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ON COMMON AND SEQUENTIAL FIXED POINTS VIA ASYMPTOTIC REGULARITY

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Abstract. In this paper, we introduce some new classes of generalized mappings and prove some common fixed point theorems for a pair of asymptotically regular mappings. Our results extend and improve various well-known results due to Kannan, Reich, Wong, Hardy and Rogers, Ćirić, Jungck, Górnicki and many others. In addition to it, a sequential fixed point for a mapping which is the point-wise limit of a sequence of functions satisfying Ćirić-Proinov-Górnicki type mapping has been proved. Supporting examples have been given in strengthening hypotheses of our established theorems.

1. Introduction and preliminaries

Fixed point theory is a trending research area for its numerous applications in different branches of mathematics such as boundary value problems, nonlinear differential and integral equations, non-linear matrix equations, homotopy theory, stability of fixed point problems and many more (see [5, 15, 19–21] and references therein). Researchers in this particular area deal with several nonlinear mappings and topological spaces to investigate fixed points.

Asymptotic regularity is an important tool to find fixed points of mappings. The definition of an asymptotically regular mapping is given as follows.

Definition ([1, 4])**.** In a metric space (X, d), a mapping $T : X \to X$ is said to be asymptotically regular at $x \in X$, if $\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0$. If T is asymptotically regular at all $x \in X$, then T is said to be asymptotically regular.

To show the Picard iterating sequence to be Cauchy for a mapping T, sometimes T has to be assumed as asymptotically regular at some point in the underlying space. Though in a metric space (X, d) for any mapping T, $d(T^n\xi, T^{n+1}\xi) \to 0$ as $n \to \infty$ for some ξ belongs to X may not imply the

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sequence $\{T^n\xi\}$ to be a Cauchy sequence. The concept of asymptotic regularity has been used to find fixed points and common fixed points of mappings in several research papers of fixed point theory.

In the year 1968, Kannan [11] proved the following common fixed point theorem for a pair of contractive mappings:

Theorem 1.1 ([11]). *Let $T, S : X \to X$, (X, d) a complete metric space, be mappings such that*

$$d(Tx, Sy) \leq k\{d(x, Tx) + d(y, Sy)\} \text{ for all } x, y \in X, \text{ where } k \in \left[0, \frac{1}{2}\right). \tag{1}$$

Then T and S have a unique common fixed point.

Putting $S = T$, in the above theorem, we get the famous Kannan fixed point theorem. Any Kannan contractive mapping in a metric space (X, d) is an asymptotically regular mapping having exactly one fixed point.

Wong [24] proved the following common fixed point theorem:

Theorem 1.2 ([24]). *Let $T, S : X \to X$, (X, d) a complete metric space, be mappings such that*

$$\begin{aligned} d(Tx, Sy) \leq\ & \alpha d(x, Tx) + \beta d(y, Sy) + \gamma d(x, Sy) \\ & + \delta d(y, Tx) + \rho d(x, y) \text{ for all } x, y \in X, \end{aligned} \tag{2}$$

where $\alpha, \beta, \gamma, \delta$ and ρ are non-negative real numbers such that

(i) $\alpha + \beta + \gamma + \delta + \rho < 1$, (ii) $\alpha = \beta$ *or* $\gamma = \delta$.

Then T and S have a unique common fixed point.

In 2018, Jo [8] obtained the following result for a pair of mappings:

Theorem 1.3 ([8]). *Let $T, S : X \to X$, (X, d) a complete metric space, be mappings satisfying (2) of Theorem 1.2, where $\alpha, \beta, \gamma, \delta$ and ρ are nonnegative real numbers such that*

(i) $\alpha + \beta + 2\gamma + \rho < 1$, (ii) $\alpha + \beta + 2\delta + \rho < 1$.

Then T and S have a unique common fixed point.

Putting $S = T$, in Theorem 1.2, we get a fixed point theorem of Hardy and Rogers [7]. For $S = T$ and $\gamma = \delta = 0$, Theorem 1.2 reduces to a fixed point theorem due to Reich [17]. If we consider $\gamma = \delta = \rho = 0$ in Theorem 1.2, then we get a common fixed point theorem of Srivastava and Gupta [23]. In fact, Srivastava and Gupta proved the result in a more general form with T and S, replaced by T^p and S^q, respectively, for some positive integers p and q.

Recently, Górnicki [6] studied a new class of contractive mappings (by taking the constant $k \in [0, \infty)$, in Kannan's fixed point theorem) and proved a fixed point theorem for such mappings over metric spaces with the assumption of continuity, which is as follows.

Theorem 1.4 ([6]). *In a complete metric space (X, d), a continuous and asymptotically regular map $T : X \to X$ satisfying*

$$d(Tx, Ty) \leq \alpha d(x, y) + K\{d(x, Tx) + d(y, Ty)\} \text{ for all } x, y \in X \tag{3}$$

for some $\alpha \in [0, 1)$ and for some $K \geq 0$ has a unique fixed point $u \in X$ and for each $x \in X$, $T^n x \to u$ as $n \to \infty$.

In [2] Bisht has shown that Theorem 1.4 pertains to both continuous and discontinuous mappings. It is important to note that condition (3) was first appeared in [16].

