

ON SOME NEW TYPE OF GENERATING FUNCTIONS OF GENERALIZED POISSON-CHARLIER POLYNOMIALS

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ABSTRACT. The present paper concerns with a study of certain generating functions and summation formulas of generalized Poisson-Charlier polynomials. Some special cases are also discussed.

1. Introduction

Recently, Khan and Ahmed [3] studied several variables analogue of Poisson-Charlier polynomials. The results obtained in this paper are analogous to those obtained in [3]. Let the sequence of functions $\{S_n(x) \mid n = 0, 1, 2, \dots\}$ be generated by ([5])

$$(1.1) \quad \sum_{n=0}^{\infty} A_{m,n} S_{m+n}(x) t^n = \frac{f(x,t)}{[g(x,t)]^m} S_m[h(x,t)],$$

where m is a non negative integer, the $A_{m,n}$ are arbitrary constants and f, g, h are suitable functions of x and t . The importance of a generating function of the form (1.1) in obtaining the bilateral and trilateral generating relations for the functions $S_n(x)$ was realized by several authors. For instance, using the generating functions of the type (1.1) for Hermite, Laguerre and Gegenbauer polynomials, Rainville [4], derived some bilateral and bilinear generating functions due to Mehler Rainville [4] (1960, p. 198, Eq. (2)). Brafman [1] (Rainville [4], 1960, pp. 198, 213). Hardy-Hille (Rainville [4], 1960, p. 212, Theorem 69) and Meixner (Rainville [4], 1960, p. 281, Eq. (24)).

The following results are required in this paper.

Lagrange's Expansion Formula ([6], p. 355). If $\phi(z)$ is holomorphic at $z = z_0$ and $\phi(z_0) \neq 0$, and if

$$(1.2) \quad z = z_0 + w\phi(z),$$

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then an analytic function $f(z)$, which is holomorphic at $z = z_0$, can be expanded as a power series in w by the Lagrange formula [Whittaker and Watson (1927), p. 133]

$$(1.3) \quad f(z) = f(z_0) + \sum_{n=1}^{\infty} \frac{w^n}{n!} D_z^{n-1} \{f'(z)[\phi(z)]^n\} |_{z=z_0},$$

where $D_z = d/dz$.

If we differentiate both sides of (1.3) with respect to w , using the relationship (1.2), and replace $f'(z)\phi(z)$ in the resulting equation by $f(z)$, we can write (1.3) in the form [cf. Polya and Szego (1972), p. 146, Problem 207]:

$$(1.4) \quad \frac{f(z)}{1 - w\phi'(z)} = \sum_{n=0}^{\infty} \frac{w^n}{n!} D_z^n \{f(z)[\phi(z)]^n\} |_{z=z_0},$$

which is usually more suitable to apply than (1.3).

For $\phi(z) \equiv 1$, both (1.3) and (1.4) evidently yield Taylor's expansion

$$(1.5) \quad f(z) = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{n!} f^{(n)}(z_0),$$

where

$$(1.6) \quad f^{(n)}(z_0) = D_z^n \{f(z)\} |_{z=z_0}.$$

The following basic lemmas (see Rainville [4, pp. 56–58]) given below are useful.

Lemma 1.

$$(1.7) \quad \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^n A(k, n - k)$$

and

$$(1.8) \quad \sum_{n=0}^{\infty} \sum_{k=0}^n B(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n + k).$$

Lemma 2.

$$(1.9) \quad \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n) = \sum_{n=0}^{\infty} \sum_{k=\lfloor \frac{n}{2} \rfloor}^n A(k, n - 2k)$$

and

$$(1.10) \quad \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} B(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B(k, n + 2k).$$

Lemma 3.

$$(1.11) \quad \sum_{n=0}^{\infty} f(n) = \sum_{n=0}^{\infty} f(2n) + \sum_{n=0}^{\infty} f(2n + 1).$$

Poisson-Charlier polynomials: The Poisson-Charlier polynomials $C_n(x; \alpha)$ are defined by ([6], p. 425)

$$(1.12) \quad C_n(x; \alpha) = \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{x}{k} k! \alpha^{-k},$$

where $\alpha > 0$, $x \in N_0$.

