) reformed as

$$(\nabla_X K)Y = t(X)LY + l(Y)AX + g(AX, Y)L\xi, \tag{3.17}$$

If we differentiate (2.8) covariantly and use (2.5), (2.6), (2.7), (2.14) and the last equation, then we find

$$(\nabla_X L)Y = -t(X)KY + m(Y)AX - g(AX, Y)K\xi,$$

which implies that

$$0 = \mathrm{Tr}\nabla_X L = -kt(X) + 2m(AX). \tag{3.18}$$

Differentiating (3.16) covariantly and using (3.15), we have

$$\begin{aligned} X(Yk) =& 2l((\nabla_X A)Y) + 2(t(X)m(AY) \\ &- g(KAX, AY) - kg(A^2X, Y)) + 2l(A\nabla_X Y). \end{aligned}$$

From which, taking the skew-symmetric part and making use of (2.14), (2.18), (3.3), (3.9), (3.12) and (3.18), we have

$$(\theta - 2c)(m(X)\eta(Y) - m(Y)\eta(X)) = 0.$$

Thus, it follows that $(\theta - 2c)(m(X) + k\eta(X)) = 0$ and hence $(\theta - 2c)l(X) = 0$ by virtue of (2.12) and (2.14). Therefore, by the assumption, we have $l = 0$. □

In the rest of this paper, we assume that M satisfies (3.1) with $\theta - 2c \neq 0$. Then we have $l = 0$ and hence

$$m(X) = -k\eta(X) \tag{3.19}$$

because of (2.14). Hence (2.9) and (2.10) reformed respectively as

$$K\xi = k\xi, \quad L\xi = 0. \tag{3.20}$$

It is, using (3.19), clear that (2.7), (2.8), (2.16) and (3.2) are reduced respectively to

$$KX = \phi LX + k\eta(X)\xi, \tag{3.21}$$

$$LX = -\phi KX, \tag{3.22}$$

$$LK + KL = 0, \tag{3.23}$$

$$g(LKX, Y) = -(\theta - c)g(\phi X, Y). \tag{3.24}$$

From the last relationship, we obtain

$$L^2X = (\theta - c)(X - \eta(X)\xi). \tag{3.25}$$

Further, if we take account of (3.19) and the fact that $l = 0$, then the structure equations (2.18)–(2.22) reformed respectively as

$$\begin{aligned} (\nabla_X A)Y - (\nabla_Y A)X = & k(\eta(Y)LX - \eta(X)LY) \\ & + c(\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi), \end{aligned} \tag{3.26}$$

$$(\nabla_X K)Y - (\nabla_Y K)X = t(X)LY - t(Y)LX, \tag{3.27}$$

$$\begin{aligned}&(\nabla_X L)Y - (\nabla_Y L)X\\ &= k(\eta(X)AY - \eta(Y)AX) - t(X)KY + t(Y)KX,\end{aligned} \tag{3.28}$$

$$g((KA - AK)X, Y) = k(\eta(X)t(Y) - t(X)\eta(Y)), \tag{3.29}$$

$$g((LA - AL)X, Y) = (Xk)\eta(X) - (Yk)\eta(X) + k((\phi A - A\phi)X, Y). \tag{3.30}$$

Putting $X = \xi$ in (3.29) and using (3.20), we find

$$g(KA\xi, Y) = kg(A\xi, Y) + k(t(Y) - t(\xi)\eta(Y)). \tag{3.31}$$

If we replace Y by ϕX in this and make use of (2.25) and (3.22), then we get

$$g(KU, X) = k(t(\phi X) - u(X)), \tag{3.32}$$

where $u(X) = g(U, X)$ for any vector fields X.

Replacing X by ξ in (3.30) and using (2.5), (3.20) and (3.22), then we find

$$KU = (\xi k)\xi - \nabla k + kU, \tag{3.33}$$

which together with (3.32) gives

$$Xk = (\xi k)\eta(X) + k(2u(X) - t(\phi X)). \tag{3.34}$$

If we replace Y by ϕY in (3.30) and make use of (3.21) and the last relationship, then we find

$$\begin{aligned}&g(\phi ALX - KAX, Y)\\ =&\ - k\{(t(Y) - t(\xi)\eta(Y))\eta(X) + 2\eta(X)(g(A\xi, Y) - \alpha\eta(Y))\\ &+ 2g(A\xi, X)\eta(Y) - g(AX, Y + g(\phi A\phi X, Y)\}.\end{aligned}$$

Taking the skew-symmetric part of this, we have

$$\phi ALX = -LA\phi X. \tag{3.35}$$

Since θ is constant if $n > 2$, differentiating (3.25) covariantly, we get

$$L(\nabla_X L)Y + (\nabla_X L)LY = -(\theta - c)(g(\phi AX, Y)\xi + \eta(Y)\phi AX).$$

Using the quite same method as that used to (3.17) from (3.13), we can derive from the last equation the following:

$$\begin{aligned}2L(\nabla_X L)Y = &\ (\theta - c)\{2t(X)\phi Y - \eta(Y)(A\phi + \phi A)X\\ &+ g((A\phi - \phi A)X, Y)\xi - \eta(X)(\phi A - A\phi)Y\}\\ &- k\{\eta(Y)(LA + AL)X - g((AL + LA)X, Y)\xi\\ &- \eta(X)(LA - AL)Y\},\end{aligned} \tag{3.36}$$

where we have used (3.24) and (3.28). Putting $Y = \xi$ in this, taking the inner product with Y and using (3.20) and (3.34), we have

$$\begin{aligned}&(\theta - c)g((A\phi - \phi A)X, Y) + (k^2 + \theta - c)(\eta(X)u(Y) + u(X)\eta(Y))\\ &+ k\{g((LA + AL)X, Y) - k(t(\phi X)\eta(Y) + \eta(X)t(\phi Y))\} = 0.\end{aligned} \tag{3.37}$$

In the following we consider the case where (2.24) with $\mu = 0$, that is, we have $A\xi = \alpha\xi$. Differentiating this covariantly along M, taking the inner product with Y and using (2.5), we find

$$g((\nabla_X A)\xi, Y) = -g(A\phi AX, Y) + \alpha g(\phi AX, Y) + (X\alpha)\eta(Y).$$

Taking the skew-symmetric part of this and using (3.20) and (3.26), we have

$$-2A\phi AX + \alpha(A\phi + \phi A)X + 2c\phi X = \eta(X)\nabla\alpha - (X\alpha)\xi. \tag{3.38}$$

If we put $X = \xi$ in this and using the fact that $A\xi = \alpha\xi$, then we have

$$Y\alpha = (\xi\alpha)\eta(Y). \tag{3.39}$$

From this we have $X(Y\alpha) = X(\xi\alpha)\eta(Y) + (\xi\alpha)g(\phi AX, Y) + (\xi\alpha)g(\xi, \nabla_X Y)$, where we have used (2.5). Thus, if we take the skew-symmetric part of this, then we get

$$X(\xi\alpha)\eta(Y) - Y(\xi\alpha)\eta(X) + (\xi\alpha)g((A\phi + \phi A)X, Y) = 0.$$

