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INFINITELY MANY HOMOCLINIC SOLUTIONS FOR DIFFERENT CLASSES OF FOURTH-ORDER DIFFERENTIAL EQUATIONS

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Abstract. In this article, we study the existence and multiplicity of homoclinic solutions for the following fourth-order differential equation

$$u^{(4)}(x) + \omega u''(x) + a(x)u(x) = f(x, u(x)), \ \forall x \in \mathbb{R} \tag{1}$$

where $a(x)$ is not required to be either positive or coercive, and $F(x, u) = \int_0^u f(x, v)dv$ is of subquadratic or superquadratic growth as $|u| \to \infty$, or satisfies only local conditions near the origin (i.e., it can be subquadratic, superquadratic or asymptotically quadratic as $|u| \to \infty$). To the best of our knowledge, there is no result published concerning the existence and multiplicity of homoclinic solutions for (1) with our conditions. The proof is based on variational methods and critical point theory.

1. Introduction

Consider the following fourth-order differential equation

$$u^{(4)}(x) + \omega u''(x) + a(x)u(x) = f(x, u(x)), \ \forall x \in \mathbb{R} \tag{$\mathcal{F}$}$$

where ω is a constant, $a \in C(\mathbb{R}, \mathbb{R})$ and $f \in C(\mathbb{R}^2, \mathbb{R})$ are two given functions. It is well-known that the mathematical modeling of important questions in different fields of research, such as mechanical engineering, control systems, economics and many others, leads naturally to the consideration of the nonlinear differential equations. In particular, the fourth-order differential equations, like $(\mathcal{F})$ have been put forward as mathematical model for the study of pattern formation in physics and mechanics. For example, the well-known extended Fisher-Kolmogorov equation proposed by Coulet et al. in 1987 [3], in the study of phase transitions, the fourth-order elastic beam equation in describing a large class of elastic deflection [12], the Swift-Hohenberg equation which is a general model for pattern-forming process derived in [4] to describe vandom thermal fluctuations in the Boussinesque equation and the propagation of lazers [6].

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Here as usual, we say that a solution u of $(\mathcal{F})$ is homoclinic (to 0) if $u \in C^4(\mathbb{R}, \mathbb{R})$ and $u(x) \to 0$ as $|x| \to \infty$. In addition, if $u \neq 0$, then u is called a nontrivial homoclinic solution. In the past years, based on variational methods and critical point theory, many researchers are interested in the existence of homoclinic solutions to equation $(\mathcal{F})$. The main difficulty in handling system $(\mathcal{F})$ arises from the fact that $H^2(\mathbb{R})$ is not compactly embedded in $L^p(\mathbb{R})$ for $p \in [2, \infty]$. To overcome this difficulty, many authors have considered the case where $a(x)$ and $f(x, u)$ are independent of x or periodic in x, see [1, 2, 7, 14, 17] and the references listed therein. In this case, the existence of homoclinic solutions can be obtained by going to the limit of periodic solutions of approximating problems. If $a(x)$ and $f(x, u)$ are neither autonomous in x nor periodic in x, the existence of homoclinic solutions of $(\mathcal{F})$ is quite different from the ones just described because of the lack of compactness of the Sobolev embedding. Notice that for the nonperiodic case, to obtain the existence of homoclinic solutions, the following coercive condition on a is often needed:

There exists a constant $a_0 > 0$ such that

$$a_0 \leq a(x) \to +\infty \quad \text{as} \quad |x| \to \infty \tag{1.1}$$

and

$$\omega \leq 2\sqrt{a_0},$$

which is used to establish the corresponding compact embedding lemmas on suitable functional spaces, see [8–11, 15, 16, 19–22] and the references cited therein. Recently, the author [18] strengthened condition (1.1) by (1.2):

There exists a constant $\sigma < 0$ such that

$$|x|^{\sigma-1} a(x) \to +\infty \quad \text{as} \quad |x| \to \infty; \tag{1.2}$$

Under condition (1.2) and some locally conditions near the origin on $f(x, u)$, the author [18] proved the existence of infinitely many homoclinic orbits for equation $(\mathcal{F})$. However, if the function a is not coercive, the conditions (1.1) and (1.2) do not hold.

Inspired by the previous results, in the present paper, we are interested in the existence of infinitely many solutions for $(\mathcal{F})$ under some weaker conditions than (1.1), (1.2) and we discuss three classes of potentials. The remainder of this article consists of four sections. After presenting some preliminaries and proving some compactness results which will aid in our analysis, we establish in Sections 3 and 4 the existence of homoclinic orbits respectively for the subquadratic and superquadratic cases. The last Section is devoted to the case where the nonlinearity still only satisfies locally conditions near the origin (i.e., it can be subquadratic, superquadratic or asymptotically quadratic at infinity).

2. Preliminaries

Let $H^2(\mathbb{R})$ be the Sobolev space with inner product and norm given respectively by

$$\langle u,v\rangle_{H^2} = \int_{\mathbb{R}} [u''(x)v''(x) + u'(x)v'(x) + u(x)v(x)]dx,$$
$$\|u\|_{H^2} = \langle u,u\rangle_{H^2}^{\frac{1}{2}}$$

for all $u,v \in H^2(\mathbb{R})$.

In the following, we shall use $\|\cdot\|_{L^s}$ to denote the norm of $L^s(\mathbb{R})$ for any $s \in [1,\infty]$. Let χ be the self-adjoint extension of the operator $\frac{d^4}{dx^4}+\omega\frac{d^2}{dx^2}+a(x)$ with the domain $\mathcal{D}(\chi) \subset L^2(\mathbb{R})$. Let $\{\mathcal{E}(\lambda) : -\infty < \lambda < \infty\}$ denotes the resolution of χ, and $U = I - \mathcal{E}(0) - \mathcal{E}(-0)$. It is well known that U commutes with χ, $|\chi|$ and $|\chi|^{\frac{1}{2}}$, and $\chi = |\chi|\, U$ is the polar decomposition of χ. Set $E = \mathcal{D}(|\chi|^{\frac{1}{2}})$ and define on E the inner product

$$\langle u,v\rangle_0 = \langle |\chi|^{\frac{1}{2}} u, |\chi|^{\frac{1}{2}} v\rangle_{L^2} + \langle u,v\rangle_{L^2}$$

and the corresponding norm

$$\|u\|_0 = \langle u,u\rangle_0^{\frac{1}{2}}.$$

Lemma 2.1 ([2, Lemma 8]). *Assume that there exists $a_0 > 0$ such that $\omega \leq 2\sqrt{a_0}$, $a_0 \leq a(x)$, $\forall x \in \mathbb{R}$. Then there exists a constant $c_0 > 0$ such that*

$$\int_{\mathbb{R}} [u^{''}(x)^2 - \omega u^{'}(x)^2 + a(x)u(x)^2]dx \geq c_0 \|u\|_{H^2}^2, \ \forall u \in H^2(\mathbb{R}).$$

The main difficulty in dealing with the existence of solutions for $(\mathcal{F})$ is the lack of compactness of the Sobolev embedding. To overcome this difficulty consider the following assumption:

$(\mathcal{A}_\sigma)$ a is bounded from below and there exists a constant $\sigma > 1$ such that

$$meas(\left\{x \in \mathbb{R} : |x|^{-\sigma} a(x) < b\right\}) < +\infty, \ \forall b > 0$$

where *meas* denotes the Lebesgue's measure on $\mathbb{R}$.