The following generating relations holds for (1.7):

$$(1.13) \quad \sum_{n=0}^{\infty} C_n(x; \alpha) \frac{t^n}{n!} = \left(1 - \frac{t}{\alpha}\right)^x e^t,$$

and

$$(1.14) \quad \sum_{n=0}^{\infty} (\lambda)_n C_n(x; \alpha) \frac{t^n}{n!} = (1-t)^{-\lambda} {}_2F_0 \left[\begin{matrix} \lambda, -x; \\ -; \end{matrix} \frac{t}{\alpha(1-t)} \right].$$

2. Generalized Poisson-Charlier polynomials

In this section, we define and study the generalized Poisson-Charlier polynomials, where $j, k \in \mathbb{N}$ ($0 \leq j < k$) as

$$(2.1) \quad \sum_{n=0}^{\infty} C_n^{(j,k)}(x; \alpha) \frac{t^n}{n!} = \left(1 - \frac{t}{\alpha}\right)^x E_j(t, k),$$

where

$$(2.2) \quad E_j(t, k) = \sum_{n=0}^{\infty} \frac{t^{kn+j}}{(kn+j)!}$$

is the Pseudo-Bessel function introduced in [2].

Note that

$$(2.3) \quad E_0(t, 1) = e^t$$

and

$$(2.4) \quad C_n^{(0,1)}(x; \alpha) = C_n(x; \alpha).$$

3. Generating functions of generalized Poisson-Charlier polynomials

The following generating functions hold for generalized Poisson-Charlier Polynomials given by (2.1)

$$(3.1) \quad C_m^{(j,k)}(x; \alpha) = m! \sum_{n=0}^{\lfloor \frac{m-j}{k} \rfloor} \frac{(-x)_{m-kn-j} \alpha^{kn-m+j}}{(m-kn-j)!(kn+j)!}.$$

To prove (3.1), we proceed as follows.

First we write,

$$(3.2) \quad \sum_{m=0}^{\infty} C_m^{(j,k)}(x; \alpha) \frac{t^m}{m!} = \sum_{m=0}^{\infty} \frac{(-x)_m t^m}{m! \alpha^m} \sum_{n=0}^{\infty} \frac{t^{kn+j}}{(kn+j)!}.$$

Setting $m \rightarrow m - kn - j$, so that $n \leq \frac{m-j}{k}$, we get

$$(3.3) \quad \sum_{m=0}^{\infty} C_m^{(j,k)}(x; \alpha) \frac{t^m}{m!} = \sum_{m=0}^{\infty} t^m \sum_{n=0}^{\lfloor \frac{m-j}{k} \rfloor} \frac{(-x)_{m-kn-j} \alpha^{kn-m+j}}{(m-kn-j)!(kn+j)!}.$$

Finally, comparing the coefficients of t , we get the result (3.1).

For $j = 0, k = 1$, (3.1) reduces to a known result:

$$(3.4) \quad C_m^{(0,1)}(x; \alpha) = C_m(x; \alpha) = m! \sum_{n=0}^m \frac{(-1)^n x! \alpha^{n-m}}{n!(m-n)!(x-n)!}.$$

For $j = 0, k = 2$, (3.1) gives

$$(3.5) \quad C_m^{(0,2)}(x; \alpha) = m! \sum_{n=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(-x)_{m-2n} \alpha^{2n-m}}{(m-2n)!(2n)!}.$$

Setting $x \rightarrow x - 1$ in (2.1), we have

$$\left(1 - \frac{t}{\alpha}\right) \sum_{n=0}^{\infty} C_n^{(j,k)}(x-1; \alpha) \frac{t^n}{n!} = \sum_{n=0}^{\infty} C_n^{(j,k)}(x; \alpha) \frac{t^n}{n!}$$

or

$$\sum_{n=0}^{\infty} C_n^{(j,k)}(x-1; \alpha) \frac{t^n}{n!} - \frac{1}{\alpha} \sum_{n=1}^{\infty} C_{n-1}^{(j,k)}(x-1; \alpha) \frac{t^n}{(n-1)!} = \sum_{n=0}^{\infty} C_n^{(j,k)}(x; \alpha) \frac{t^n}{n!},$$

comparing the coefficients of t^n , we get

$$(3.6) \quad C_n^{(j,k)}(x-1; \alpha) - \frac{n}{\alpha} C_{n-1}^{(j,k)}(x-1; \alpha) = C_n^{(j,k)}(x; \alpha).$$

Again from (2.1), we have

$$\left(1 - \frac{t}{\alpha}\right)^{-x} \sum_{n=0}^{\infty} C_n^{(j,k)}(x; \alpha) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{t^{kn+j}}{(kn+j)!}$$

or

$$\sum_{m=0}^{\infty} \frac{(x)_m t^m}{m! \alpha^m} \sum_{n=0}^{\infty} C_n^{(j,k)}(x; \alpha) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{t^{kn+j}}{(kn+j)!}.$$