Putting $Y = \xi$ in this, we get $X(\xi\alpha) = \xi(\xi\alpha)\eta(X)$. Thus, the last equation can be written as

$$(\xi\alpha)(A\phi + \phi A) = 0. \tag{3.40}$$

In the next place, differentiating (3.19) covariantly and using (2.5), we find $\nabla_X m = -(Xk)\xi + k\phi AX$, from which taking the skew-symmetric part and making use of (2.22) with $l = 0$,

$$LAX - ALX - k(A\phi + \phi A)X = (Xk)\xi - \eta(X)\nabla k.$$

Since $A\xi = \alpha\xi$ was assumed, we then have

$$\nabla k = (\xi k)\xi \tag{3.41}$$

because of (3.20). From the last two relationships, it follows that

$$LAX - ALX = k(A\phi + \phi A)X. \tag{3.42}$$

Differentiating (3.41) covariantly, and taking the skew-symmetric part obtained, we get

$$(\xi k)(A\phi + \phi A)X = 0, \tag{3.43}$$

where we have used (2.5). From this and (3.39), we can write (3.38) as

$$(\xi k)(A^2\phi X + c\phi X) = 0. \tag{3.44}$$

In the previous paper [14], the following proposition was proved for the case where $c > 0$.

Proposition 3.3. *Let M be a real $(2n-1)$-dimensional $(n > 2)$ semi-invariant submanifold of codimension 3 in a complex space form $M_{n+1}(c), c \neq 0$. If it satisfies $dt = 2\theta\omega$ for a scalar $\theta \neq 2c$ and $\mu = g(A\xi, W) = 0$, then $k = 0$.*

Sketch of Proof. This fact was proved for $c > 0$ (see Proposition 3.5 of [14]). But, regardless of the sign of c this one is established. However, according to [14], only $\xi k = 0$ and $\xi\alpha = 0$ need to be proved. We are now going to prove, using (3.41), that $\xi k = 0$.

Now, let Ω_1 be a set of points such that $\xi k \neq 0$ on M and suppose that Ω_1 be nonvoid. Then we have

$$A\phi + \phi A = 0, \quad LA = AL \tag{3.45}$$

on Ω_1 because of (3.42) and (3.43). We discuss our arguments on Ω_1.

From (3.44), we have $A^2\phi X + c\phi X = 0$, which connected to properties of the almost contact metric structure yields

$$A^2X + cX = (\alpha^2 + c)\eta(X)\xi. \tag{3.46}$$

Since $A\xi = \alpha\xi$ was assumed, we can write (3.37) as $(\theta - c)A\phi X + kALX = 0$. Applying this by ϕ and using (3.21), we obtain

$$(\theta - c)AX - kAKX = \alpha(\theta - c - k^2)\eta(X)\xi.$$

Combining this to (3.46), we find $-kKX + (\theta - c)X = (\theta - c - k^2)\eta(X)\xi$, which shows $(n-1)(\theta - c) = 0$. Thus we have $\theta - c = 0$ if $n > 2$. This contradicts Lemma 3.1. Therefore we conclude that $\Omega_1 = \emptyset$.

By the same way as above we can prove $\xi\alpha = 0$ by virtue of (3.40) and (3.45). This completes the proof. □

4. The structure Jacobi operator satisfying $R_\xi A = AR_\xi$

We will continue our arguments under the same hypotheses $dt = 2\theta\omega$ for a scalar $\theta(\neq 2c)$ as in Section 3. Further, we assume that $R_\xi A = AR_\xi$ on a semi-invariant submanifold of codimension 3 in $M_{n+1}(c), c \neq 0$.

By virtue of (3.19) and (3.20) we can write (2.33) as

$$R_\xi X = c(X - \eta(X)\xi) + \alpha AX - \eta(AX)A\xi + kKX - k^2\eta(X)\xi.$$

Thus, $R_\xi A = AR_\xi$ gives

$$\begin{aligned} &g(A^2\xi, X)g(A\xi, Y) - g(A^2\xi, Y)g(A\xi, X) \\ &\quad + (k^2 + c)\{g(A\xi, X)\eta(Y) - g(A\xi, Y)\eta(X)\} \\ &= k^2(t(Y)\eta(X) - t(X)\eta(Y)), \end{aligned}$$

where we have used (3.29). Putting $X = \xi$ in this, we find

$$-\alpha A^2\xi + (\beta - k^2 - c)A\xi = k^2 t - \{k^2 t(\xi) + \alpha(k^2 + c)\}\xi. \tag{4.1}$$

Combining the last two equations, we obtain

$$g(A^2\xi, X)(A\xi - \alpha\xi) - (g(A\xi, X) - \alpha\eta(X))A^2\xi = \beta(\eta(X)A\xi - g(A\xi, X)\xi).$$

If we put $X = A\xi$ in this, and take account of (2.26), then we have

$$\mu^2 A^2\xi = (\gamma - \beta\alpha)A\xi + (\beta^2 - \alpha\gamma)\xi. \tag{4.2}$$

Now, let

$$\Omega = \{p \in M; k(p) \neq 0\}.$$

In the rest of this section, we suppose that Ω is nonvoid. Then from Proposition 3.3 and (4.2) we have

$$A^2\xi = \rho A\xi + (\beta - \rho\alpha)\xi, \tag{4.3}$$

where we have defined the function ρ by $\mu^2\rho = \gamma - \beta\alpha$. Hence, (4.1) is reformed as

$$(\beta - \rho\alpha - k^2 - c)(g(A\xi, X) - \alpha\eta(X)) = k^2(t(X) - t(\xi)\eta(X)),$$

which shows

$$(c + \rho\alpha - \beta + k^2)u(X) = k^2 t(\phi X) \tag{4.4}$$

From this and (3.32) we have

$$KU = \tau U \tag{4.5}$$

on Ω, where we have put

$$k\tau = c + \rho\alpha - \beta. \tag{4.6}$$

Accordingly (4.3) turns out to be

$$A^2\xi = \rho A\xi + (c - k\tau)\xi. \tag{4.7}$$

Applying (4.5) by ϕ and using (3.22), we find

$$LU = \mu\tau W. \tag{4.8}$$

Because of (3.24) and taking account of (4.5) and (4.8), we see that

$$\tau^2 = \theta - c. \tag{4.9}$$

Thus, τ is constant on Ω if $n > 2$. From (3.22) we have

$$LX = K\phi X \tag{4.10}$$

because L is a symmetric tensor. If we put $X = \mu W$ in (4.10) and make use of (2.25) and (4.5), then we obtain

$$\mu LW = \tau U. \tag{4.11}$$

In the next step, we see from (2.24) and (4.7) that

$$AW = \mu\xi + (\rho - \alpha)W \tag{4.12}$$

since we have $\mu \neq 0$ on Ω, where we have used (2.26). Differentiating this covariantly along Ω, we find

$$(\nabla_X A)W + A\nabla_X W = (X\mu)\xi + \mu\nabla_X\xi + X(\rho - \alpha)W + (\rho - \alpha)\nabla_X W. \tag{4.13}$$