Definition. In a metric space (X, d), let $T, S : X \to X$ be two mappings. Then,

- (i) T is said to be asymptotic regular with respect to S at a point $x_0 \in X$ [18] if there exists a sequence $\{x_n\}_{n=0,1,...}$ in X such that $Tx_n = Sx_{n+1}$ for all $n = 0, 1, \ldots$ and $d(Sx_{n+1}, Sx_{n+2}) \to 0$ as $n \to \infty$.
- (ii) Let $\{x_n\}$ be a sequence in X such that $Tx_n = Sx_{n+1}$ for all $n = 0, 1, \ldots$ and $Tx_n \to z$ as $n \to \infty$ for some $z \in X$. Then T (resp. S) is said to be (T, S)-orbitally continuous [14, 22] if $TTx_n \to Tz$ as $n \to \infty$ (resp. $STx_n \to Sz$ as $n \to \infty$).

Definition ([10]). In a metric space (X, d), two maps $T, S : X \to X$ are said to be compatible if $\lim_{n\to\infty} d(TSx_n, STx_n) = 0$, whenever $\{x_n\}$ is a sequence in X with $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$.

Bisht and Singh [3] have studied Jungck type common fixed point theorem for a pair of self mappings over a metric space satisfying condition (4).

Theorem 1.5 ([3]). *Let (X, d) be a complete metric space and $T, S : X \to X$ be two mappings. Suppose that T is asymptotic regular with respect to S and satisfy the following condition*

$$d(Tx, Ty) \leq \alpha d(Sx, Sy) + K\{d(Tx, Sx) + d(Ty, Sy)\} \tag{4}$$

for all $x, y \in X$, for some $\alpha \in [0, 1)$ and for some $K \geq 0$. Then T and S have a unique common fixed point in X, provided T and S are (T, S)-orbitally continuous and compatible.

More recently, Khan and Oyetunbi [12] have proved another type of common fixed point theorem for a pair of mappings, which is as follows.

Theorem 1.6. *Let (X, d) be a complete metric space and $T, S : X \to X$ be two asymptotically regular mappings satisfying the following condition*

$$d(Tx, Sy) \leq \lambda d(x, y) + K\{d(x, Tx) + d(y, Sy)\} \tag{5}$$

for all $x, y \in X$, for some $\lambda \in [0, 1)$ and $K \geq 0$. Then T and S have a unique common fixed point in X, provided T and S are either k-continuous for some $k \geq 1$ or orbitally continuous.

Panja et al. [13] have generalized the contractive condition (3) and introduced a new type of contractive mapping called Ćirić–Proinov-Górnicki type mapping.

Let us consider the class $\mathscr{F}$ of all functions $F : [0,\infty) \times [0,\infty) \to [0,\infty)$ satisfying the following conditions:
(i) $F(0,0) = 0$,
(ii) F is continuous at $(0,0)$.

Definition ([13], Ćirić-Proinov-Górnicki type mapping)**.** In a metric space (X,d), a mapping $T : X \to X$ is said to be Ćirić-Proinov-Górnicki type mapping if there exists $\alpha \in [0,1)$ such that

$$d(Tx,Ty) \leq \alpha \max\{d(x,y), d(x,Ty), d(y,Tx)\} + F(d(x,Tx), d(y,Ty)) \tag{6}$$

for all $x, y \in X$ and for some $F \in \mathscr{F}$.

Theorem 1.7 ([13])**.** *Let (X,d) be a complete metric space and $T : X \to X$ be an asymptotically regular Ćirić-Proinov-Górnicki type mapping. Then T has a unique fixed point provided either T is k-continuous for $k \geq 1$ or T is orbitally continuous.*

We now define three new classes of contractive mappings for a pair of mappings which extend definitions of Panja et al. [13].

Definition. In a metric space (X,d), mappings $T, S : X \to X$ are said to be

(i) Ćirić-Proinov-Górnicki (CPG) type mapping if there exists $\lambda \in [0,1)$ such that

$$d(Tx,Sy) \leq \lambda \max\{d(x,y), d(x,Sy), d(y,Tx)\} + F(d(x,Tx), d(y,Sy)). \tag{7}$$

(ii) Hardy-Rogers-Proinov-Górnicki (HRPG) type mapping if there exist $\alpha, \beta, \gamma \in [0,1)$ with $\alpha + \beta + \gamma < 1$ such that

$$d(Tx,Sy) \leq \alpha d(x,y) + \beta d(x,Sy) + \gamma d(y,Tx) + F(d(x,Tx), d(y,Sy)). \tag{8}$$

(iii) Reich-Proinov-Górnicki (RPG) type mapping if there exist $\alpha \in [0,1)$ such that

$$d(Tx,Sy) \leq \alpha d(x,y) + F(d(x,Tx), d(y,Sy)) \tag{9}$$

for all $x, y \in X$ and for some $F \in \mathscr{F}$.

2. Main results

We start with the following common fixed point theorem for a pair of mappings satisfying CPG type contractive condition.

Theorem 2.1. *Let (X,d) be a complete metric space and $T, S : X \to X$ be two asymptotically regular mappings satisfying (7) for all $x, y \in X$, for some $\lambda \in [0,1)$ and for some $F \in \mathscr{F}$. Then T and S have a unique common fixed point provided T and S are either k-continuous for some $k \geq 1$ or orbitally continuous.*

Proof. First we claim that $d(T^n x, S^n x) \to 0$ as $n \to \infty$ for all $x \in X$.