Setting $n \rightarrow nk + j - m \geq 0 \rightarrow m \leq nk + j$

$$\sum_{n=0}^{\infty} \sum_{m=0}^{nk+j} \frac{(x)_m t^m}{m! \alpha^m} C_{nk+j-m}^{(j,k)}(x; \alpha) \frac{t^{nk+j-m}}{(nk+j-m)!} = \sum_{n=0}^{\infty} \frac{t^{kn+j}}{(kn+j)!},$$

comparing the coefficients of t^{nk+j} , we have

$$(3.7) \quad \sum_{m=0}^{nk+j} \frac{(x)_m C_{nk+j-m}^{(j,k)}(x; \alpha)}{m! \alpha^m (nk+j-m)!} = \frac{1}{(kn+j)!}.$$

For $j = 0, k = 1$, it reduces to

$$(3.8) \quad \sum_{m=0}^n \frac{(x)_m C_{n-m}(x; \alpha)}{m! (n-m)! \alpha^m} = \frac{1}{n!}.$$

Multiplying both sides of (2.1) by $(1 - \frac{t}{\alpha})^{-x+y}$, we have

$$\left(1 - \frac{t}{\alpha}\right)^{-x+y} \sum_{n=0}^{\infty} C_n^{(j,k)}(x; \alpha) \frac{t^n}{n!} = \left(1 - \frac{t}{\alpha}\right)^y E_j(t, k)$$

or

$$\sum_{m=0}^n \frac{(x-y)_m t^m}{m! \alpha^m} \sum_{n=0}^{\infty} C_n^{(j,k)}(x; \alpha) \frac{t^n}{n!} = \sum_{n=0}^{\infty} C_n^{(j,k)}(y; \alpha) \frac{t^n}{n!}.$$

Replacing $n \rightarrow n - m$ and comparing the coefficient of t^n , we get

$$(3.9) \quad m! \sum_{m=0}^n \frac{(x-y)_m}{m! (n-m)! \alpha^m} C_{n-m}^{(j,k)}(x; \alpha) = C_n^{(j,k)}(y; \alpha).$$

For $y = 0, j = 0, k = 1$, it reduces to

$$(3.10) \quad m! \sum_{m=0}^n \frac{(x)_m}{m! (n-m)! \alpha^m} C_{n-m}(x; \alpha) = C_n(\alpha).$$

Replacing x by $x + y$, we get from (3.9)

$$(3.11) \quad m! \sum_{m=0}^n \frac{(x)_m}{m! (n-m)! \alpha^m} C_{n-m}^{(j,k)}(x+y; \alpha) = C_n^{(j,k)}(y; \alpha).$$

4. Summation formulae

In this section we obtain the summation formula for generalized Poisson-Charlier polynomials $C_n^{(j,k)}(x, \alpha)$. We start with the definition of generalized Poisson-Charlier polynomials $C_n^{(j,k)}(x, \alpha)$ and write by using Lemma 1.1,

$$\begin{aligned} \sum_{n=0}^{\infty} C_n^{(j,k)}(x; \alpha) \frac{t^n}{n!} &= \sum_{n=0}^{\infty} C_{2n}^{(j,k)}(x; \alpha) \frac{t^{2n}}{(2n)!} + \sum_{n=0}^{\infty} C_{2n+1}^{(j,k)}(x; \alpha) \frac{t^{2n+1}}{(2n+1)!} \\ &= \left[\sum_{k=0}^{\infty} \frac{(-x)_{2k}}{(2k)!} \left(\frac{t}{\alpha}\right)^{2k} + \sum_{k=0}^{\infty} \frac{(-x)_{2k+1}}{(2k+1)!} \left(\frac{t}{\alpha}\right)^{2k+1} \right] e^t. \end{aligned}$$

Replacing t by it , we get

$$\sum_{n=0}^{\infty} C_{2n}^{(j,k)}(x; \alpha) \frac{t^{2n}}{2n!} (-1)^n + i \sum_{n=0}^{\infty} C_{2n+1}^{(j,k)}(x; \alpha) \frac{t^{2n+1}}{(2n+1)!} (-1)^n$$

$$= \left[\sum_{k=0}^{\infty} \frac{(-x)_{2k}}{(2k)!} \left(\frac{t}{\alpha}\right)^{2k} (-1)^k + i \sum_{k=0}^{\infty} \frac{(-x)_{2k+1}}{(2k+1)!} \left(\frac{t}{\alpha}\right)^{2k+1} (-1)^k \right] (\cos t + i \sin t).$$