If we take the inner product W to this and using (2.29) and (4.12), we find

$$g((\nabla_X A)W, W) = -2g(AU, X) + X\rho - X\alpha \tag{4.14}$$

because W is orthogonal to ξ. Taking the inner product with ξ to (4.13) and using (2.29), we also find

$$\mu g((\nabla_X A)W, \xi) = (\rho - 2\alpha)g(AU, X) + \mu(X\mu), \tag{4.15}$$

or, using (3.26)

$$\mu(\nabla_\xi A)W = (\rho - 2\alpha)AU + \mu\nabla\mu - k\mu LW - cU. \tag{4.16}$$

From this and (3.26) we verify that

$$\mu(\nabla_W A)\xi = (\rho - 2\alpha)AU - 2cU + \mu\nabla\mu. \tag{4.17}$$

Putting $X = \xi$ in (4.14) and using (4.15), we get

$$W\mu = \xi\rho - \xi\alpha. \tag{4.18}$$

Replacing X by ξ in (4.13) and using (4.11) and (4.16), we find

$$\begin{aligned} &(\rho - 2\alpha)AU - k\tau U - cU + \mu\nabla\mu + \mu(A\nabla_\xi W - (\rho - \alpha)\nabla_\xi W) \\ &= \mu(\xi\mu)\xi + \mu^2 U + \mu(\xi\rho - \xi\alpha)W. \end{aligned} \tag{4.19}$$

Now, it is, using (3.32) and (4.5), verified that

$$t(\phi X) = \left(1 + \frac{\tau}{k}\right) u(X). \tag{4.20}$$

Replacing X by ϕX and using properties of the almost contact metric structure, we obtain

$$t(X) = t(\xi)\eta(X) - \mu\left(1 + \frac{\tau}{k}\right) w(X), \tag{4.21}$$

where $w(X) = g(W, X)$ for any vector fields X.

Using (2.24) and (3.20), we can write (3.31) as

$$\mu g(KW, X) = k\mu w(X) + k(t(X) - t(\xi)\eta(X)),$$

which together with (4.21) implies that

$$KW = -\tau W \tag{4.22}$$

because of $\mu \neq 0$ on Ω.

Because of (3.20), we can write (2.31) as

$$(\nabla_\xi A)\xi = 2AU + \nabla\alpha. \tag{4.23}$$

If we put $X = \xi$ in (2.32) and make use of (2.24) and (2.26), then we find

$$\phi(\nabla_\xi A)\xi = \nabla_\xi U + \beta\xi - \alpha A\xi - \phi AU,$$

which together with (4.23) yields

$$\nabla_\xi U = 3\phi AU + \alpha A\xi - \beta\xi + \phi\nabla\alpha. \tag{4.24}$$

Differentiating (4.5) covariantly along Ω, we find

$$(\nabla_X K)U + K\nabla_X U = \tau\nabla_X U. \tag{4.25}$$

Taking the inner product with Y to this, and taking the skew-symmetric part with respect to X and Y, we obtain

$$\begin{aligned}&\mu\tau(t(X)w(Y)-t(Y)w(X))+g(K\nabla_X U,Y)-g(K\nabla_Y U,X)\\ &=\tau(g(\nabla_X U,Y)-g(\nabla_Y U,X)),\end{aligned}\tag{4.26}$$

where we have used (3.27) and (4.8).

On the other hand, from (2.25) we have $\phi U=-\mu W$. Differentiating this covariantly and using (2.6), we find

$$g(AU,X)\xi-\phi\nabla_X U=(X\mu)W+\mu\nabla_X W.\tag{4.27}$$

Putting $X=\xi$ in this and using (4.24), we get

$$\mu\nabla_\xi W=3AU-\alpha U+\nabla\alpha-(\xi\alpha)\xi-(\xi\mu)W,\tag{4.28}$$

which together with (4.12)

$$W\alpha=\xi\mu.\tag{4.29}$$

Substituting (4.28) and (4.29) into (4.19), we obtain

$$\begin{aligned}&3A^2U-2\rho AU+(\alpha\rho-\beta-k\tau-c)U+A\nabla\alpha+\frac{1}{2}\nabla\beta-\rho\nabla\alpha\\ &=2\mu(W\alpha)\xi+(2\alpha-\rho)(\xi\alpha)\xi+\mu(\xi\rho)W.\end{aligned}\tag{4.30}$$

5. Semi-invariant submanifolds satisfying $\nabla_\xi R_\xi=0$

We will continue our arguments under the same hypotheses $dt=2\theta\omega$ for a scalar $\theta(\neq 2c)$ and $R_\xi A=AR_\xi$ as in Section 4. Further, we assume that $\nabla_\xi R_\xi=0$ holds on a semi-invariant submanifold of codimension 3 in $M_{n+1}(c)$, $c\neq 0$.

Differentiating the first equation of Section 4 covariantly along M and using (2.5), we find

$$\begin{aligned}&g((\nabla_X R_\xi)Y,Z)\\ =&-(k^2+c)(\eta(Z)g(\nabla_X\xi,Y)+\eta(Y)g(\nabla_X\xi,Z))+(X\alpha)g(AY,Z)\\ &+\alpha g((\nabla_X A)Y,Z)-g(A\xi,Z)(g((\nabla_X A)\xi,Y)-g(A\phi AY,X))\\ &-g(A\xi,Y)(g((\nabla_X A)\xi,Z)-g(A\phi AZ,X))+(Xk)g(KY,Z)\\ &+kg((\nabla_X K)Y,Z)-2k(Xk)\eta(Y)\eta(Z).\end{aligned}$$

Replacing X by ξ in this and using (2.5) and (4.23), we find

$$\begin{aligned}&g((\nabla_\xi R_\xi)Y,Z)\\ =&-(k^2+c)(u(Y)\eta(Z)+u(Z)\eta(Y))+(\xi\alpha)g(AY,Z)+\alpha g((\nabla_\xi A)Y,Z)\\ &-g(A\xi,Z)(3g(AU,Y)+Y\alpha)-g(A\xi,Y)(3g(AU,Z)+Z\alpha)\\ &+(\xi k)g(KY,Z)+kg((\nabla_\xi K)Y,Z)-2k(\xi k)\eta(Y)\eta(Z),\end{aligned}$$

which shows

$$\begin{aligned}(\nabla_\xi R_\xi)X = &-(k^2+c)(u(X)\xi+\eta(X)U)+(\xi\alpha)AX+\alpha(\nabla_\xi A)X\\ &-(3AU+\nabla\alpha)g(A\xi,X)-(3g(AU,X)+X\alpha)A\xi+(\xi k)KX\\ &+k(\nabla_\xi K)X-2k(\xi k)\eta(X)\xi.\end{aligned}$$

Thus, $\nabla_\xi R_\xi=0$ gives

$$\begin{aligned}&\alpha(\nabla_\xi A)X+k(\nabla_\xi K)X+(\xi\alpha)AX+(\xi k)KX\\ &=(k^2+c)(u(X)\xi+\eta(X)U)+(3AU+\nabla\alpha)g(A\xi,X)\\ &\quad+(3g(AU,X)+X\alpha)A\xi+2k(\xi k)\eta(X)\xi.\end{aligned}\tag{5.1}$$

Replacing X by ξ in this and using (3.20), we find

$$\alpha(\nabla_\xi A)\xi+k(\nabla_\xi K)\xi=(k^2+c)U+\alpha(3AU+\nabla\alpha)+k(\xi k)\xi.\tag{5.2}$$

Differentiating (3.20) covariantly with respect to ξ and using (2.5) and (4.5), we have $(\nabla_\xi K)\xi=(\xi k)\xi+(k-\tau)U$. Thus, the equation (5.2) can be written as

$$\alpha AU+(k\tau+c)U=0,\tag{5.3}$$

where we have used (3.20) and (4.23).