Now if $S = T$, then there is nothing to prove. So suppose that $S \neq T$. Then using (7),

$$\begin{aligned} d(T^n x, S^n x) &= d(T(T^{n-1}x), S(S^{n-1}x)) \\ &= \lambda \max\{d(T^{n-1}x, S^{n-1}x), d(T^{n-1}x, S^n x), d(S^{n-1}x, T^n x)\} \\ &\quad + F(d(T^{n-1}x, T^n x), d(S^{n-1}x, S^n x)) \\ &= \lambda \Gamma_n + F(t_n, s_n), \qquad (10) \end{aligned}$$

where $\Gamma_n = \max\{d(T^{n-1}x, S^{n-1}x), d(T^{n-1}x, S^n x), d(S^{n-1}x, T^n x)\}$, $t_n = d(T^{n-1}x, T^n x)$ and $s_n = d(S^{n-1}x, S^n x)$ for all $n \in \mathbb{N}$. Since S and T are asymptotic regular so $t_n \to 0$ and $s_n \to 0$ as $n \to \infty$.

Now if $\Gamma_n = d(T^{n-1}x, S^{n-1}x)$, then using triangle inequality,

$$\Gamma_n \leq d(T^{n-1}x, T^n x) + d(T^n x, S^n x) + d(S^n x, S^{n-1}x) \text{ and hence from (10),}$$

we have

$$d(T^n x, S^n x) \leq \frac{\lambda}{1-\lambda}\left\{d(T^{n-1}x, T^n x) + d(S^{n-1}x, S^n x)\right\} + \frac{1}{1-\lambda} F(t_n, s_n). \qquad (11)$$

Next if $\Gamma_n = d(T^{n-1}x, S^n x)$ or $\Gamma_n = d(S^{n-1}x, T^n x)$, then in a similar way we can get the followings, respectively,

$$d(T^n x, S^n x) \leq \frac{\lambda}{1-\lambda} d(T^{n-1}x, T^n x) + \frac{1}{1-\lambda} F(t_n, s_n), \qquad (12)$$

$$d(T^n x, S^n x) \leq \frac{\lambda}{1-\lambda} d(S^{n-1}x, S^n x) + \frac{1}{1-\lambda} F(t_n, s_n). \qquad (13)$$

Combining (11), (12) and (13) we can write,

$$d(T^n x, S^n x) \leq \frac{\lambda}{1-\lambda}\left\{d(T^{n-1}x, T^n x) + d(S^{n-1}x, S^n x)\right\} + \frac{1}{1-\lambda} F(t_n, s_n).$$

Now by using asymptotic regularity of S and T and the properties of the function F we get, $\alpha_n = d(T^n x, S^n x) \to 0$ as $n \to \infty$ for all $x \in X$.

Next let $x_0 \in X$ be arbitrary and consider the sequence $x_n = T^n x_0$ for all $n = 0, 1, 2, \ldots$. Then for $n, m \in \mathbb{N}$ and $m > n$ we have,

$$\begin{aligned} & d(x_n, x_m) \\ &= d(T^n x_0, T^m x_0) \\ &\leq d(T^n x_0, S^n x_0) + d(S^n x_0, T^m x_0) \\ &= d(T^n x_0, S^n x_0) + d(T^m x_0, S^n x_0) \\ &\leq d(T^n x_0, S^n x_0) + F\left(d(T^{m-1}x_0, T^m x_0), d(S^{n-1}x_0, S^n x_0)\right) \\ &\quad + \lambda \max\left\{d(T^{m-1}x_0, S^{n-1}x_0), d(T^{m-1}x_0, S^n x_0), d(T^m x_0, S^{n-1}x_0)\right\} \\ &= \alpha_n + F(\beta_m, \gamma_n) + \lambda K_{m,n}, \qquad (14) \end{aligned}$$

where $K_{m,n} = \max\left\{d(T^{m-1}x_0, S^{n-1}x_0), d(T^{m-1}x_0, S^n x_0), d(T^m x_0, S^{n-1}x_0)\right\}$ and $\beta_n = d(T^{n-1}x_0, T^n x_0)$, $\gamma_n = d(S^{n-1}x_0, S^n x_0)$. Since S and T are asymptotic regular it follows that $\beta_n, \gamma_n \to 0$ as $n \to \infty$.

Case 1: If $K_{m,n} = d(T^{m-1}x_0, S^{n-1}x_0)$, then (14) gives

$$\begin{aligned} & d(x_n, x_m) \\ \leq\ & \alpha_n + F(\beta_m, \gamma_n) + \lambda d(T^{m-1}x_0, S^{n-1}x_0) \\ \leq\ & \alpha_n + F(\beta_m, \gamma_n) \\ & + \lambda\left[d(S^{n-1}x_0, T^{n-1}x_0) + d(T^{n-1}x_0, T^n x_0) + d(T^n x_0, T^m x_0) + d(T^m x_0, T^{m-1}x_0)\right], \end{aligned}$$

which implies

$$d(x_n, x_m) \leq \frac{1}{1-\lambda}\left\{\alpha_n + F(\beta_m, \gamma_n)\right\} + \frac{\lambda}{1-\lambda}\left[\alpha_{n-1} + \beta_n + \beta_m\right]. \tag{15}$$

Case 2: If $K_{m,n} = d(T^{m-1}x_0, S^n x_0)$, then (14) gives

$$\begin{aligned} & d(x_n, x_m) \\ \leq\ & \alpha_n + F(\beta_m, \gamma_n) + \lambda d(T^{m-1}x_0, S^n x_0) \\ \leq\ & \alpha_n + F(\beta_m, \gamma_n) + \lambda\left[d(T^{m-1}x_0, T^m x_0) + d(T^m x_0, T^n x_0) + d(T^n x_0, S^n x_0)\right], \end{aligned}$$

implying that

$$d(x_n, x_m) \leq \frac{1}{1-\lambda}\left\{\alpha_n + F(\beta_m, \gamma_n)\right\} + \frac{\lambda}{1-\lambda}\left[\beta_m + \alpha_n\right]. \tag{16}$$