Lemma 2.2. *Assume that $(\mathcal{A}_\sigma)$ is satisfied. Then E is compactly embedded in $L^s(\mathbb{R})$ for all $s \in [1,\infty]$.*

Proof. First, we will assume that the function a satisfies the following condition:

There exists a constant $a_0 > 0$ such that

$$\omega \leq 2\sqrt{a_0}, \ a_0 \leq a(x), \ \forall x \in \mathbb{R}. \tag{2.1}$$

For any $\epsilon > 0$, by condition $(\mathcal{A}_\sigma)$ we can choose $r_\epsilon \geq 1$ such that $meas(B_\epsilon) \leq \epsilon$, where

$$B_\epsilon = \left\{x \in \mathbb{R} \setminus [-r_\epsilon, r_\epsilon] : |x|^{-\sigma} a(x) < \frac{1}{\epsilon}\right\}.$$

Let

$$D_\epsilon = \mathbb{R} \setminus (B_\epsilon \cup]-r_\epsilon, r_{\epsilon[})$$

and

$$l_\epsilon = \inf_{x\in D_\epsilon} |x|^{-\sigma} a(x).$$

Then $\frac{1}{l_\epsilon} \leq \epsilon$. Let (u_k) be a sequence such that $u_k \rightharpoonup u$ weakly in E. The Banach-Steinhauss Theorem implies that

$$M = \sup_{k\in\mathbb{N}} \|u_k - u\|_0 < \infty.$$

Since $E \subset H^2(\mathbb{R}) \subset L^p(\mathbb{R})$ for all $p \in [2, \infty]$ with continuous embedding, there exists a constant $M_0 > 0$ such that

$$\|u_k - u\|_{L^\infty} \leq M_0, \ \forall k \in \mathbb{N}.$$

Since $a(x) \geq a_0$ in $I_\epsilon =]-r_\epsilon, r_\epsilon[$, the operator $P : E \to H^2(I_\epsilon)$, $u \longmapsto u_{|I_\epsilon}$ is a continuous linear operator, where $H^2(I_\epsilon)$ denotes the weighted Sobolev space

$$H^2(I_\epsilon) = \left\{ u : I_\epsilon \to \mathbb{R} : \int_{I_\epsilon} [u^{''}(x)^2 - \omega u^{'}(x)^2 + a(x)u(x)^2]dx < +\infty \right\}.$$

Sobolev's embedding Theorem implies that $u_k \to u$ uniformly in $\overline{I}_\epsilon$.

Step 1. We claim that E is compactly embedded in $L^\infty(\mathbb{R})$. In fact, firstly, let us remark that for any $n \in \mathbb{N}$ and $x \in \mathbb{R}$, $|x| \geq r_\epsilon$, one has for $v \in E$

$$v(x) = \int_x^{x+1} [-\dot{v}(s)(x + 1 - s)^{n+1} + v(s)(n + 1)(x + 1 - s)^n]ds.$$

So by Hölder's inequality

$$|v(x)| \leq \frac{1}{\sqrt{2n+3}} (\int_x^{x+1} |\dot{v}(s)|^2 ds)^{\frac{1}{2}} + \frac{n+1}{\sqrt{2n+1}} (\int_x^{x+1} |v(s)|^2 ds)^{\frac{1}{2}}$$

for any $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Then for any $n, k \in \mathbb{N}$ and $|x| \geq r_\epsilon$ one has

$$\begin{aligned}
&|u_k(x) - u(x)| \\
\leq\ & \frac{1}{\sqrt{2n+3}} \Big(\int_{|s|\geq r_\epsilon} |\dot{u}_k(s) - \dot{u}(s)|^2 ds \Big)^{\frac{1}{2}} + \frac{n+1}{\sqrt{2n+1}} \Big(\int_{|s|\geq r_\epsilon} |u_k(s) - u(s)|^2 ds \Big)^{\frac{1}{2}} \\
\leq\ & \frac{M}{\sqrt{2n+3}\sqrt{\omega}} + \frac{n+1}{\sqrt{2n+1}} (meas(B_\epsilon))^{\frac{1}{2}} \|u_k - u\|_\infty \\
&+ \frac{n+1}{\sqrt{2n+1}} \Big(\frac{1}{l_\epsilon} \int_{D_\epsilon} |s|^{-\sigma} a(s) \left| u_k - u^2 \right| ds \Big)^{\frac{1}{2}} \\
\leq\ & \frac{M}{\sqrt{2}\sqrt{n+1}\sqrt{\omega}} + \sqrt{2}(M_0 + M)\sqrt{n}\epsilon^{\frac{1}{2}}.
\end{aligned}$$

Choose $n = [\epsilon^{-\frac{1}{2}}]$ (the integer part of $\epsilon^{-\frac{1}{2}}$), then $n \leq \epsilon^{-\frac{1}{2}} < n + 1$. Hence

$$|u_k(x) - u(x)| \leq [\frac{M}{\sqrt{2\omega}} + M_0\sqrt{2}]\epsilon^{\frac{1}{4}}, \ \forall |x| \geq r_\epsilon. \tag{2.2}$$

On the other hand, since $u_k \to u$ uniformly on I_ϵ, there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$

$$\|u_k - u\|_{L^\infty(I_\epsilon)} < \epsilon. \tag{2.3}$$

Combining (2.2) and (2.3), we get $u_k \to u$ in $L^\infty(\mathbb{R})$.

Step 2: E is compactly embedded in $L^2(\mathbb{R})$. In fact, we have

$$\begin{aligned}\int_{|x|\geq r_\epsilon} |u_k(x) - u(x)|^2 \, dx &= \int_{B_\epsilon} |u_k(x) - u(x)|^2 \, dx + \int_{D_\epsilon} |u_k(x) - u(x)|^2 \, dx \\ &\leq meas(B_\epsilon) M_0^2 + \frac{1}{l_\epsilon} \int_{D_\epsilon} a(x) \, |u_k(x) - u(x)|^2 \, dx \\ &\leq (M^2 + M_0^2)\epsilon.\end{aligned}$$

Since $u_k \to u$ uniformly on I_ϵ, we get $\|u_k - u\|_{L^2} \to 0$ as $k \to \infty$.

Step 3: $p \in]2, \infty[$. We claim that E is compactly embedded in $L^p(\mathbb{R})$. In fact, we have

$$\|u_k - u\|_{L^p}^p \leq \|u_k - u\|_{L^\infty}^{p-2} \|u_k - u\|_{L^2}^2 .$$

By Steps 1 and 2, we deduce that $u_k \to u$ in $L^p(\mathbb{R})$.