In the rest of this section, we suppose that Ω is nonvoid.

Remark 5.1. $\alpha\neq 0$ on Ω.

In fact, if not, then we have $\alpha=0$ on this subset. We discuss our arguments on such a place. Then (4.6) and (5.3) are reformed respectively as $\beta-c+k\tau=0$, $k\tau+c=0$ and hence $\beta-2c=0$. Therefore we have $c>0$.

On the other hand, from (3.33) and (4.5) we have

$$\nabla k=(\xi k)\xi+(k-\tau)U.\tag{5.4}$$

Since $k\tau+c=0$ and τ is some constant, we have $k=$ const. on the set. Thus, $k-\tau=0$ and consequently $\tau^2+c=0$, a contradiction because of $c>0$. Therefore $\alpha=0$ is not impossible on Ω.

From (5.3) and Remark 5.1 we have

$$AU=\lambda U,\quad \alpha\lambda+k\tau+c=0.\tag{5.5}$$

Remark 5.2. $\tau\neq 0$ on Ω.

In fact, if not from (4.9) we have $\theta-c=0$. So (3.25) yields $L=0$. Consequently (3.21) is reformed as $KX=k\eta(X)\xi$. From (3.28) we also have

$$k(\eta(X)g(AY,\xi)-\eta(Y)g(AX,\xi)+\eta(X)t(Y)-\eta(Y)t(X))=0.$$

Putting $Y=\xi$ and $\sigma=\alpha+t(\xi)$ in this, we have $t(X)+g(A\xi,X)-\sigma\eta(X)=0$. Combining the last two equations, it follows that

$$AX=\eta(X)A\xi+g(A\xi,X)\xi-\alpha\eta(X)\xi,$$

which tells us that $AU=0$, which together with (5.3) gives $k\tau+c=0$, a contradiction. Thus, Remark 5.2 is established.

Lemma 5.3. *$\xi\lambda = 0$ and $W\lambda = 0$ on Ω.*

Proof. Differentiating $AU = \lambda U$ covariantly along Ω, we find

$$g((\nabla_X A)U, Y) + g(A\nabla_X U, Y) = (X\lambda)u(Y) + \lambda g(\nabla_X U, Y),$$

from which, taking the skew-symmetric part,

$$\begin{aligned} &\mu(k\tau - c)(\eta(Y)w(X) - \eta(X)w(Y)) + g(A\nabla_X U, Y) - g(A\nabla_Y U, X) \\ &= (X\lambda)u(Y) - (Y\lambda)u(X) + \lambda(g(\nabla_X U, Y) - g(\nabla_Y U, X)), \end{aligned}$$

where we have used (2.27), (3.26) and (4.8). Replacing X by U in this and taking account of $AU = \lambda U$, we get

$$A\nabla_U U - \lambda\nabla_U U = (U\lambda)U - \mu^2\nabla\lambda. \tag{5.6}$$

If we take the inner product with W and remember (4.12), then we have

$$\mu g(\nabla_U U, \xi) + \mu^2(W\lambda) + (\rho - \alpha - \lambda)g(\nabla_U U, W) = 0. \tag{5.7}$$

By the way, (4.25) implies that $g((\nabla_X K)U, U) = 0$. Because of (3.27), (4.8), (4.21) and the last equation gives $(\nabla_U K)U = 0$, which connected to (4.4), (4.22) and Remark 5.2 yields $g(W, \nabla_U U) = 0$. Thus, (5.7) reformed as $\mu g(\nabla_U U, \xi) + \mu^2(W\lambda) = 0$. However, the first term of this vanishes identically by virtue of (2.28), (4.12) and Remark 5.2, which shows $\mu(W\lambda) = 0$ and hence

$$W\lambda = 0.$$

In same the way, we verify, using (2.28) and (4.12), that

$$\xi\lambda = 0.$$

□

Now, if we take account of (4.9), (4.20) and Lemma 5.3, then (3.37) turns out to be

$$\tau^2(A\phi - \phi A)X + \tau(\tau - k)(u(X)\xi + \eta(X)U) + k(AL + LA)X = 0.$$

Putting $X = A\xi$ in this and using (4.7), (4.11), (4.12) and (5.5), we find

$$(k - \tau)(\rho - \alpha) + \lambda(k + \tau) = 0 \tag{5.8}$$

because $\tau \neq 0$ on Ω.

Lemma 5.4. *$\xi k = 0$ on Ω.*

Proof. Applying (5.4) by W, we have $Wk = 0$. Differentiating the second equation of (5.5) with respect to W and using Lemma 5.3 and above fact, we find $\lambda W\alpha = 0$, which together with (5.5) yields $(k\tau + c)W\alpha = 0$.

Now, $\xi k \neq 0$ on Ω, then we have $k\tau + c \neq 0$. Thus, $W\alpha = 0$ on this subset. We discuss our arguments on such a place. Differentiating (5.8) with respect to W and using Lemma 5.3 and these facts, we find $(k - \tau)W\rho = 0$. Since $\xi k \neq 0$, we have $k - \tau \neq 0$ and hence $W\rho = 0$. Differentiation (4.6) with respect to W gives $W\beta = 0$. Since $W\alpha = 0$, we see from (2.26) that $W\mu = 0$ which connected to (4.18) yields $\xi\rho - \xi\alpha = 0$.

If we differentiate (5.8) with respect to ξ and remember Lemma 5.3 and the last relationship, then we obtain $\lambda + \rho - \alpha = 0$, which together with (5.8) yields $\rho - \alpha = 0$. Hence (5.8) becomes $\lambda(k + \tau) = 0$, which implies $\lambda(\xi k) = 0$ and therefore $\lambda = 0$. Thus (5.5) is reformed as $k\tau + c = 0$, a contradiction. Accordingly we proved that $\xi k = 0$ on Ω. □

Lemma 5.5. *$k - \tau \neq 0$ on Ω.*

Proof. If not, then we have $k - \tau = 0$ on an open subset of Ω. We discuss our argument on such a place. Then we have $\lambda = 0$ because of (5.8) and Remark 5.2. So (4.6) and (5.5) turns out respectively to

$$\beta - \rho\alpha + 2\tau^2 = 0, \tag{5.9}$$

$$AU = 0, \quad \tau^2 + c = 0 \tag{5.10}$$

which together with (4.9) gives $\theta = 0$.