Case 3: If $K_{m,n} = d(T^m x_0, S^{n-1}x_0)$, then from (14) we get

$$\begin{aligned} & d(x_n, x_m) \\ \leq\ & \alpha_n + F(\beta_m, \gamma_n) + \lambda d(T^m x_0, S^{n-1}x_0) \\ \leq\ & \alpha_n + F(\beta_m, \gamma_n) + \lambda\left[d(T^m x_0, T^n x_0) + d(T^n x_0, S^n x_0) + d(S^n x_0, S^{n-1}x_0)\right], \end{aligned}$$

which yields

$$d(x_n, x_m) \leq \frac{1}{1-\lambda}\left\{\alpha_n + F(\beta_m, \gamma_n)\right\} + \frac{\lambda}{1-\lambda}\left[\alpha_n + \gamma_n\right]. \tag{17}$$

Now combining (15), (16) and (17) we see that

$$d(x_n, x_m) \leq \frac{1}{1-\lambda}\left\{\alpha_n + F(\beta_m, \gamma_n)\right\} + \alpha_{n-1} + \beta_n + \beta_m + \alpha_n + \gamma_n \to 0$$

as $n, m \to \infty$, since F is continuous at $(0,0)$ and $F(0,0) = 0$.

Hence $\{x_n\}$ is a Cauchy sequence in X and by completeness of X, let $x_n \to u \in X$ as $n \to \infty$, i.e., $T^n x_0 \to u$ as $n \to \infty$.

Again since $d(S^n x_0, u) \leq d(S^n x_0, T^n x_0) + d(T^n x_0, u)$, therefore $S^n x_0 \to u$ as $n \to \infty$.

Suppose S and T are k-continuous: Since $\lim_{n\to\infty} x_{n+1} = u$, so $\lim_{n\to\infty} Tx_n = u$. Moreover, for each $k \geq 1$ we have $\lim_{n\to\infty} T^k x_n = u$. Since $\lim_{n\to\infty} T^{k-1}x_n = u$ due to k-continuity of T, we get $\lim_{n\to\infty} T^k x_n = Tu$. Thus $u = Tu$, i.e., $u \in X$ is a fixed point of T. In a similar way using k-continuity of S, we can get $u = Su$, i.e., $u \in X$ is a fixed point of S. Hence $u \in X$ is a common fixed point of S and T.

Next suppose S and T are orbitally continuous: We have $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = u$. Again from orbital continuity of T, $\lim_{n\to\infty} x_n = u$ implies $\lim_{n\to\infty} Tx_n = Tu$. Hence $u = Tu$, i.e., $u \in X$ is a fixed point of T. Similarly using orbital continuity of S, we get $Su = u$. That is, $u \in X$ is a common fixed point of S and T.

To show uniqueness of the common fixed point let us suppose that $v(\neq u) \in X$ be another common fixed point of S and T, i.e., $Tv = v = Sv$. Then, (7) turns into

$$\begin{aligned} d(u,v) &= d(Tu,Tv) \\ &\leq \lambda \max\{d(u,v), d(u,Sv), d(v,Tu)\} + F\left(d(u,Tu), d(v,Sv)\right) \\ &= \lambda d(u,v) + F(0,0). \end{aligned}$$

Since $F(0,0) = 0$, we have $(1-\lambda)d(u,v) \leq 0$ which yields that $d(u,v) = 0$, a contradiction. Hence the common fixed point of S and T is unique. □

The following example illustrates Theorem 2.1.

Example 2.2. Let $X = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ be the metric space endowed with the usual metric. Let $T, S : X \to X$ be defined by $T(0) = 0$, $T(\frac{1}{n}) = \frac{1}{n+1}$ and $S(0) = 0$, $S(\frac{1}{n}) = \frac{1}{n+2}$ for all $n \geq 1$. If we choose $\lambda = \frac{1}{2}$ and $F(x,y) = 3[\sqrt{x} + \sqrt{y}]$ for all $x, y \in [0,\infty)$, then we have the following three cases:

Case-I: For $x = 0$ and $y = \frac{1}{n}$, $n \geq 1$, we see that

$$\begin{aligned} d(Tx,Sy) = \frac{1}{n+2} &\leq \frac{1}{2n} + 3\left(\sqrt{\frac{2}{n(n+2)}}\right) \\ &= \lambda \max\{d(x,y), d(x,Sy), d(y,Tx)\} + F(d(x,Tx), d(y,Sy)). \end{aligned}$$

Case-II: For $x = \frac{1}{n}$, $n \geq 1$ and $y = 0$, we get

$$\begin{aligned} d(Tx,Sy) = \frac{1}{n+1} &\leq \frac{1}{2n} + 3\left(\frac{1}{\sqrt{n(n+1)}}\right) \\ &= \lambda \max\{d(x,y), d(x,Sy), d(y,Tx)\} + F(d(x,Tx), d(y,Sy)). \end{aligned}$$

Case-III: For $x = \frac{1}{n}$ and $y = \frac{1}{m}$, $n, m \geq 1$, it follows that

$$\begin{aligned} d(Tx,Sy) &= |\frac{1}{n+1} - \frac{1}{m+2}| \\ &\leq \frac{1}{2}\max\{|\frac{1}{n} - \frac{1}{m}|, |\frac{1}{n} - \frac{1}{m+2}|, |\frac{1}{n+1} - \frac{1}{m}|\} \\ &\quad + 3\left(\frac{1}{\sqrt{n(n+1)}} + \sqrt{\frac{2}{m(m+2)}}\right) \\ &= \lambda \max\{d(x,y), d(x,Sy), d(y,Tx)\} + F(d(x,Tx), d(y,Sy)). \end{aligned}$$