Step 4: $p \in [1, 2[$. We claim that $u_k \to u$ in $L^p(\mathbb{R})$. Let $s = \frac{\sigma}{2-p}$. Then $p > \frac{2}{1+\sigma}$ and $sp > 1$. For $v \in L^p(\mathbb{R})$, we have

$$\begin{aligned}&\int_{|x|\geq r_\epsilon} |v(x)|^p \, dx \\ &= \int_{B_\epsilon} |v(x)|^p \, dx + \int_{\{x\in D_\epsilon : |x|^s|v(x)|\leq 1\}} |v(x)|^p \, dx + \int_{\{x\in D_\epsilon : |x|^s|v(x)|\geq 1\}} |v(t)|^p \, dx \\ &\leq (meas(B_\epsilon))^{\frac{1}{2}} \|v\|_{L^{2p}}^p + \int_{|x|\geq r_\epsilon} |x|^{-\sigma} \, dx + \int_{D_\epsilon} (|x|^s \, |v(x)|)^2 \, |x|^{-sp} \, dx \\ &\leq \epsilon^{\frac{1}{2}} \|v\|_{L^{2p}}^p + \int_{|x|\geq r_\epsilon} |x|^{-\sigma} \, dx + \int_{|x|\geq r_\epsilon} |x|^{\sigma} \, |v(x)|^2 \, dx \\ &\leq \epsilon^{\frac{1}{2}} \|v\|_{L^{2p}}^p + \int_{|x|\geq r_\epsilon} |x|^{-\sigma} \, dx + \frac{1}{l_\epsilon} \|v\|_0^2 .\end{aligned}$$

Choose r_ϵ such that $\int_{|x|\geq r_\epsilon} |x|^{-\sigma} \, dx \leq \epsilon^{\frac{1}{2}}$, we obtain

$$\int_{|x|\geq r_\epsilon} |v(x)|^p \, dx \leq \epsilon^{\frac{1}{2}} \|v\|_{L^{2p}}^p + \epsilon^{\frac{1}{2}} + \epsilon^{\frac{1}{2}} \|v\|_0^2 .$$

Hence, we have

$$\int_{|x|\geq r_\epsilon} |u_k(x) - u(x)|^p \, dx \leq \epsilon^{\frac{1}{2}} \|u_k - u\|_{L^{2p}}^p + \epsilon^{\frac{1}{2}} + \epsilon^{\frac{1}{2}} \|u_k - u\|_0^2 .$$

Since $2p \geq 2$, we deduce from Steps 2,3, the existence of a constant $M_1 > 0$ such that

$$\int_{|x|\geq r_\epsilon} |u_k(x) - u(x)|^p \, dx \leq \epsilon^{\frac{1}{2}} (M_1^p + 1 + M^2).$$

As above $\int_{I_\epsilon} |u_k(x) - u(x)|^p \, dx \to 0$ as $k \to \infty$. Hence $u_k \to u$ in $L^p(\mathbb{R})$.

Now, by a standard argument, we prove the general case which does not need the condition (2.1) for all $t \in \mathbb{R}$. The proof of Lemma 2.2 is completed. □

By Lemma 2.2, we see, since the selfadjoint operator χ in $L^2(\mathbb{R})$ is bounded from below, it possesses a compact resolvent. Therefore, the spectrum $\sigma(\chi)$ consists of eigenvalues numbered in $\lambda_1 \le \lambda_2 \le \cdots \to +\infty$ (counted in their multiplicities), and a corresponding system of eigenfunctions $(e_j)_{j\in\mathbb{N}}$, $(\chi e_j = \lambda_j e_j)$, forms an orthonormal basis in $L^2(\mathbb{R})$. Let k^- (resp. k^0) be the number of $\lambda_j < 0$ (resp. $\lambda_j = 0$), $\bar{k} = k^- + k^0$ and let $E^- = span\{e_1, \dots, e_{k^-}\}$, $E^0 = span\{e_{k^-+1}, \dots, e_{\bar{k}}\}$ and $E^+ = \overline{span\{e_{\bar{k}+1}, \dots\}}$. Then $E = E^- \oplus E^0 \oplus E^+$. We introduce on E the following inner product

$$\langle u, v\rangle = \langle |\chi|^{\frac{1}{2}} u, |\chi|^{\frac{1}{2}} v\rangle_{L^2} + \langle u^0, v^0\rangle_{L^2}$$

and the corresponding norm

$$\|u\|^2 = \left\||\chi|^{\frac{1}{2}} u\right\|_{L^2}^2 + \left\|u^0\right\|_{L^2}^2,$$

where $u = u^- + u^0 + u^+$, $v = v^- + v^0 + v^+ \in E^- \oplus E^0 \oplus E^+$. Clearly, $\|u\|_{L^2}^2 \le \lambda \|u\|^2$ for all $u \in E$, where $\lambda = \max\left\{1, \lambda_{\bar{k}+1}^{-1}, |\lambda_{k^-}|^{-1}\right\}$. Since $\|u\|_0^2 = \left\|u^- + u^0\right\|_{L^2}^2 + \|u\|^2$ for all $u \in E$, one has $\|u\|^2 \le \|u\|_0^2 \le (1+\lambda)\|u\|^2$, i.e., the norms $\|.\|_0$ and $\|.\|$ are equivalent. From now on the norm $\|.\|$ on E will be used. By Lemma 2.2, for all $p \in [1, \infty]$, there exists a constant $\eta_p > 0$ such that

$$\|u\|_{L^p} \le \eta_p \|u\|, \ \forall u \in E. \tag{2.4}$$

For later use, let

$$a(u, v) = \langle |\chi|^{\frac{1}{2}} Uu, |\chi|^{\frac{1}{2}} v\rangle_{L^2}, \ \forall u, v \in E$$

be the quadratic form associated with χ. For any $u \in \mathcal{D}(\chi)$ and $v \in E$, we have

$$a(u, v) = \int_{\mathbb{R}} [u^{''}(x)v^{''}(x) - \omega u^{'}(x)v^{'}(x) + a(x)u(x)v(x)]dx \tag{2.5}$$

and so, since $\mathcal{D}(\chi)$ is dense in E, (2.5) holds for all $u \in E$. Moreover, by definition

$$a(u, u) = \left\|u^+\right\|^2 - \left\|u^-\right\|^2 \tag{2.6}$$

for all $u = u^- + u^0 + u^+ \in E = E^- \oplus E^0 \oplus E^+$.

3. Subquadratic case

In this section, we are interested in the existence of infinitely many homoclinic solutions for equation $(\mathcal{F})$ when the potential $F(x,u) = \int_0^u f(x,u)dx$ is subquadratic at infinity with respect to u. More precisely, we make the following assumptions.

(F_1) $F(x,u) \geq 0$ for all $(x,u) \in \mathbb{R}^2$ and there exist constants $0 < \mu < 2$ and $R > 0$ such that

$$f(x,u)u \leq \mu F(x,u),\ \forall x \in \mathbb{R},\ |u| \geq R,$$

and

$$f(x,u)u \leq 2F(x,u),\ \forall x \in \mathbb{R},\ |u| \leq R;$$

(F_2) $\lim_{|u|\to 0} \frac{F(x,u)}{|u|^2} = +\infty$ uniformly for $x \in \mathbb{R}$;

(F_3) There exists a constant $c > 0$ such that

$$|f(x,u)| \leq c\,|u|\,,\ \forall x \in \mathbb{R},\ |u| \leq R,$$

where R is the constant in (F_1);

(F_4) $\liminf_{|u|\to\infty} \frac{F(x,u)}{|u|} \geq a$, where a is a positive constant.