In the next step, differentiating (4.22) covariantly, we find

$$g((\nabla_X K)W, Y) + g(K\nabla_X W, Y) + \tau g(\nabla_X W, Y) = 0,$$

from which, taking the skew-symmetric part and using (3.27) and (4.11),

$$\begin{aligned} &\frac{\tau}{\mu}(t(X)u(Y) - t(Y)u(X)) + g(K\nabla_X W, Y) - g(K\nabla_Y W, X) \\ &= \tau((\nabla_Y W)X - (\nabla_X W)Y). \end{aligned} \tag{5.11}$$

By the way, differentiating (2.24) covariantly and using (2.5), we find

$$(\nabla_X A)\xi + A\phi AX = (X\alpha)\xi + \alpha\phi AX + (X\mu)W + \mu\nabla_X W.$$

If we put $X = \mu W$ in this and make use of (4.12), (4.17) and (5.10), then we find

$$\mu^2\nabla_W W - \mu\nabla\mu = (\alpha^2 - \alpha\rho - 2c)U - \mu(W\alpha)\xi - \mu(W\mu)W.$$

By (4.21) we have $t(W) = -2\mu$. Thus, replacing X by W in (5.11) and making use of the last equation, we have

$$\mu(K\nabla\mu + \tau\nabla\mu) = 2\tau(\mu^2 - \alpha^2 + \rho\alpha + 2c)U + 2\mu\tau(W\alpha)\xi.$$

If we take the inner product with U to this and take account of (4.5), then we obtain $U\mu = (\mu^2 - \alpha^2 + \rho\alpha + 2c)\mu$, which together with (2.26) and (5.9) gives

$$U\mu = 2(\mu^2 + \tau^2 + c)\mu. \tag{5.12}$$

On the other hand, differentiating (5.10) covariantly with respect to ξ, we find $(\nabla_\xi A)U + A\nabla_\xi U = 0$, which together with (4.3), (4.24) and (5.10) implies that

$$(\nabla_\xi A)U + (\alpha\rho - \beta)A\xi + \alpha(\beta - \rho\alpha)\xi + A\phi\nabla\alpha = 0.$$

Applying by ϕ, we have

$$\phi(\nabla_\xi A)U + (\alpha\rho - \beta)U + \phi A\phi\nabla\alpha = 0. \tag{5.13}$$

Putting $X = U$ and $Y = \xi$ in (3.26), we see

$$(\nabla_U A)\xi = (\nabla_\xi A)U \tag{5.14}$$

by virtue of (2.27), (3.20), (4.8) and (5.10).

From (2.30) we have

$$\nabla_X U + g(A^2\xi, X)\xi = \phi(\nabla_X A)\xi + \phi A\phi AX + \alpha AX - \alpha\eta(AX)\xi,$$

which connected to (5.10) gives $\nabla_U U = \phi(\nabla_U A)\xi$. Thus, (5.14) becomes $\nabla_U U = \phi(\nabla_\xi A)U$. If we take account (5.9), (5.10) and this, then (5.13) can be written as

$$\nabla_U U = 2cU - \phi A\phi A\nabla\alpha.$$

Now, taking the inner product with U to this and making use of (2.24), (2.25), (2.27) and (4.12), then we obtain

$$\mu(U\mu) = 2c\mu^2.$$

From (5.12) and this, we see that $\mu^2 = c$. It follows from (5.10) that $\tau^2 + \mu^2 = 0$ and hence $\tau = 0$, a contradiction by virtue of Remark 5.2. This completes the proof of Lemma 5.5. □

By Remark 5.2 and Lemma 5.5, we may only discuss our arguments where $\tau \neq 0$ and $k - \tau \neq 0$ on Ω.

By virtue of Lemma 5.4, (5.4) is reduced to

$$Xk = (k - \tau)u(X) \tag{5.15}$$

for any vector fields X. Differentiating (5.15) covariantly along Ω, we find

$$Y(Xk) = (Yk)u(X) + (k - \tau)(g(\nabla_Y U, X) + u(\nabla_Y X)).$$

If we take the skew-symmetric part and take account of (5.15) and Lemma 5.5, then we obtain

$$g(\nabla_Y U, X) = g(\nabla_X U, Y). \tag{5.16}$$

Putting $Y = \xi$ in this, we find $g(\nabla_\xi U, X) + g(U, \nabla_X \xi) = 0$, which together with (2.28) and (4.24) implies that

$$3\phi AU + \alpha A\xi - \beta\xi + \phi\nabla\alpha + \mu AW = 0.$$

By virtue of (2.24), (2.27), (4.12) and (5.5), this is reformed as

$$\phi\nabla\alpha + (\rho - 3\lambda)\mu W = 0,$$

which tells us that

$$\nabla\alpha = (\xi\alpha)\xi + (\rho - 3\lambda)U. \tag{5.17}$$

Differentiating the second equation of (5.5) with respect to ξ and taking account of Lemma 5.3 and Lemma 5.4, we find $\lambda\xi\alpha = 0$. But the function λ dose not vanish by virtue of (5.5), (5.15), Lemma 5.5 and Remark 5.2. Thus, we have $\xi\alpha = 0$ on Ω. Accordingly (5.17) turns out to be

$$\nabla\alpha = (\rho - 3\lambda)U. \tag{5.18}$$

In the next step, from (4.26) and (5.16) we have

$$g(K\nabla_X U, Y) - g(K\nabla_Y U, X) + \mu\tau\{t(X)w(Y) - t(Y)w(X)\} = 0.$$

Putting $X = \xi$ in this and using (2.28), (4.24) and the fact that $AU = \lambda U$, we find

$$K(3\lambda\phi U + \alpha A\xi - \beta\xi + \phi\nabla\alpha) + k\mu AW + \mu\tau t(\xi)W = 0,$$

which connected to (2.24), (2.27), (4.12), (4.22) and (5.18) gives

$$\tau t(\xi) + (\rho - \alpha)(k + \tau) = 0. \tag{5.19}$$

On the other hand, differentiating (4.21) covariantly along Ω, and taking account of (2.5), (2.6), (4.27), (5.5) and (5.15), we find

$$\begin{aligned} X(t(Y)) = X(t(\xi))\eta(Y) + t(\xi)g(\phi AX, Y) + \frac{\tau}{k^2}(k-\tau)\mu u(X)w(Y) \\ - \left(1 + \frac{\tau}{k}\right)(\lambda u(X)\eta(Y) - g(\phi\nabla_X U, Y)) + t(\nabla_X Y), \end{aligned}$$

from which taking the skew-symmetric part and using (3.1),

$$\begin{aligned} &2\theta g(\phi X, Y) + \frac{\tau}{k^2}(k-\tau)\mu(u(Y)w(X) - u(X)w(Y)) \\ &= X(t(\xi))\eta(Y) - Y(t(\xi))\eta(X) + t(\xi)(g(\phi AX, Y) - g(\phi AY, X)) \\ &\quad - \left(1 + \frac{\tau}{k}\right)\{\lambda(u(X)\eta(Y) - u(Y)\eta(X)) - (g(\phi\nabla_X U, Y) - g(\phi\nabla_Y U, X))\}. \end{aligned}$$

By the way, taking the inner product with Y to (2.30) and taking the skew-symmetric part, we get

$$\begin{aligned} &g(\phi\nabla_X U, Y) - g(\phi\nabla_Y U, X) - (\rho - 2\lambda)(u(X)\eta(Y) - u(Y)\eta(X)) \\ &= 2cg(\phi X, Y) - 2g(A\phi AX, Y) + \alpha(g(\phi AX, Y) - g(\phi AY, X)), \end{aligned}$$

where we have used (3.20), (3.26), (5.5) and (5.18).