Hence T and S satisfy the contractive condition (7). If T and S satisfy the contractive condition (5), then for $x = 0$ and $y = \frac{1}{n}$, $n \geq 1$ we have

$$d(Tx, Sy) = \frac{1}{n+2} \leq \lambda d(x, y) + K\{d(x, Tx) + d(y, Sy)\} = \frac{\lambda}{n} + \frac{2K}{n(n+2)}$$

for some $\lambda \in [0, 1)$ and $K \in [0, +\infty)$. From which it follows that $1 \leq \lambda \frac{n+2}{n} + \frac{2K}{n}$, taking $n \to \infty$ we see that $\lambda \geq 1$, a contradiction. Therefore, T and S do not satisfy the contractive condition (5). Here both T and S are asymptotically regular, orbitally continuous and 0 is the unique common fixed point of T and S.

Corollary 2.3 ([12])**.** *Let (X, d) be a complete metric space and $T, S : X \to X$ be two asymptotically regular maps satisfying* (5)*. Then T and S have a unique common fixed point provided T and S are either k-continuous for some $k \geq 1$ or orbitally continuous.*

Proof. We have

$$\begin{aligned} & d(Tx, Sy) \\ \leq\ & \lambda d(x, y) + K\{d(x, Tx) + d(y, Sy)\} \\ =\ & \lambda \max\{d(x, y), d(x, Sy), d(y, Tx)\} + F(d(x, Tx), d(y, Sy)) \text{ for all } x, y \in X. \end{aligned}$$

Where $F(x, y) = K(x + y)$ is continuous at $(0, 0) \in [0, \infty)^2$ and $F(0, 0) = 0$, i.e., $F \in \mathscr{F}$.

Hence from Theorem 2.1, S and T have a unique common fixed point in X. ☐

The following results are easy consequences of Theorem 2.1.

Corollary 2.4. *Let (X, d) be a complete metric space and $T, S : X \to X$ be two asymptotically regular mappings satisfying* (8) *for all $x, y \in X$. Then T and S have a unique common fixed point provided T and S are either k-continuous for some $k \geq 1$ or orbitally continuous.*

Corollary 2.5. *Let (X, d) be a complete metric space and $T, S : X \to X$ be two asymptotically regular mappings satisfying* (9) *for all $x, y \in X$. Then T and S have a unique common fixed point provided T and S are either k-continuous for some $k \geq 1$ or orbitally continuous.*

In the following we give a Jungck type common fixed point theorem for a pair of mappings.

Theorem 2.6. *Let (X, d) be a complete metric space and $T, S : X \to X$ be two mappings such that T is asymptotic regular with respect to S and satisfy the following condition*

$$\begin{aligned} d(Tx, Ty) \leq\ & \lambda \max\{d(Sx, Sy), d(Tx, Sy), d(Sx, Ty)\} \\ & + F(d(Tx, Sx), d(Ty, Sy)) \end{aligned} \tag{18}$$

for all $x, y \in X$, for some $\lambda \in [0,1)$ and for some $F \in \mathscr{F}$. Then T and S have a unique common fixed point provided T and S are (T,S)-orbitally continuous and compatible.

Proof. Uniqueness of fixed point clearly follows form the equation (18). Now we prove the existence of common fixed point of T and S.

Since T is asymptotic regular with respect to S at $x_0 \in X$, so there exists a sequence $\{x_n\}$ in X such that $Tx_n = Sx_{n+1} = y_n$ (say) for all $n = 1, 2, \ldots$ and $d(Sx_{n+1}, Sx_{n+2}) \to 0$ as $n \to \infty$, i.e., $d(y_n, y_{n+1}) \to 0$ as $n \to \infty$.

First we will show that $\{y_n\}$ is a Cauchy sequence in X. For $p = 1, 2, 3, \ldots$ we have,

$$\begin{aligned} &d(y_{n+p}, y_n) \\ &\leq d(y_{n+p}, y_{n+p+1}) + d(y_{n+p+1}, y_{n+1}) + d(y_{n+1}, y_n) \\ &\leq d(y_{n+p}, y_{n+p+1}) + \lambda M_{n,p} + F\left(d(y_{n+p+1}, y_{n+p}), d(y_{n+1}, y_n)\right) \\ &\quad + d(y_{n+1}, y_n), \end{aligned} \tag{19}$$

by using (18) where $M_{n,p} = \max\{d(y_{n+p}, y_n), d(y_{n+p+1}, y_n), d(y_{n+p}, y_{n+1})\}$.