Our main result in this Section reads as follows.

Theorem 3.1. *Suppose that $(\mathcal{A}_\sigma)$, (F_1)-(F_4) hold and $F(x,u)$ is even in u for all $x \in \mathbb{R}$. Then $(\mathcal{F})$ possesses infinitely many nontrivial homoclinic solutions.*

Example 3.2. Let

$$F(x,u) = h(x)\,|u|^\mu$$

where $0 < \inf_{x\in\mathbb{R}} h(x) \leq \sup_{x\in\mathbb{R}} h(x) < \infty$ for all $x \in \mathbb{R}$ and $\mu \in [1,2[$. It is easy to check that F satisfies conditions (F_1)-(F_4).

Proof of Theorem 3.1

In the following, c_n, $n \in \mathbb{N}$ denotes some various positive constants. For equation $(\mathcal{F})$, we associate the following functional defined on the space E introduced in Section 2 by

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}} [u^{''}(x)^2 - \omega u^{'}(x)^2 + a(x)u(x)^2]dx - \int_{\mathbb{R}} F(x,u(x))dx.$$

Then by (2.6), Φ can be rewritten as

$$\Phi(u) = \frac{1}{2}(\left\|u^+\right\|^2 - \left\|u^-\right\|^2) - \varphi(u),\ u = u^- + u^0 + u^+ \in E,$$

where

$$\varphi(u) = \int_{\mathbb{R}} F(x,u(x))dx,\ u \in E.$$

It is well known that under assumptions of Theorem 3.1 the functional Φ is continuously differentiable on E and its critical points on E are exactly the homoclinic solutions of the equation $(\mathcal{F})$. Moreover φ' is compact and for all $u, v \in E$

$$\Phi'(u)v = \int_{\mathbb{R}} \Big[u''(x)v''(x) - \omega u'(x)v'(x) + a(x)u(x)v(x)\Big]dx - \int_{\mathbb{R}} f(x, u(x))v(x)dx.$$

To prove Theorem 3.1, the following Variant Fountain Theorem developed by Zou [22] will be needed. Let E be a Banach space with the norm $\|\cdot\|$ and $E = \overline{\oplus_{m\in\mathbb{N}} X_m}$ with $\dim X_m < \infty$ for any $m \in \mathbb{N}$. Set

$$Y_k = \oplus_{m=1}^{k} X_m,\ Z_k = \overline{\oplus_{m=k}^{\infty} X_m}.$$

Consider a family of functionals $f_\lambda \in C^1(E, \mathbb{R})$ defined by

$$\Phi_\lambda(u) = A(u) - \lambda B(u),\ u \in E,\ \lambda \in [1, 2].$$

Lemma 3.3 ([22]). *Assume that the functionals f_λ defined previously, satisfy*

(T_1) *Φ_λ maps bounded sets into bounded sets uniformly for all $\lambda \in [1, 2]$ and*

$$\Phi_\lambda(-u) = \Phi_\lambda(u),\ \forall(\lambda, u) \in [1, 2] \times E;$$

(T_2) *$B(u) \geq 0$ for all $u \in E$ and $B(u) \to +\infty$ as $\|u\| \to \infty$ on any finite dimensional subspace of E;*

(T_3) *There exist $\rho_k > r_k > 0$ such that*

$$a_k(\lambda) = \inf_{u\in Z_k, \|u\|=\rho_k} \Phi_\lambda(u) \geq 0 > b_k(\lambda) = \max_{u\in Y_k, \|u\|=r_k} \Phi_\lambda(u)$$

for all $\lambda \in [1, 2]$ and

$$d_k(\lambda) = \inf_{u\in Z_k, \|u\|\leq\rho_k} \Phi_\lambda(u) \to 0 \text{ as } k \to \infty \text{ uniformly for } \lambda \in [1, 2].$$

Then there exist $\lambda_n \to 1$, $u_{\lambda_n} \in Y_n$ such that

$$(\Phi_{\lambda_n|Y_n})'(u_{\lambda_n}) = 0,\ \Phi_{\lambda_n}(u_{\lambda_n}) \to c_k \in [d_k(2), b_k(1)].$$

Particularly, if (u_{λ_n}) has a convergent subsequence for every k, then Φ_1 has infinitely many nontrivial critical points $u_k \in E \setminus \{0\}$ satisfying $\Phi_1(u_k) \to 0^-$ as $k \to \infty$.

For $m \in \mathbb{N}$, let $X_m = \mathbb{R}e_m$, where (e_m) is the sequence defined in Section 2, then Y_k and Z_k are defined as above. In order to apply the above Variant Fountain Theorem for proving our result, we introduce the family of functionals

$$\Phi_\lambda(u) = A(u) - \lambda B(u),\ (\lambda, u) \in [1, 2] \times E,$$

where

$$A(u) = \frac{1}{2}\left\|u^+\right\|^2,\ B(u) = \frac{1}{2}\left\|u^-\right\|^2 + \int_{\mathbb{R}} F(x, u(x))dx,$$

for $u = u^- + u^0 + u^+ \in E = E^- \oplus E^0 \oplus E^+$.

Now, for $|u| \leq R$, we have by (F_3) and the Mean Value Theorem

$$F(x, u) = \int_0^1 f(x, su)xds \leq \frac{c}{2}|u|^2 \leq \frac{cR}{2}|u|. \tag{3.1}$$

For $|u| \geq R$, set
$$\psi(\xi) = F(x, \frac{Rx}{\xi\,|u|})\xi^{\mu},\ \xi \in]0,1].$$
By (F_1), it holds
$$\begin{aligned}\psi'(\xi) &= -f(x, \frac{Ru}{\xi\,|u|})\frac{Ru}{\xi^2\,|u|}\xi^{\mu} + \mu F(x, \frac{Ru}{\xi\,|u|})\xi^{\mu-1} \\ &= \xi^{\mu-1}[-f(x, \frac{Ru}{\xi\,|u|})\frac{Ru}{\xi\,|u|} + \mu F(x, \frac{Ru}{\xi\,|u|})] \geq 0.\end{aligned}$$
So ψ is nondecreasing in $]0,1]$, and since $\xi = \frac{R}{|u|} \leq 1$, then
$$F(x,u)(\frac{R}{|u|})^{\mu} \leq F(x, \frac{Ru}{|u|}). \tag{3.2}$$
Combining (3.1) and (3.2) yields
$$F(x,u)(\frac{R}{|u|})^{\mu} \leq \frac{c}{2}R^2$$
and then
$$F(x,u) \leq \frac{c}{2}R^{2-\mu}\,|u|^{\mu}$$
which with (3.1) gives
$$F(x,u) \leq c_1(|u| + |u|^{\mu}),\ \forall (x,u) \in \mathbb{R}^2. \tag{3.3}$$
It follows from (2.4) and (3.3) that for any $\lambda \in [1,2]$ and $u \in E$
$$|\Phi_\lambda(u)| \leq \frac{1}{2}\,\|u\|^2 + 2c_1(\eta_1\,\|u\| + \eta_\mu\,\|u\|^{\mu})$$
which implies that Φ_λ maps bounded sets into bounded sets uniformly for $\lambda \in [1,2]$. Note that $F(x,-u) = F(x,u)$, so we have $\Phi_\lambda(-u) = \Phi_\lambda(u)$ for all $(\lambda,u) \in [1,2] \times E$. Thus the condition (T_1) of Lemma 3.3 holds.