Combining the last two equations, it follows that

$$\begin{aligned} &2\theta g(\phi X, Y) + \frac{\tau}{k^2}(k-\tau)\mu(u(Y)w(X) - u(X)w(Y)) \\ &- t(\xi)(g(\phi AX, Y) - g(\phi AY, X)) \\ &= X(t(\xi))\eta(Y) - Y(t(\xi))\eta(X) \\ &+ \left(1 + \frac{\tau}{k}\right)\{2cg(\phi X, Y) + (\rho - 3\lambda)(u(X)\eta(Y) - u(Y)\eta(X)) \\ &\qquad - 2g(A\phi AX, Y) + \alpha(g(\phi AX, Y) - g(\phi AY, X))\}. \end{aligned} \tag{5.20}$$

Putting $Y = \xi$ in this and using (2.5) and (5.5), we find

$$X(t(\xi)) = \xi(t(\xi))\eta(X) + \left\{t(\xi) + \left(1 + \frac{\tau}{k}\right)(\lambda + \alpha - \rho)\right\}u(X). \tag{5.21}$$

Differentiating (5.19) and using (5.15), we have

$$\tau X(t(\xi)) = (\alpha - \rho)(k - \tau)u(X) + (k + \tau)(X\alpha - X\rho).$$

From this and (5.8) it follows that

$$\tau X(t(\xi)) = (k+\tau)(X\alpha - X\rho + \lambda u(X)). \tag{5.22}$$

By the way, if we differentiate (5.8) with respect to ξ and make use of Lemma 5.3, Lemma 5.4, Lemma 5.5 and the fact that $\xi\alpha = 0$, then we obtain $\xi\rho = 0$ and hence $\xi(t(\xi)) = 0$ because of (5.19) and Remark 5.2. Thus, (5.21) reformed as

$$k\tau X(t(\xi)) = (k+\tau)\left\{\tau(\lambda+\alpha-\rho) - k(\rho-\alpha)\right\}u(X),$$

where we have used (5.19). From this and (5.22) we have

$$(k+\tau)(\nabla\alpha - \nabla\rho + \lambda U) = \left(1+\frac{\tau}{k}\right)\left\{(k+\tau)(\alpha-\rho) + \tau\lambda\right\}U,$$

where we have used (5.8). According to Lemma 5.5 and (5.15), it is clear that $k+\tau \neq 0$ on Ω. Thus, the last equation turns out to be

$$k(\nabla\alpha - \nabla\rho) = \{(k+\tau)(\alpha-\rho) + \lambda(\tau-k)\}U,$$

or, using (5.8),

$$(k-\tau)(\nabla\alpha - \nabla\rho) = 4\tau\lambda U. \tag{5.23}$$

On the other hand, differentiating (5.8) covariantly and using (5.15) and itself, we obtain $(k-\tau)(\nabla\rho - \nabla\alpha) + (k+\tau)\nabla\lambda - 2\tau\lambda U = 0$, or using (5.23),

$$(k+\tau)\nabla\lambda = 6\tau\lambda U. \tag{5.24}$$

Finally, we are going to prove that Ω is empty. If we put $X = U$ and $Y = W$ in (5.20), then by (4.12), (5.5) and (5.8), we have

$$2\theta k(k-\tau) + \frac{\tau}{k}(k-\tau)^2\mu^2 + 2\tau\lambda k t(\xi) = 2c(k^2-\tau^2) + 2\lambda^2(k+\tau)^2 - 2\tau(k+\tau)\alpha\lambda.$$

By (2.26), (4.6), (4.9), (5.5) and (5.8), we have $(k-\tau)\mu^2 = 2k\theta$. Thus, from this, using (4.9), (5.19) and the fact that $k+\tau \neq 0$, we have

$$\theta(k-\tau) - \lambda k(\rho-\alpha) = k\theta + \lambda^2(k+\tau),$$

which implies that

$$-\tau\theta(k-\tau) - \lambda k(\rho-\alpha)(k-\tau) = \lambda^2(k^2-\tau^2).$$

It follows from (5.8) that

$$\theta(k-\tau) = \lambda^2(k+\tau).$$

Differentiating this covariantly along Ω, and using (5.4) and (5.24), we have $\tau\lambda^2 = 0$, which connected to Remark 5.2 gives $\lambda = 0$ on Ω. Thus, $k\tau + c = 0$ because of (5.5) and consequently k is constant on Ω. Hence $U = 0$ by virtue of (5.15) and Lemma 5.5. By Proposition 3.3, we conclude that $k = 0$ and hence $\Omega = \emptyset$.

Developed above, we have:

Lemma 5.6. *Let M be a real $(2n-1)$-dimensional $(n>2)$ semi-invariant submanifold of codimension 3 in $M_{n+1}(c)$, $c\neq 0$ such that $dt=2\theta\omega$ for a scalar $\theta(\neq 2c)$. If M satisfies $R_\xi A=AR_\xi$ and at the same time $\nabla_\xi R_\xi=0$, then $k=0$ on M.*

6. Theorems

In this section we will suppose that M be a real $(2n-1)$-dimensional $(n>2)$ semi-invariant submanifold of codimension 3 in $M_{n+1}(c)$, $c\neq 0$ such that $dt=2\theta\omega$ for a scalar $\theta(\neq 2c)$. If M satisfies $R_\xi A=AR_\xi$ and at the same time $\nabla_\xi R_\xi=0$. Then by Lemma 5.6, we have $k=0$ on M and hence $m(X)=0$ for any vector fields X on M because of (3.19). We also have $K\xi=0$ by virtue of (3.20).

Because of (4.9), we can write (3.37) and (3.8) as

$$(\theta-c)\{g((A\phi-\phi A)X,Y)+\eta(X)u(Y)+\eta(Y)u(X)\}=0, \tag{6.1}$$

$$K^2X=(\theta-c)(X-\eta(X)\xi), \tag{6.2}$$

respectively.