Now if $M_{n,p} = d(y_{n+p}, y_n)$, then from (19) we get,

$$\begin{aligned} &(1-\lambda)d(y_{n+p}, y_n) \\ &\leq d(y_{n+p}, y_{n+p+1}) + F\left(d(y_{n+p+1}, y_{n+p}), d(y_{n+1}, y_n)\right) + d(y_{n+1}, y_n). \end{aligned} \tag{20}$$

Again if $M_{n,p} = d(y_{n+p+1}, y_n)$, then by triangle inequality,

$$M_{n,p} \leq d(y_{n+p+1}, y_{n+p}) + d(y_{n+p}, y_n) \text{ and then from (19)}$$

we get,

$$\begin{aligned} &(1-\lambda)d(y_{n+p}, y_n) \\ &\leq (1+\lambda)d(y_{n+p}, y_{n+p+1}) + F\left(d(y_{n+p+1}, y_{n+p}), d(y_{n+1}, y_n)\right) \\ &\quad + d(y_{n+1}, y_n). \end{aligned} \tag{21}$$

Finally if $M_{n,p} = d(y_{n+p}, y_{n+1})$, then by triangle inequality,

$$M_{n,p} \leq d(y_{n+p}, y_n) + d(y_n, y_{n+1}) \text{ and then from (19),}$$

we get

$$\begin{aligned} &(1-\lambda)d(y_{n+p}, y_n) \\ &\leq d(y_{n+p}, y_{n+p+1}) + F\left(d(y_{n+p+1}, y_{n+p}), d(y_{n+1}, y_n)\right) \\ &\quad + (1+\lambda)d(y_{n+1}, y_n). \end{aligned} \tag{22}$$

Now combining (20), (21) and (22) we can write,

$$\begin{aligned} &(1-\lambda)d(y_{n+p}, y_n) \\ &\leq (1+\lambda)\{d(y_{n+p}, y_{n+p+1}) + d(y_{n+1}, y_n)\} \\ &\quad + F(d(y_{n+p+1}, y_{n+p}), d(y_{n+1}, y_n)). \end{aligned} \tag{23}$$

Therefore using the properties of F from (23) it follows that, $d(y_{n+p}, y_n) \to 0$ whenever $n \to \infty$ for any $p = 1, 2, \ldots$, i.e., $\{y_n\}$ is a Cauchy sequence in X and due to completeness of X, $\lim_n y_n = \lim_n Tx_n = \lim_n Sx_{n+1} = u \in X$ (say).

Since T and S are (T,S)-orbitally continuous so,

$$\lim_n TTx_n = \lim_n TSx_n = Tu \quad \text{and} \quad \lim_n STx_n = \lim_n SSx_n = Su.$$

Since T and S are compatible, so $d(TSx_n, STx_n) \to 0$ as $n \to \infty$, which implies $Tu = Su$. Again by compatibility of T and S we have, $T(Tu) = T(Su) = S(Tu) = S(Su)$.
Now using (18) we have,

$$\begin{aligned} & d(Tu, TTu) \\ \leq\ & \lambda \max\{d(Su, STu), d(Tu, STu), d(Su, TTu)\} + F(d(Tu, Su), d(TTu, STu)) \\ =\ & \lambda d(Tu, TTu) + F(0, 0). \end{aligned}$$

Since $F(0,0) = 0$ and $\lambda \in [0,1)$ we have, $d(Tu, TTu) = 0$. Therefore, $Tu = T(Tu) = S(Tu)$, i.e., $Tu \in X$ is a common fixed point of T and S. □

We now give an example which illustrates Theorem 2.6.

Example 2.7. Let us take $X = [0, \infty)$ equipped with the usual metric of $\mathbb{R}$. Let $T, S : X \to X$ be defined by $Tx = \frac{x}{x+1}$ and $Sx = \frac{2x}{x+2}$ for all $x \geq 0$. Then T and S satisfy the contractive condition (18) for $\lambda = \frac{1}{2}$ and $F(x,y) = \sqrt{x} + \sqrt{y}$ for all $x, y \in [0, \infty)$. But it can be easily verified that T, S do not satisfy the contractive condition (4). Here T is asymptotically regular with respect to S, T and S both are (T,S)-orbitally continuous and compatible and 0 is the unique common fixed point of T and S.

From Theorem 2.6 we get the immediate corollary.

Corollary 2.8 ([3])**.** *Let (X,d) be a complete metric space and $T, S : X \to X$ be two mappings such that T is asymptotic regular with respect to S satisfying (4). Then T and S have a unique common fixed point provided T and S are (T,S)-orbitally continuous and compatible.*

From Theorems 2.1 and 2.6 we get our next corollary.

Corollary 2.9 ([13])**.** *Let (X,d) be a complete metric space and $T : X \to X$ be an asymptotic regular map satisfying*

$$d(Tx, Ty) \leq \lambda \max\{d(x,y), d(y,Tx), d(x,Ty)\} + F(d(x,Tx), d(y,Ty))$$

for all $x, y \in X$, for some $\lambda \in [0,1)$ and for some $F \in \mathscr{F}$. Then T has a unique fixed point provided T is orbitally continuous in X.

Proof. In Theorem 2.1 if we consider $S = T$, then we get our required result.

On the other hand in Theorem 2.6, by taking $S \equiv I$, where I is the identity map on X the result follows immediately. □

The following common fixed point theorem, an extension of Jungck's result [9] was given in [8].

Theorem 2.10 ([8])**.** *Let S be a continuous self-mapping on a complete metric space (X, d). Then S has a fixed point if and only if there exist constants $a, b, c \in [0, 1)$ with $a + b + c < 1$ and a continuous self-mapping T on X satisfying the following conditions*:

(a) $TX \subseteq SX$;

(b) $T \circ S = S \circ T$;

(c) $d(Tx, Ty) \leq ad(Tx, Sx) + bd(Ty, Sy) + cd(Sx, Sy)$ *for all* $x, y \in X$.

Indeed, S and T have a unique common fixed point if the conditions defined above hold.

Theorem 2.10 is a particular case of Theorem 2.6 as condition (c) of Theorem 2.10 implies (18). Continuities of S and T imply orbital continuity of S and T. Also, compatibility of S and T implies commutativity of S and T [condition (b)].