Comparing real and imaginary parts, we get

$$\begin{aligned} (4.1) \quad & \sum_{n=0}^{\infty} C_{2n}^{(j,k)}(x; \alpha) \frac{t^{2n}}{2n!} (-1)^n \\ &= \cos t \left[\sum_{k=0}^{\infty} \frac{(-x)_{2k}}{(2k)!} \left(\frac{t}{\alpha}\right)^{2k} (-1)^k \right] - \sin t \left[\sum_{k=0}^{\infty} \frac{(-x)_{2k+1}}{(2k+1)!} \left(\frac{t}{\alpha}\right)^{2k+1} (-1)^k \right] \\ &= \cos t {}_3F_2 \left[\begin{matrix} -\frac{1-x}{2}, \frac{2-x}{2}, 1; \\ 1, \frac{3}{2}; \end{matrix} -\left(\frac{t}{\alpha}\right)^2 \right] \\ &\quad + \frac{xt \sin t}{\alpha} {}_3F_2 \left[\begin{matrix} -\frac{x}{2}, \frac{1-x}{2}, 1; \\ \frac{1}{2}, \frac{3}{2}; \end{matrix} -\left(\frac{t}{\alpha}\right)^2 \right] \end{aligned}$$

and

$$\begin{aligned} (4.2) \quad & \sum_{n=0}^{\infty} C_{2n+1}^{(j,k)}(x; \alpha) \frac{t^{2n+1}}{(2n+1)!} (-1)^n \\ &= \cos t \left[\sum_{k=0}^{\infty} \frac{(-x)_{2k+1}}{(2k+1)!} \left(\frac{t}{\alpha}\right)^{2k+1} (-1)^k \right] + \sin t \left[\sum_{k=0}^{\infty} \frac{(-x)_{2k}}{(2k)!} \left(\frac{t}{\alpha}\right)^{2k} (-1)^k \right] \\ &= \sin t {}_3F_2 \left[\begin{matrix} -\frac{x}{2}, \frac{1-x}{2}, 1; \\ \frac{1}{2}, \frac{3}{2}; \end{matrix} -\left(\frac{t}{\alpha}\right)^2 \right] \\ &\quad - \frac{xt \cos t}{\alpha} {}_3F_2 \left[\begin{matrix} -\frac{1-x}{2}, \frac{2-x}{2}, 1; \\ 1, \frac{3}{2}; \end{matrix} -\left(\frac{t}{\alpha}\right)^2 \right]. \end{aligned}$$

From (2.1), we have

$$\sum_{n=0}^{\infty} C_n^{(j,k)}(x; \alpha) \frac{t^n}{n!} = \left(1 - \frac{t}{\alpha}\right)^x E_j(t, k).$$

Therefore,

$$\sum_{n=0}^{\infty} C_{2n}^{(j,k)}(x; \alpha) \frac{t^{2n}}{(2n)!} + \sum_{n=0}^{\infty} C_{2n+1}^{(j,k)}(x; \alpha) \frac{t^{2n+1}}{(2n+1)!}$$

$$\begin{aligned}
&= \left(1 - \frac{t}{\alpha}\right)^x \left[\sum_{n=0}^{\infty} \frac{t^{2nk+j}}{(2nk+j)!} + \sum_{n=0}^{\infty} \frac{t^{(2n+1)k+j}}{(2n+1)k+j)!} \right] \\
&= \sum_{m=0}^{\infty} \frac{(-x)_m}{\alpha^m m!} t^m \left[\sum_{n=0}^{\infty} \frac{t^{2nk+j}}{(2nk+j)!} + \sum_{n=0}^{\infty} \frac{t^{(2n+1)k+j}}{(2n+1)k+j)!} \right].
\end{aligned}$$

Replacing t by it , we get

$$\begin{aligned}
(4.3) \quad & \sum_{n=0}^{\infty} C_{2n}^{(j,k)}(x; \alpha) \frac{t^{2n}}{2n!} (-1)^n + i \sum_{n=0}^{\infty} C_{2n+1}^{(j,k)}(x; \alpha) \frac{t^{2n+1}}{(2n+1)!} (-1)^n \\
&= \left[\sum_{m=0}^{\infty} a_{2m} (-1)^m t^{2m} + i \sum_{m=0}^{\infty} a_{2m+1} t^{2m+1} \right] \\
&\quad \times \left[\sum_{n=0}^{\infty} \frac{t^{2nk+j} (-1)^n}{(2nk+j)!} + i \sum_{n=0}^{\infty} \frac{t^{(2n+1)k+j}}{(2n+1)k+j)!} \right],
\end{aligned}$$

where $a_m = \frac{(-x)_m}{\alpha^m m!}$.