Because of (3.33) and Lemma 5.6, we have $KU=0$, which together with (6.2) gives $(\theta-c)U=0$. Thus, (6.1) reformed as

$$(\theta-c)(A\phi-\phi A)X=0 \tag{6.3}$$

In what follows we assume that $\theta-c\neq 0$ on M, then from (6.3) we have $A\phi=\phi A$. From this fact and (3.39), the equation (3.38) can be written as

$$A^2X=\alpha AX+c(X-\eta(X)\xi). \tag{6.4}$$

Further, (3.36) turns out to be

$$L(\nabla_X L)Y=(\theta-c)(t(X)\phi Y-\eta(Y)A\phi X-\eta(X)\phi AY). \tag{6.5}$$

From (3.24), (3.25) and the assumption, we have

$$L\phi X=-KX. \tag{6.6}$$

which together with (3.21) gives

$$L\phi=-\phi L, \quad K\phi=-\phi K. \tag{6.7}$$

Differentiating (2.8) covariantly, we have $\eta((\nabla_X L)Y)=g(AX,KY)$. If we take account (3.35), (6.6) and this, then (6.5) can be written as

$$\begin{aligned} g((\nabla_X L)Y,Z)=&-t(X)g(KY,Z)+\eta(X)g(AKY,Z)\\ &+\eta(Y)g(AKX,Z)+g(AX,KY)\eta(Z). \end{aligned} \tag{6.8}$$

In the same way using the differentiation of (6.2), we have

$$\begin{aligned} g((\nabla_X K)Y,Z)=&\ t(X)g(LY,Z)-\eta(X)g(ALY,Z)\\ &-\eta(Y)g(ALX,Z)-g(AX,LY)\eta(Z). \end{aligned} \tag{6.9}$$

By Theorem 4.3 of [17], we have

$$(\nabla_X A)Y = -c(\eta(Y)\phi X + g(\phi X, Y)\xi). \tag{6.10}$$

Differentiating (6.9) covariantly along M and making use of (2.5), (3.20), (6.4), (6.8), (6.10) and itself, we find

$$\begin{aligned}(\nabla_Z \nabla_X K)(Y) = {} & Z(t(X)) - c(\eta(X)g(Z, KY)\xi + \eta(X)\eta(Y)KZ) + T(Z, X, Y)\\ & - \alpha(\eta(X)\eta(Y)AKZ + g(AZ, KY)\eta(X)\xi)\\ & - g(\nabla_Z X, \xi)ALY - g(X, \phi AZ)ALY - \eta(Y)AL\nabla_Z X\\ & - g(Y, \phi AZ)ALX - g(A\nabla_Z X, LY)\xi - g(AX, LY)\phi AZ,\end{aligned}$$

where $T(Z, X, Y)$ is a certain vector field with $T(Z, X, Y) = T(X, Z, Y)$, from which, taking the skew-symmetric part with respect to Z and X, and making use of (3.1), (6.3) with $\tau \neq 0$ and the Ricci identity for K,

$$\begin{aligned}& (R(Z, X) \cdot K)(Y)\\ = {} & 2\theta g(\phi Z, X)LY\\ & - c(\eta(X)g(Z, KY)\xi - \eta(Z)g(X, KY)\xi\\ & + \eta(Y)(\eta(X)KZ - \eta(Z)KX))\\ & - \alpha\{\eta(Y)(\eta(X)AKZ - \eta(Z)AKX)\\ & + g(AZ, KY)\eta(X)\xi - g(AX, KY)\eta(Z)\xi\}\\ & + 2g(Z, \phi AX)ALY - g(Y, \phi AZ)ALX + g(Y\phi AX)ALZ\\ & - g(AX, LY)\phi AZ + g(AZ, LY)\phi AX.\end{aligned} \tag{6.11}$$

Putting $Z = \phi e_i$ and $X = e_i$ and summing for i, and using (3.1), (3.21), (3.22), (6.3) and (6.4), we find

$$\sum_{i=0}^{2n-1} (R(\phi e_i, e_i) \cdot K)(Y) = 4\{c - (n-1)\theta\}LY + 2(h + \alpha)LAY. \tag{6.12}$$

On the other hand, from (2.17) we see, using (3.22), (6.2)–(6.4), (6.6) and (6.7), that

$$\sum_{i=0}^{2n-1} (R(\phi e_i, e_i) \cdot K)(Y) = 4\{2\theta - (2n+3)c\}LY - 4\alpha LAY,$$

which connected to (6.12) implies that

$$(h + 3\alpha)LAX = 2\{(n+1)\theta - 2(n+2)c\}LX,$$

which together with (3.25) yields

$$(h + 3\alpha)(g(AX, Y) - \alpha\eta(X)\eta(Y)) = 2\{(n+1)\theta - 2(n+2)c\}(X - \eta(X)\eta(Y)).$$

If we put $X = Y = e_i$ and summing up to $i = 0, 1, \ldots, 2n-2$, we have

$$(h + 3\alpha)(h - \alpha) = 4(n-1)\{(n+1)\theta - 2(n+2)c\},$$

which implies

$$(h-\alpha)^2 + 4\alpha(h-\alpha) = \delta, \tag{6.13}$$

where we have put $\delta = 4(n-1)\{(n+1)\theta - 2(n+2)c\}$.

Since we have $\mathrm{Tr}L = 0 = \mathrm{Tr}K$, $K\xi = 0 = L\xi$ and $A\xi = \alpha\xi$, if we put $X = \xi$, $Y = Z = e_i$ in (6.8) and (6.9), and summing up to $i = 0, 1, \ldots, 2n-2$, then we have

$$\mathrm{Tr}(AK) = 0, \quad \mathrm{Tr}(AL) = 0, \tag{6.14}$$

respectively, and hence

$$\mathrm{Tr}(A^2K) = 0, \quad \mathrm{Tr}(A^2L) = 0 \tag{6.15}$$

by virtue of (6.4).

If we put $Y = e_i$, $Z = Ae_i$ in (6.11) and summing up to $i = 0, 1, \ldots, 2n-2$, then from (6.14) and (6.15) we have

$$\sum_{i=0}^{2n-1} (R(Ae_i, X)\cdot K)(e_i) = (2\theta - 3\alpha^2 - 4c)AKX - 3c\alpha KX.$$

By (2.17), (6.14) and (6.15) we have

$$\sum_{i=0}^{2n-1} (R(Ae_i, X)\cdot K)(e_i) = (8c - 2\theta + h_{(2)})AKX - \{(\theta - 2c)(h-\alpha) - c\alpha\}KX.$$

Thus, above two equations gives

$$(4\theta - 12c - h_{(2)} - 3\alpha^2)AKX = \{4c\alpha - (\theta - 2c)(h-\alpha)\}KX,$$

which connected to (6.2) yields

$$(4\theta - 12c - h_{(2)} - 3\alpha^2)(h-\alpha) = 2(n-1)\{4c\alpha - (\theta-2c)(h-\alpha)\}. \tag{6.16}$$

Since we have from (6.4)

$$h_{(2)} = \alpha h + 2(n-1)c, \tag{6.17}$$

if we combine (6.13) to (6.16), then we obtain

$$(\theta - 3c)(h-\alpha) = 2\alpha(n-1)(\theta - 2c). \tag{6.18}$$

If $c < 0$ or $\theta - 2c < 0$ for $c > 0$, then $\theta - 3c \neq 0$ because $\theta - c$ is nonnegative. So we can write (6.18) as

$$h - \alpha = \frac{2(n-1)}{\theta - 3c}(\theta - 2c)\alpha.$$

Substituting this into (6.13), we have

$$4(n-1)(\theta - 2c)\alpha^2\{(n+1)\theta - 2(n+2)c\} = \delta(\theta - 3c)^2.$$

Thus, it follows that

$$\{\alpha^2(\theta - 2c) - (\theta - 3c)^2\}\{(n+1)(\theta - 2c) - 2c\} = 0. \tag{6.19}$$