In the following theorem we obtain a sequential fixed point for a mapping which is the point-wise limit of a sequence of functions satisfying a particular contractive condition.

Let us consider the class $\mathscr{F}'$ of all functions $F : [0, \infty) \times [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(i) $F(0, 0) = 0$;

(ii) F is everywhere continuous in $[0, \infty) \times [0, \infty)$.

Theorem 2.11. *Let (X, d) be a complete metric space and $\{T_n\}_{n\in\mathbb{N}}$, T are self mappings on X. Suppose that $\{T_n\}_{n\in\mathbb{N}}$ and T satisfy the following conditions*:

(i) *there exists $F \in \mathscr{F}'$ and for each $n \in \mathbb{N}$ there exists $0 \leq \lambda_n < 1$ such that*

$$d(T_n x, T_n y) \leq \lambda_n \max\{d(x, y), d(x, T_n y), d(y, T_n x)\} + F(d(x, T_n x), d(y, T_n y)) \tag{24}$$

for all $x, y \in X$, where $\sup_n \lambda_n = \lambda < 1$, i.e., each T_n, $n \in \mathbb{N}$, is Ćirić-Proinov-Górnicki type mapping;

(ii) $\{T_n\}_{n\geq 1}$ *converges point-wise to the mapping* T;

(iii) *mappings T and for each $n \in \mathbb{N}$, T_n are either k-continuous for $k \geq 1$ or orbitally continuous and asymptotically regular.*

Then T is also a Ćirić-Proinov-Górnicki type mapping for $\lambda < 1$ and $F \in \mathscr{F}'$. Also the sequence of unique fixed points $\{u_n\}_{n\geq 1}$ of $\{T_n\}_{n\geq 1}$ converges to the unique fixed point of T.

Proof. Let $x, y \in X$ be chosen as arbitrary. Then for any $n \in \mathbb{N}$

$$\begin{aligned} d(Tx, Ty) &\leq d(Tx, T_n x) + d(T_n x, T_n y) + d(T_n y, Ty) \\ &\leq d(Tx, T_n x) + d(T_n y, Ty) + \lambda_n \max\{d(x, y), d(x, T_n y), d(y, T_n x)\} \\ &\quad + F(d(x, T_n x), d(y, T_n y)) \end{aligned}$$

$$\begin{aligned}
&\le d(Tx, T_nx) + d(T_ny, Ty) + F(d(x, T_nx), d(y, T_ny)) \\
&\quad + \lambda \max\{d(x, y), d(x, Ty) + d(Ty, T_ny), d(y, Tx) + d(Tx, T_nx)\}. \qquad (25)
\end{aligned}$$

By taking $n \to \infty$, due to continuity of d, from (25) we get

$$d(Tx, Ty) \le \lambda \max\{d(x, y), d(x, Ty), d(y, Tx)\} + F(d(x, Tx), d(y, Ty)). \qquad (26)$$

Since x, y are arbitrary, T is a Ćirić-Proinov-Górnicki type mapping for $\lambda < 1$ and $F \in \mathscr{F}'$. Since condition (iii) holds it follows that T has a unique fixed point u in X. Now,

$$\begin{aligned}
&d(T_nu_n, T_nu) \\
&\le \lambda_n \max\{d(u_n, u), d(u, T_nu_n), d(u_n, T_nu)\} + F(d(u_n, T_nu_n), d(u, T_nu)) \\
&= \lambda_n \max\{d(u_n, u), d(u_n, T_nu)\} + F(0, d(u, T_nu)) \\
&\le \lambda \max\{d(u_n, u), d(u_n, T_nu)\} + F(0, d(u, T_nu)) \\
&\le \lambda[d(u_n, u) + d(T_nu_n, T_nu)] + F(0, d(u, T_nu)) \text{ for all } n \in \mathbb{N}. \qquad (27)
\end{aligned}$$

Thus $d(T_nu_n, T_nu) \le \frac{\lambda}{1-\lambda} d(u_n, u) + \frac{1}{1-\lambda} F(0, d(u, T_nu))$ for all $n \in \mathbb{N}$. Now due to continuity of d it follows that $d(T_nu_n, T_nu) \to 0$ as $n \to \infty$. Therefore we have

$$\begin{aligned}
d(u_n, u) &= d(T_nu_n, Tu) \\
&\le d(T_nu_n, T_nu) + d(T_nu, Tu) \to 0 \text{ as } n \to \infty. \qquad (28)
\end{aligned}$$

Hence the sequence of unique fixed points $\{u_n\}_{n\ge1}$ of $\{T_n\}_{n\ge1}$ converges to the unique fixed point u of T. □

The following example illustrates Theorem 2.11.

Example 2.12. Let us consider the metric space $X = [0, \infty)$ equipped with the usual metric. Let $T_n : X \to X$ be defined by $T_n(x) = \frac{nx+x}{nx+n+1}$ for all $x \in X$ and for all $n \in \mathbb{N}$. Then $\{T_n\}_{n\ge1}$ converges point-wise to the function $T(x) = \frac{x}{x+1}$ for all $x \in X$. For each $n \ge 1$, T_n is a Ćirić-Proinov-Gónicki type mapping with $\lambda_n = \frac{1}{2}$ and $F(x, y) = \sqrt{x} + \sqrt{y}$ for all $x, y \in X$ without being Górnicki type mapping (the contractive condition (3)). The mappings T and T_n, $n \in \mathbb{N}$, are continuous and asymptotically regular. Clearly T is a Ćirić-Proinov-Górnicki type mapping with $\lambda = \frac{1}{2}$ and $F(x, y) = \sqrt{x} + \sqrt{y}$ for all $x, y \in X$ with the unique fixed point 0 which is the limit of the sequence of unique fixed points $\{0\}$ of $\{T_n\}_{n\ge1}$.