Lemma 3.4. *Assume that $(\mathcal{A}_\sigma)$, (F_1) and (F_4) hold. Then $B(u) \geq 0$ for all $u \in E$ and $B(u) \to +\infty$ as $\|u\| \to \infty$ on any finite-dimensional subspace of E.*

Proof. By assumption (F_1), it is clear that $B(u) \geq 0$ for all $u \in E$. We claim that for any finite-dimensional subspace F of E, there exists a constant $\epsilon_0 > 0$ such that
$$meas(\{x \in \mathbb{R} : |u(x)| \geq \epsilon_0\,\|u\|\}) \geq \epsilon_0,\ \forall u \in F \setminus \{0\}\,, \tag{3.4}$$
where *meas* denotes the Lebesgue's measure in $\mathbb{R}$. If not, for any $k \in \mathbb{N}$, there exists $u_k \in F \setminus \{0\}$ such that
$$meas(\left\{x \in \mathbb{R} : |u_k(x)| \geq \frac{1}{k}\,\|u_k\|\right\}) < \frac{1}{k}.$$
Let $v_k = \frac{u_k}{\|u_k\|} \in F$, then $\|v_k\| = 1$ and
$$meas(\left\{x \in \mathbb{R} : |v_k(x)| \geq \frac{1}{k}\right\}) < \frac{1}{k},\ \forall k \in \mathbb{N}. \tag{3.5}$$

Since F is finite dimensional, then by taking a subsequence if necessary, we can assume that $v_k \to v_0$ in F for some $v_0 \in F$, $\|v_0\| = 1$. Recall that any two norms on F are equivalent, so one has

$$\int_{\mathbb{R}} |v_k(x) - v_0(x)|\, dx \to 0 \quad \text{as} \quad k \to \infty. \tag{3.6}$$

Since $\|v_0\| = 1$, then $\|v_0\|_{L^\infty} = \sup_{t\in\mathbb{R}} |v_0(x)| > 0$. Hence there exists a constant $\sigma_0 > 0$ such that

$$meas(\{x \in \mathbb{R} : |v_0(x)| \geq \sigma_0\}) \geq \sigma_0. \tag{3.7}$$

For any $k \in \mathbb{N}$, let

$$\Omega_k = \left\{x \in \mathbb{R} : |v_k(x)| < \frac{1}{k}\right\},\ \Omega_0 = \{x \in \mathbb{R} : |v_0(x)| \geq \sigma_0\}.$$

By (3.5) and (3.7), for any $k \in \mathbb{N}$ large enough, it holds

$$meas(\Omega_0 \cap \Omega_k) = meas(\Omega_0 \setminus \Omega_k^c) \geq meas(\Omega_0) - meas(\Omega_k^c) \geq \sigma_0 - \frac{1}{k} \geq \frac{\sigma_0}{2}.$$

Then for k large enough

$$\begin{aligned}\int_{\mathbb{R}} |v_k(x) - v_0(x)|\, dx &\geq \int_{\Omega_0\cap\Omega_k} |v_k(x) - v_0(x)|\, dx \\ &\geq (\sigma_0 - \frac{1}{k}) meas(\Omega_0 \cap \Omega_k) \geq \frac{\sigma_0^2}{4} > 0\end{aligned}$$

which contradicts (3.6). Therefore (3.4) holds.

For the ϵ_0 given in (3.4), denote

$$\Omega_u = \{x \in \mathbb{R} : |u(x)| \geq \epsilon_0 \|u\|\},\ \forall u \in F \setminus \{0\}.$$

Then by (3.4), we obtain

$$meas(\Omega_u) \geq \epsilon_0,\ \forall u \in F \setminus \{0\}. \tag{3.8}$$

By (F_4), there exists a constant $R_1 > R$ such that

$$F(x, u) \geq a\frac{|u|}{2},\ \forall x \in \mathbb{R},\ |u| \geq R_1. \tag{3.9}$$

Let $u \in F$ such that $\|u\| \geq \frac{R_1}{\epsilon_0}$, we have

$$|u(x)| \geq \epsilon_0 \|u\| \geq R_1,\ \forall x \in \Omega_u. \tag{3.10}$$

Combining (3.9) and (3.10) yields for all $u \in F$, $\|u\| \geq \frac{R_1}{\epsilon_0}$

$$B(u) \geq \int_{\Omega_u} F(x, u)dx \geq \frac{a\epsilon_0}{2} \|u\|\, meas(\Omega_u) \geq \frac{a\epsilon_0^2}{2} \|u\|.$$

This implies that $B(u) \to +\infty$ as $\|u\| \to \infty$ on any finite-dimensional subspace F of E. The proof of Lemma 3.4 is completed. □

Lemma 3.5. *Suppose that* $(\mathcal{A}_\sigma)$ *holds. Then for any* $p \in [2, \infty]$

$$l_p(k) = \sup_{u \in Z_k, \|u\|=1} \|u\|_{L^p} \to 0 \ \textit{as} \ \ k \to \infty.$$

Proof. It is clear that $0 < l_p(k+1) \le l_p(k)$, so that $l_p(k) \to \bar{l}_p$ as $k \to \infty$. For every $k \ge 1$, there exists $u_k \in Z_k$ such that $\|u_k\| = 1$ and $\|u_k\|_{L^p} > \frac{1}{2} l_p(k)$. For any $v \in E$, let $v = \sum_{i=1}^{\infty} v_i e_i$. By the Cauchy-Schwartz inequality, one has

$$|\langle u_k, v\rangle| = \left| \langle u_k, \sum_{i=k+1}^{\infty} v_i e_i \rangle \right| \le \sum_{i=k+1}^{\infty} |v_i| \, \|e_i\| \to 0 \text{ as } k \to \infty$$

which implies that $u_k \rightharpoonup 0$. Without loss of generality, Lemma 2.2 implies that $u_k \to 0$ in $L^p(\mathbb{R})$. Thus we have proved that $\bar{l}_p = 0$. The proof of Lemma 3.5 is completed. □