For the case where $c > 0$ if $\theta - 2c < 0$, then we have $(n+1)(\theta - 2c) - 2c < 0$ and hence it follows from (6.19) that $\theta = 3c$ and $\alpha = 0$, a contradiction. Hence, we conclude that $\theta = c$ on M because our discussions were in the case where $\theta \neq c$, that is, the normal connection of M is L-flat. Therefore from (3.25) and (6.2) we have $K = L = 0$. Thus, we have

Lemma 6.1. *Let M be a real $(2n-1)$-dimensional $(n > 2)$ semi-invariant submanifold of codimension 3 satisfying $dt = 2\theta\omega$ for a certain scalar $\theta(< 2c)$ in a complex projective space $P_{n+1}\mathbb{C}$. If M satisfies $\nabla_\xi R_\xi = 0$ and at the same time $AR_\xi = R_\xi A$, then $K = L = 0$ and the normal connection of M is L-flat.*

From (2.17), the Ricci tensor S is given by

$$SX = c\{(2n+1)X - 3\eta(X)\xi\} + hAX - A^2X,$$

because of $k = 0$. Thus, the scalar curvature s of M is given by

$$s = 4(n^2-1)c + h^2 - h_{(2)} - 4(n-1)(\theta - c),$$

where we have used (3.25) and (6.2), which together with (6.17) gives

$$s = 2(n-1)(2n+1)c - 4(n-1)(\theta - c) + h(h - \alpha). \tag{6.20}$$

If we assume $c < 0$, then by (3.5) we have $(n+1)(\theta - 2c) - 2c > 0$. Thus, it follows from (6.19) that $\alpha^2(\theta - 2c) = (\theta - 3c)^2$, which connected to (6.18) gives $\alpha(h - \alpha) = 2(n-1)(\theta - 3c)$. Hence (6.13) is reformed as

$$h(h - \alpha) = 2(n-1)(2n-1)(\theta - c) - 2(n-1)c.$$

Thus, the scalar curvature s is given by

$$s = 2(n-1)(2n-3)(\theta - c) + 2(n-1)c. \tag{6.21}$$

Hence if we assume $s - 2(n-1)c \leq 0$, then by (3.4) we have $\theta - c = 0$, a contradiction.

Therefore we conclude that:

Lemma 6.2. *Let M be a real $(2n-1)$-dimensional $(n > 2)$ semi-invariant submanifold of codimension 3 satisfying $dt = 2\theta\omega$ for a scalar θ in a complex hyperbolic space $H_{n+1}\mathbb{C}$. If M satisfies $\nabla_\xi R_\xi = 0$ and at the same time $AR_\xi = R_\xi A$, then $K = L = 0$ and the normal connection of M is L-flat provided that the scalar curvature s of M holds $s - 2(n-1)c \leq 0$.*

As a consequence of Lemma 6.1, we have $K = L = 0$ and $\nabla^\perp C = 0$, namely $\nabla^\perp_X C = 0$ for any vector field X on M. Hence, $H_2(p)$ appeared in Theorem 2.1 is spanned by the distinguished normal C, and C is parallel in normal bundle. Thus, by Theorem 2.1 we have:

Theorem 6.3. *Let M be a real $(2n-1)$-dimensional $(n > 2)$ semi-invariant submanifold of codimension 3 in a complex space form $M_{n+1}(c)$, $c \neq 0$ such that the third fundamental form t satisfies $dt = 2\theta\omega$ for a scalar θ, where $\omega(X,Y) = g(\phi X, Y)$ for any vector fields X and Y on M. If M satisfies $R_\xi A = AR_\xi$ and at the same time $\nabla_\xi R_\xi = 0$, then M is a real hypersurface* *in a complex space form* $M_n(c)$, $c \neq 0$ *provided that* $\theta - 2c < 0$ *for* $c > 0$, *or* $s \leq 2(n-1)c$ *for* $c < 0$.

From Lemma 6.1, 6.2 we can write (5.1) as

$$\begin{aligned} \alpha(\nabla_\xi A)X + (\xi\alpha)AX = c(u(X)\xi + \eta(X)U) + \eta(AX)(3AU + \nabla\alpha) \\ + (3g(AU, X) + X\alpha)A\xi. \end{aligned} \tag{6.22}$$

Here, the distinguished normal C can be regarded as a unit normal vector field N on M in $M_n(c)$. Thus, the second fundamental form A with respect to C can also be regarded as that of N. Using (3.26) with $k = 0$ and (6.22), we can verify that $U = 0$, that is, $A\xi = \alpha\xi$ (see [7]). So α is a constant and hence $\alpha(\nabla_\xi A)X = 0$ because of (6.22), which together with (2.6) and (3.36) yield $\alpha(A\phi - \phi A) = 0$ and hence $A\xi = 0$ or $A\phi = \phi A$. Therefore, owing to Theorem 6.3 and Theorem O, we have:

Theorem 6.4. *Let* M *be a real* $(2n-1)$*-dimensional* $(n > 2)$ *semi-invariant submanifold of codimension* 3 *in a complex projective space* $P_{n+1}\mathbb{C}$ *such that the third fundamental form* t *satisfies* $dt = 2\theta\omega$ *for a certain scalar* $\theta(< 2c)$, *where* $\omega(X, Y) = g(\phi X, Y)$ *for any vector fields* X *and* Y *on* M, *and* M *satisfies* $R_\xi A = AR_\xi$ *if* $\eta(A\xi) \neq 0$. *Then* $\nabla_\xi R_\xi = 0$ *holds on* M *if and only if* M *is locally congruent to one of the following hypersurface*:

- (A_1) *a geodesic hyperplane of radius* r, *where* $0 < r < \pi/2$ *and* $r \neq \pi/4$,
- (A_2) *a tube of radius* r *over a totally geodesic* $P_k\mathbb{C}$ *for some* $k \in \{1, \ldots, n-2\}$, *where* $0 < r < \pi/2$ *and* $r \neq \pi/4$.

In the same way as above, we verify from Lemma 6.2 and Theorem MR

Theorem 6.5. *Let* M *be a real* $(2n-1)$*-dimensional* $(n > 2)$ *semi-invariant submanifold of codimension* 3 *satisfying* $dt = 2\theta\omega$ *for a scalar* θ *in a complex hyperbolic space* $H_{n+1}\mathbb{C}$, *where* $\omega(X, Y) = g(\phi X, Y)$ *for any vector fields* X *and* Y *on* M, *and satisfies* $R_\xi A = AR_\xi$. *Then* $\nabla_\xi R_\xi = 0$ *holds on* M *if and only if* M *is locally congruent to one of the following hypersurface provided that the scalar curvature* s *of* M *holds* $s - 2(n-1)c \leq 0$:

- (A_0) *a horosphere in* $H_n\mathbb{C}$,
- (A_1) *a geodesic hypersphere or a tube over a complex hyperplane* $H_{n-1}\mathbb{C}$,
- (A_2) *a tube over a totally geodesic* $H_k\mathbb{C}$ *for some* $k \in \{1, \ldots, n-2\}$.

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