Comparing real and imaginary parts, we get

$$(4.4) \quad C_{2n}^{(j,k)}(x; \alpha) \frac{t^{2n}}{(2n)!} (-1)^n = AC - BD$$

and

$$(4.5) \quad C_{2n+1}^{(j,k)}(x; \alpha) \frac{t^{2n+1}}{(2n+1)!} (-1)^n = BC + AD,$$

where

$$\begin{aligned}
A &= \sum_m a_{2m} (-1)^m t^{2m}, \quad B = i \sum_m a_{2m+1} t^{2m+1}, \\
C &= \sum_n \frac{t^{2nk+j} (-1)^n}{(2nk+j)!} \quad \text{and} \quad D = i \sum_n \frac{t^{(2n+1)k+j}}{(2n+1)k+j!}.
\end{aligned}$$

5. The generalized Poisson-Charlier polynomials of two variables

The generalized Poisson-Charlier polynomials of two variables $C_n^{(j,k)}(x, y; \alpha, \beta)$ are defined as follows:

$$(5.1) \quad C_n^{(j,k)}(x, y; \alpha, \beta) = \sum_{r=0}^{\lfloor \frac{n-j}{k} \rfloor} \sum_{s=0}^{\lfloor \frac{n-r-j}{k} \rfloor} (-1)^{r+s} (r+s)! \alpha^{-r} \beta^{-s} \binom{\frac{n}{k}}{r+s} \binom{x}{r} \binom{y}{s},$$

where $\alpha, \beta > 0$, $x, y \in N_0$.

The following generating relations hold for (5.1):

$$(5.2) \quad \sum_{n=0}^{\infty} C_n^{(j,k)}(x, y; \alpha, \beta) \frac{t^n}{n!} = \left(1 - \frac{t}{\alpha}\right)^x \left(1 - \frac{t}{\beta}\right)^y E_j(t, k)$$

and

$$(5.3) \quad \begin{aligned} & \sum_{n=0}^{\infty} (\lambda)_n C_n^{(j,k)}(x, y; \alpha, \beta) \frac{t^n}{n!} \\ &= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\lambda+j)_{r+s} (\lambda)_j}{r!s!} \left(\frac{t}{\alpha}\right)^r \left(\frac{t}{\beta}\right)^s (x)_r (-y)_s \phi_{r,s}, \end{aligned}$$

where

$$(5.4) \quad \phi_{r,s} = \sum_{n=0}^{\infty} \frac{(\lambda+r+s+j)_{nk} t^{nk+j}}{(nk+j)!}.$$

Proof of (5.2). We have from L.H.S,

$$\begin{aligned} & \left(1 - \frac{t}{\alpha}\right)^x \left(1 - \frac{t}{\beta}\right)^y E_j(t, k) \\ &= \sum_{r=0}^{\infty} \frac{(-x)_r}{r!} \left(\frac{t}{\alpha}\right)^r \sum_{s=0}^{\infty} \frac{(-y)_s}{s!} \left(\frac{t}{\beta}\right)^s \sum_{n=0}^{\infty} \frac{t^{kn+j}}{(kn+j)!} \\ &= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^r x! (-1)^s y! (\alpha)^{-r} (\beta)^{-s} t^{kn+j+r+s}}{(x-r)! (y-s)! r!s! (kn+j)!}. \end{aligned}$$

Replacing $n \rightarrow \frac{n-r-j}{k}$, then $n \rightarrow n-s$, we get

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{\lfloor \frac{n-j}{k} \rfloor} \sum_{s=0}^{\lfloor \frac{n-r-j}{k} \rfloor} \frac{(-1)^{r+s} x! y! (\alpha)^{-r} (\beta)^{-s} t^n}{(x-r)! (y-s)! r!s! \left(\frac{n}{k} - r - s - j\right)!},$$

from which the result follows. \square

Proof of (5.3). We have

$$\begin{aligned} & \sum_{n=0}^{\infty} (\lambda)_n C_n^{(j,k)}(x, y; \alpha, \beta) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{r=0}^{\lfloor \frac{n-j}{k} \rfloor} \sum_{s=0}^{\lfloor \frac{n-r-j}{k} \rfloor} (\lambda)_n (-1)^{r+s} \binom{n}{r+s} \binom{x}{r} \binom{y}{s} (r+s)! \alpha^{-r} \beta^{-s} \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{r=0}^{\lfloor \frac{n-j}{k} \rfloor} \sum_{s=0}^{\lfloor \frac{n-r-j}{k} \rfloor} (\lambda)_n \frac{(-x)_r (-y)_s}{r!s! (n-r-s)! (\alpha)^r (\beta)^s} t^n. \end{aligned}$$