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References

[1] J. B. Baillon, R. E. Bruck, and S. Reich, *On the asymptotic behavior of nonexpansive mappings and semigroups in Banach spaces*, Houston J. Math. **4** (1978), no. 1, 1–9.

[2] R. K. Bisht, *A note on the fixed point theorem of Górnicki*, J. Fixed Point Theory Appl. **21** (2019), no. 2, Paper No. 54, 3 pp. https://doi.org/10.1007/s11784-019-0695-x

[3] R. K. Bisht and N. K. Singh, *On asymptotic regularity and common fixed points*, J. Anal. **28** (2020), no. 3, 847–852. https://doi.org/10.1007/s41478-019-00213-0

[4] F. E. Browder and W. V. Petryshyn, *The solution by iteration of nonlinear functional equations in Banach spaces*, Bull. Amer. Math. Soc. **72** (1966), 571–575. https://doi.org/10.1090/S0002-9904-1966-11544-6

[5] D. Dey, R. Fierro, and M. Saha, *Well-posedness of fixed point problems*, J. Fixed Point Theory Appl. **20** (2018), no. 2, Paper No. 57, 12 pp. https://doi.org/10.1007/s11784-018-0538-1

[6] J. Górnicki, *Remarks on asymptotic regularity and fixed points*, J. Fixed Point Theory Appl. **21** (2019), no. 1, Paper No. 29, 20 pp. https://doi.org/10.1007/s11784-019-0668-0

[7] G. E. Hardy and T. D. Rogers, *A generalization of a fixed point theorem of Reich*, Canad. Math. Bull. **16** (1973), 201–206. https://doi.org/10.4153/CMB-1973-036-0

[8] J. H. Jo, *Some generalizations of fixed point theorems and common fixed point theorems*, J. Fixed Point Theory Appl. **20** (2018), no. 4, Paper No. 144, 17 pp. https://doi.org/10.1007/s11784-018-0631-5

[9] G. Jungck, *Commuting mappings and fixed points*, Amer. Math. Monthly **83** (1976), no. 4, 261–263. https://doi.org/10.2307/2318216

[10] G. Jungck, *Compatible mappings and common fixed points*, Internat. J. Math. Math. Sci. **9** (1986), no. 4, 771–779. https://doi.org/10.1155/S0161171286000935

[11] R. Kannan, *Some results on fixed points*, Bull. Calcutta Math. Soc. **60** (1968), 71–76.

[12] A. R. Khan and D. M. Oyetunbi, *On some mappings with a unique common fixed point*, J. Fixed Point Theory Appl. **22** (2020), no. 2, Paper No. 47, 7 pp. https://doi.org/10.1007/s11784-020-00781-w

[13] S. Panja, K. Roy, M. Saha, and R. K. Bisht, *Some fixed point theorems via asymptotic regularity*, Filomat **34** (2020), no. 5, 1621–1627. https://doi.org/10.2298/FIL2005621P

[14] A. Pant and R. P. Pant, *Orbital continuity and fixed points*, Filomat **31** (2017), no. 11, 3495–3499. https://doi.org/10.2298/fil1711495p

[15] V. Pata, *Fixed Point Theorems and Applications*, Unitext, 116, Springer, Cham, 2019. https://doi.org/10.1007/978-3-030-19670-7

[16] P. D. Proinov, *Fixed point theorems in metric spaces*, Nonlinear Anal. **64** (2006), no. 3, 546–557. https://doi.org/10.1016/j.na.2005.04.044

[17] S. Reich, *Some remarks concerning contraction mappings*, Canad. Math. Bull. **14** (1971), 121–124. https://doi.org/10.4153/CMB-1971-024-9

[18] B. E. Rhoades, S. Sessa, M. S. Khan, and M. D. Khan, *Some fixed point theorems for Hardy-Rogers type mappings*, Internat. J. Math. Math. Sci. **7** (1984), no. 1, 75–87. https://doi.org/10.1155/S0161171284000077

[19] K. Roy and M. Saha, *Fixed points of mappings over a locally convex topological vector space and Ulam-Hyers stability of fixed point problems*, Novi Sad J. Math. **50** (2020), no. 1, 99–112. https://doi.org/10.30755/NSJOM.09198

[20] K. Roy and M. Saha, *Generalized contractions and fixed point theorems over bipolar cone$_{tvs}$ b-metric spaces with an application to homotopy theory*, Mat. Vesnik **72** (2020), no. 4, 281–296.

[21] M. Saha and A. Ganguly, *Random fixed point theorem on a Ćirić-type contractive mapping and its consequence*, Fixed Point Theory Appl. **2012** (2012), 209, 18 pp. https://doi.org/10.1186/1687-1812-2012-209

[22] K. P. R. Sastry, S. V. R. Naidu, I. H. N. Rao, and K. P. R. Rao, *Common fixed points for asymptotically regular mappings*, Indian J. Pure Appl. Math. **15** (1984), no. 8, 849–854.
[23] P. Srivastava and V. K. Gupta, *A note on common fixed points*, Yokohama Math. J. **19** (1971), no. 2, 91–95.
[24] C. S. Wong, *Common fixed points of two mappings*, Pacific J. Math. **48** (1973), 299–312. http://projecteuclid.org/euclid.pjm/1102945724

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