Lemma 3.6. *Assume that* $(\mathcal{A}_\sigma)$, (F_2) *and* (F_3) *are satisfied. Then there exist a constant* $k_0 \in \mathbb{N}$ *and two sequences* $0 < r_k < \rho_k \to 0$ *as* $k \to \infty$ *such that*

$$a_k(\lambda) = \inf_{u \in Z_k, \|u\| = \rho_k} \Phi_\lambda(u) > 0, \ \forall k \ge k_0, \ \forall \lambda \in [1, 2], \tag{3.11}$$

$$b_k(\lambda) = \max_{u \in Y_k, \|u\| = r_k} \Phi_\lambda(u) < 0, \ \forall k \ge k_0, \ \forall \lambda \in [1, 2], \tag{3.12}$$

$$c_k(\lambda) = \inf_{u \in Z_k, \|u\| \le \rho_k} \Phi_\lambda(u) \to 0 \ \textit{as} \ k \to \infty \ \textit{uniformly for} \ \lambda \in [1, 2]. \tag{3.13}$$

Proof. For any $u \in E$ with $\|u\| \le \frac{R}{\eta_\infty}$, (2.4) implies

$$\|u\|_{L^\infty} \le R. \tag{3.14}$$

Note that $Z_k \subset E^+$ for any $k \ge k^+ + 1$ with k^+ is defined in Section 2. It follows from (3.1) and (3.14) that for $k \ge k^+ + 1$ and $\|u\| \le \frac{R}{\eta_\infty}$, one has

$$\begin{aligned} \Phi_\lambda(u) &\ge \frac{1}{2} \|u\|^2 - 2 \int_{\mathbb{R}} \frac{cR}{2} |u| \, dx \\ &\ge \frac{1}{2} \|u\|^2 - cR \|u\|_{L^1}, \ \forall \lambda \in [1, 2], \ \forall u \in Z_k. \end{aligned} \tag{3.15}$$

Let

$$M_k = l_1(k) = \sup_{u \in Z_k \setminus \{0\}} \frac{\|u\|_{L^1}}{\|u\|}, \ \forall k \in \mathbb{N}. \tag{3.16}$$

Then by Lemma 3.5, we have

$$M_k \to 0 \ \text{ as } \ k \to \infty. \tag{3.17}$$

Combining (3.15), (3.16) yields

$$\Phi_\lambda(u) \ge \frac{1}{2} \|u\|^2 - cRM_k \|u\| \tag{3.18}$$

for all $k \geq k^+ + 1$ and $u \in Z_k$ with $\|u\| \leq \frac{R}{\eta_\infty}$. For any $k \geq k^+ + 1$, let $\rho_k = 4cRM_k$. Then by (3.17), there exists an integer $k_0 \geq k^+ + 1$ such that

$$\rho_k \leq \frac{R}{\eta_\infty},\ \forall k \geq k_0. \tag{3.19}$$

It follows from (3.18) and (3.19) that for all $k \geq k_0$

$$a_k(\lambda) = \inf_{u\in Z_k, \|u\|=\rho_k} \Phi_\lambda(u) \geq \frac{1}{2}\rho_k^2 - cRM_k\rho_k = \frac{1}{4}\rho_k^2 > 0.$$

Hence (3.11) is satisfied.

Now, for any $k \geq k_0$, (3.18) implies that for any $u \in Z_k$ with $\|u\| \leq \rho_k$

$$\Phi_\lambda(u) \geq -cRM_k\rho_k.$$

Since $\Phi_\lambda(0) = 0$, we deduce that

$$0 \geq \inf_{u\in Z_k, \|u\|\leq\rho_k} \Phi_\lambda(u) \geq -cRM_k\rho_k,\ \forall k \geq k_0,$$

which implies that (3.13) is satisfied.

It remains to prove (3.12). Since the two norms $\|\cdot\|_{L^2}$ and $\|\cdot\|$ are equivalent in finite-dimensional space Y_k, then for any $k \in \mathbb{N}$, there exists a constant $d_k > 0$ such that

$$\|u\|_{L^2} \geq d_k \|u\|,\ \forall u \in Y_k. \tag{3.20}$$

By (F_2), for any $k \in \mathbb{N}$, there exists a constant $\epsilon_k > 0$ such that

$$F(x,u) \geq \frac{|u|^2}{d_k^2},\ \forall |u| \leq \epsilon_k. \tag{3.21}$$

For any $k \in \mathbb{N}$ and $u \in E$ with $\|u\| \leq \frac{\epsilon_k}{\eta_\infty}$, one has by (2.4), $\|u\|_{L^\infty} \leq \epsilon_k$. Consequently, for all $k \in \mathbb{N}$ and $u \in Y_k$ with $\|u\| \leq \frac{\epsilon_k}{\eta_\infty}$, it follows from (3.20) and (3.21)

$$\Phi_\lambda(u) \leq \frac{1}{2}\left\|u^+\right\|^2 - \int_{\mathbb{R}} \frac{|u|^2}{d_k^2}dx \leq -\frac{1}{2}\|u\|^2, \forall\lambda \in [1,2]. \tag{3.22}$$

For any $k \in \mathbb{N}$, choose $0 < r_k < \min\left\{\rho_k, \frac{\epsilon_k}{\eta_\infty}\right\}$, then (3.22) implies

$$b_k(\lambda) = \max_{u\in Y_k, \|u\|=r_k} \Phi_\lambda(u) \leq -\frac{r_k^2}{2} < 0,\ \forall k \in \mathbb{N}.$$

The proof of Lemma 3.6 is completed. □

It follows from above that there exists a positive integer k_0 such that for all $k \geq k_0$, all the conditions of Lemma 3.3 are satisfied, therefore for all $k \geq k_0$, there exist sequences $0 < \lambda_j \to 1$, $u_j \in Y_j$ such that

$$(f_{\lambda_j|Y_j})'(u_{\lambda_j}) = 0,\ f_{\lambda_j}(u_{\lambda_j}) \to \theta_k \in [d_k(2), b_k(1)]. \tag{3.23}$$

Lemma 3.7. *Under the assumptions of Theorem 3.1, the sequence (u_{λ_j}) is bounded in E.*

Proof. Set for $j \in \mathbb{N}$

$$\Lambda_j = \left\{x \in \mathbb{R} : \left|u_{\lambda_j}(x)\right| \geq R_1\right\},$$

where R_1 is defined in (3.9) and note that by (F_1), $f(x,u)u \leq 2F(x,u)$, $\forall (x,u) \in \mathbb{R}^2$. Hence it holds from (F_1), (3.9) and (3.23) that

$$\begin{aligned}
-\Phi_{\lambda_j}(u_{\lambda_j}) &= \frac{1}{2}(\Phi_{\lambda_j|Y_j})'(u_{\lambda_j})u_{\lambda_j} - \Phi_{\lambda_j}(u_{\lambda_j}) \\
&= \lambda_j \int_{\mathbb{R}} [F(x,u_{\lambda_j}) - \frac{1}{2}f(x,u_{\lambda_j})u_{\lambda_j}]dx \\
&\geq \lambda_j \int_{\Lambda_j} [F(x,u_{\lambda_j}) - \frac{1}{2}f(x,u_{\lambda_j})u_{\lambda_j}]dx \\
&\geq \lambda_j \frac{2-\mu}{2} \int_{\Lambda_j} F(x,u_{\lambda_j})dx \\
&\geq \lambda_j \frac{a\lambda_j(2-\mu)}{4} \int_{\Lambda_j} \left|u_{\lambda_j}\right| dx, \ \forall j \in \mathbb{N}.
\end{aligned}$$