Setting $n \rightarrow nk+r+j+s$, we have

$$\begin{aligned} &= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (\lambda)_{nk+r+j+s} \frac{(-x)_r (-y)_s}{r!s! (n+j)! (\alpha)^r (\beta)^s} t^{nk+r+j+s} \\ &= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (\lambda)_{nk+r+j+s} \left[\left(\frac{t}{\alpha}\right)^r \left(\frac{t}{\beta}\right)^s \frac{(-x)_r (-y)_s}{r!s!} \frac{t^{nk+j}}{(nk+j)!} \right] \end{aligned}$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (\lambda + r + s + j)_{nk} (\lambda + j)_{r+s} (\lambda)_j \left[\left(\frac{t}{\alpha}\right)^r \left(\frac{t}{\beta}\right)^s \frac{(-x)_r (-y)_s}{r!s!} \frac{t^{nk+j}}{(nk+j)!} \right] \\
&= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\lambda + j)_{r+s} (\lambda)_j}{r!s!} \left(\frac{t}{\alpha}\right)^r \left(\frac{t}{\beta}\right)^s (x)_r (-y)_s \phi_{r,s},
\end{aligned}$$

which proves (5.3), where, $\phi_{r,s}$ is given by (5.4). \square

Special Case: For $j = 0$, and $k = 1$, (5.2) reduces to (3.2).

Generalized Poisson-Charlier polynomials of n -variables $C_n^{(j,k)}(x_1, x_2, \dots, x_n; \alpha_1, \alpha_2, \dots, \alpha_n)$ are defined as follows:

$$\begin{aligned}
(5.5) \quad &C_n^{(j,k)}(x_1, x_2, \dots, x_n; \alpha_1, \alpha_2, \dots, \alpha_n) \\
&= \sum_{r_1=0}^n \sum_{r_2=0}^{n-r_1} \sum_{r_k=0}^{n-r_1-r_2-\dots-r_{k-1}} (-1)^{r_1+r_2+\dots+r_k} \binom{\frac{n}{k}}{r_1+r_2+\dots+r_k} \binom{x_1}{r_1} \binom{x_2}{r_2} \dots \binom{x_n}{r_k} \\
&\quad \times (r_1+r_2+\dots+r_k)! \alpha_1^{-r_1} \alpha_2^{-r_2} \dots \alpha_n^{-r_k},
\end{aligned}$$

where $j, k \in \mathbb{N}$, $0 \leq j < k$, $\alpha_1, \alpha_2, \dots, \alpha_n > 0$, $x_1, x_2, \dots, x_n \in \mathbb{N}_0$.

Results for three and n -variables will follow as in case of two variables.

References

- [1] F. Brafman, *Some generating functions for Laguerre and Hermite polynomials*, Canadian J. Math. **9** (1957), 180–187. <http://doi.org/10.4153/CJM-1957-020-1>
- [2] G. Dattoli, C. Cesarano, and D. Sacchetti, *Pseudo-Bessel functions and applications*, Georgian Math. J. **9** (2002), no. 3, 473–480.
- [3] M. A. Khan and S. Ahmed, *On some new generating functions for Poisson-Charlier polynomials of several variables*, Math. Sci. Res. J. **15** (2011), no. 5, 127–136.
- [4] E. D. Rainville, *Special functions*, Reprint of 1960 first edition, Chelsea Publishing Co., Bronx, NY, 1971.
- [5] J. P. Singhal and H. M. Srivastava, *A class of bilateral generating functions for certain classical polynomials*, Pacific J. Math. **42** (1972), 755–762. <http://projecteuclid.org/euclid.pjm/1102959810>
- [6] H. M. Srivastava and H. L. Manocha, *A treatise on generating functions*, Ellis Horwood Series: Mathematics and its Applications, Ellis Horwood Ltd., Chichester, 1984.

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