It follows from (3.23) that

$$\int_{\Lambda_j} \left|u_{\lambda_j}\right| dx \leq c_2, \ \forall j \in \mathbb{N}. \tag{3.24}$$

For any $j \in \mathbb{N}$, let $\chi_j : \mathbb{R} \to \mathbb{R}$ be the indicator of Λ_j, that is

$$\chi_j(x) = \begin{cases} 1 & \text{if } t \in \Lambda_j, \\ 0 & \text{if } t \in \Lambda_j^c. \end{cases}$$

Hence by the definition of Λ_j and (3.24), we have

$$\left\|(1-\chi_j)u_{\lambda_j}\right\|_{L^\infty} \leq R_1 \text{ and } \left\|\chi_j u_{\lambda_j}\right\|_{L^1} \leq c_2, \ \forall j \in \mathbb{N}.$$

Therefore from the equivalence of any two norms on finite-dimensional space $E^- \oplus E^0$, it holds that

$$\begin{aligned}
\left\|u_{\lambda_j}^- + u_{\lambda_j}^0\right\|_{L^2}^2 &= \langle u_j^- + u_j^0, u_j\rangle_{L^2} \\
&= \langle u_{\lambda_j}^- + u_{\lambda_j}^0, (1-\chi_j)u_{\lambda_j}\rangle_{L^2} + \langle u_{\lambda_j}^- + u_{\lambda_j}^0, \chi_j u_{\lambda_j}\rangle_{L^2} \\
&\leq \left\|u_{\lambda_j}^- + u_{\lambda_j}^0\right\|_{L^1} \left\|(1-\chi_j)u_{\lambda_j}\right\|_{L^\infty} + \left\|u_{\lambda_j}^- + u_{\lambda_j}^0\right\|_{L^\infty} \left\|\chi_j u_{\lambda_j}\right\|_{L^1} \\
&\leq \left(c_3 \left\|(1-\chi_j)u_{\lambda_j}\right\|_{L^\infty} + c_4 \left\|\chi_j u_{\lambda_j}\right\|_{L^1}\right) \left\|u_{\lambda_j}^- + u_{\lambda_j}^0\right\|_{L^2} \\
&\leq (c_3 R_1 + c_4 c_2) \left\|u_{\lambda_j}^- + u_{\lambda_j}^0\right\|_{L^2}, \ \forall j \in \mathbb{N}.
\end{aligned}$$

Consequently

$$\left\|u_{\lambda_j}^- + u_{\lambda_j}^0\right\|_{L^2} \leq c_3 R_1 + c_4 c_2, \ \forall j \in \mathbb{N}$$

which together with the fact that $E^- \oplus E^0$ is of finite-dimensional, implies that

$$\left\|u_{\lambda_j}^- + u_{\lambda_j}^0\right\| \leq c_5, \ \forall j \in \mathbb{N}. \tag{3.25}$$

By noting that

$$\left\|u^+_{\lambda_j}\right\|^2 = 2\Phi_{\lambda_j}(u_{\lambda_j}) + \lambda_j \left\|u^-_{\lambda_j}\right\|^2 + 2\lambda_j \int_{\mathbb{R}} F(x, u_{\lambda_j})dx,\ \forall j \in \mathbb{N},$$

we deduce from (2.4), (3.3), (3.23) and (3.25) that

$$\begin{aligned}\|u_{\lambda_j}\|^2 &= \left\|u^-_{\lambda_j} + u^0_{\lambda_j}\right\|^2 + \left\|u^+_{\lambda_j}\right\|^2 \\ &= \left\|u^-_{\lambda_j} + u^0_{\lambda_j}\right\|^2 + 2f_{\lambda_j}(u_{\lambda_j}) + \lambda_j \left\|u^-_{\lambda_j}\right\|^2 + 2\lambda_j \int_{\mathbb{R}} F(x, u_{\lambda_j})dx \\ &\le c_6 + c_8(\|u_{\lambda_j}\| + \|u_{\lambda_j}\|^{\mu}),\ \forall j \in \mathbb{N},\end{aligned}$$

and since $\mu < 2$ this implies that (u_{λ_j}) is bounded in E. The proof of Lemma 3.7 is completed. □

It remains to prove that (u_{λ_j}) has a strongly convergent subsequence in E. Since $E^- \oplus E^0$ is finite-dimensional, then by Lemmas 2.2, 3.5, we can assume, without loss of generality, that

$$u^-_{\lambda_j} \to u^-,\ u^0_{\lambda_j} \to u^0,\ u^+_{\lambda_j} \rightharpoonup u^+ \text{ and } u_{\lambda_j} \rightharpoonup u \text{ as } j \to \infty \tag{3.26}$$

for some $u = u^- + u^0 + u^+ \in E^- \oplus E^0 \oplus E^+$. In virtue of the Riez Representation Theorem, $(\Phi_{\lambda_j|Y_j})' : Y_j \to Y_j^*$ and $\varphi' : E \to E^*$ can be viewed as $(\Phi_{\lambda_j|Y_j})' : Y_j \to Y_j$ and $\varphi' : E \to E$, where Y_j^* and E^* are the dual spaces of Y_j and E respectively. Set $P_j : E \to Y_j$ be the orthogonal projection for all $j \in \mathbb{N}$, we have

$$0 = (\Phi_{\lambda_j|Y_j})'(u_{\lambda_j}) = u^+_{\lambda_j} - \lambda_j u^-_{\lambda_j} - \lambda_j P_j \varphi'(u_{\lambda_j}),\ \forall j \in \mathbb{N}$$

that is

$$u^+_{\lambda_j} = \lambda_j[u^-_{\lambda_j} + P_j\varphi'(u_{\lambda_j})],\ \forall j \in \mathbb{N}. \tag{3.27}$$

Since $\varphi' : E \to E$ is compact, then without loss of generality, (3.26) implies that the right-hand side of (3.27) converges strongly in E and then $u^+_{\lambda_j} \to u^+$ in E. This with (3.26) implies $u_{\lambda_j} \to u$ in E.

Consequently, by Lemma 3.3, the functional $\Phi_1 = \Phi$ possesses infinitely many nontrivial critical points, which implies that $(\mathcal{F})$ has infinitely many nontrivial homoclinic orbits. The proof of Theorem 3.1 is completed.

4. Superquadratic case

In this Section, we are interested in the existence of infinitely many homoclinic orbits of $(\mathcal{F})$ when the potential $F(x, u)$ is superquadratic at infinity with respect to u. More precisely, we make the following assumptions:

(F_5) $$\frac{F(x, u)}{|u|^2} \to +\infty \text{ as } |u| \to \infty, \text{ uniformly in } x \in \mathbb{R};$$

(F_6) $$f(x, u)u \ge 2F(x, u) \ge 0,\ \forall (x, u) \in \mathbb{R}^2;$$

(F_7)
$$\frac{|f(x,u)|}{|u|} \to 0 \text{ as } |u| \to 0, \text{ uniformly in } x \in \mathbb{R};$$

(F_8) there exist constants $\alpha > 0$ and $a > 0$ such that
$$|f(x,u)| \le a(|u|^{\alpha} + 1),\ \forall (x,u) \in \mathbb{R}^2;$$

(F_9) there exist constants $\beta \ge \alpha$, $\beta > 1$, $b > 0$ and $r > 0$ such that
$$f(x,u)u - 2F(x,u) \ge b\,|u|^{\beta},\ \forall x \in \mathbb{R},\ \forall\,|u| \ge r.$$

Our main results in this Section read as follows:

Theorem 4.1. *Assume that* $(\mathcal{A}_\sigma)$ *and* (F_5)-(F_9) *hold. Then the fourth-order differential equation* $(\mathcal{F})$ *possesses at least one nontrivial homoclinic solution.*

Theorem 4.2. *Assume that* $(\mathcal{A}_\sigma)$ *and* (F_5)-(F_9) *hold and* $F(x,u)$ *is even in* $u \in \mathbb{R}$. *Then* $(\mathcal{F})$ *has infinitely many distinct homoclinic solutions.*

Example 4.3. Let
$$F(x,u) = |u|^2 \ln(1 + |u|^2).$$
A straightforward computation shows that F satisfies Theorems 4.1 and 4.2.

Now we are going to establish the corresponding variational framework to obtain homoclinic solutions for $(\mathcal{F})$. To this end, define the functional $\Phi : E \to \mathbb{R}$ as in Section 3. It is well known that under the assumptions of Theorem 4.1, the functional Φ is continuously differentiable on E and its critical points on E are exactly the homoclinic solutions of the equation $(\mathcal{F})$. For the existence and multiplicity of homoclinic solutions of $(\mathcal{F})$, we will appeal to the following abstract critical lemmas. Let E be a Banach space and $\Phi \in C^1(E,\mathbb{R})$. As usual we say Φ satisfies the Palais-Smale condition ((PS) for short) if any sequence $(u_k) \subset E$ for which $(\Phi(u_k))$ is bounded and $\Phi'(u_k) \to 0$ as $k \to \infty$ possesses a convergent subsequence.

Lemma 4.4 (Generalized Mountain Pass Theorem [13])**.** *Let E an infinite dimensional Banach space such that $E = V \oplus X$, where V is finite dimensional. If $\Phi \in C^1(E,\mathbb{R})$ and the following conditions hold:*

(Φ_1) Φ *satisfies the* (PS) *condition;*

(Φ_2) *there are constants* $\rho, \delta > 0$ *such that*
$$\Phi_{|\partial B_\rho \cap X} \ge \delta;$$
where $\partial B_\rho = \{u \in E : \|u\| = \rho\}$;

(Φ_3) *there are constants* $r > \rho$, $M > 0$ *and* $e \in X$ *with* $\|e\| = 1$ *such that*
$$\Phi_{|\partial\Lambda} \le 0 \text{ and } \Phi_{|\Lambda} \le M,$$
where
$$\Lambda = (B_r \cap V) \oplus \{se : 0 \le s \le r\}.$$
Then Φ has a critical point u with $\Phi(u) \ge \delta$.

Lemma 4.5 (Symmetric Mountain Pass Theorem [13]). *Let E be an infinite dimensional Banach space such that $E = V \oplus X$, where V is finite dimensional. If $\Phi \in C^1(E, \mathbb{R})$ is even and satisfies $\Phi(0) = 0$, (Φ_1), (Φ_2) and*

$(\Phi_3^{'})$ *for each finite dimensional subspace $\tilde{E} \subset E$, there is an $R = R(\tilde{E}) > 0$ such that $\Phi \leq 0$ on $\tilde{E} \setminus B_R$.*

Then Φ possesses an unbounded sequence of critical values.

In the following, c_n, $n \in \mathbb{N}$ denote some various constants.

Lemma 4.6. *If $(\mathcal{A}_\sigma)$, (F_6), (F_8) and (F_9) hold, then Φ satisfies the (PS) condition.*

Proof. Let $(u_k) \subset E$ be a (PS) sequence, i.e., there exists a constant $M > 0$ such that

$$|\Phi(u_k)| \leq M, \ \forall k \in \mathbb{N} \text{ and } \Phi'(u_k) \to 0 \text{ as } k \to \infty. \tag{4.1}$$

We claim that (u_k) is bounded. If not, passing to a subsequence if necessary, we may assume that $\|u_k\| \to \infty$ as $k \to \infty$. By (F_6) and (F_9), we have

$$\begin{aligned} 2\Phi(u_k) - \Phi'(u_k)u_k &= \int_{\mathbb{R}} [f(x, u_k)u_k - 2F(x, u_k)]dx \\ &\geq b \int_{\{t \in \mathbb{R}: |u_k(x)| \geq r\}} |u_k(x)|^\beta \, dx \end{aligned} \tag{4.2}$$

for all positive integer k, which implies that

$$\frac{1}{\|u_k\|} \int_{\{t \in \mathbb{R}: |u_k(x)| \geq r\}} |u_k(x)|^\beta \, dx \to 0 \tag{4.3}$$

as $k \to \infty$. Let

$$v_k(x) = \begin{cases} u_k(x) & \text{if } |u_k(x)| \leq r, \\ 0 & \text{if } |u_k(x)| > r, \end{cases} \tag{4.4}$$

and

$$w_k(x) = u_k(x) - v_k(x) \tag{4.5}$$

for all positive integer k and all $x \in \mathbb{R}$. By (4.2) and (4.5), we get

$$c_1(1 + \|u_k\|) \geq b \|w_k\|_{L^\beta}^\beta \tag{4.6}$$

for all positive integer k. It follows from Hölder's inequality, (4.4), (4.5) and the equivalence of the norms on the finite dimensional subspace $E^- \oplus E^0$ that

$$\begin{aligned} \left\|u_k^- + u_k^0\right\|_{L^2}^2 &= \langle u_k^- + u_k^0, v_k \rangle_{L^2} + \langle u_k^- + u_k^0, w_k \rangle_{L^2} \\ &\leq \left\|u_k^- + u_k^0\right\|_{L^1} \|v_k\|_{L^\infty} + \left\|u_k^- + u_k^0\right\|_{L^{\beta'}} \|w_k\|_{L^\beta} \\ &\leq c_2 \left\|u_k^- + u_k^0\right\|_{L^2} (1 + \|w_k\|_{L^\beta}) \end{aligned} \tag{4.7}$$

for all positive integer k, where $\beta' = \frac{\beta}{\beta - 1}$ $(\beta > 1)$ is the Hölder's conjugate of β.

From the equivalence of the norms on the finite dimensional subspace $E^- \oplus E^0$, (4.6) and (4.7) we obtain

$$\left\|u_k^- + u_k^0\right\| \le c_3 \left\|u_k^- + u_k^0\right\|_{L^2} \le c_4(1 + \|w_k\|_{L^\beta}) \le c_5(1 + \|u_k\|^{\frac{1}{\beta}})$$

for all positive integer k, which implies that

$$\frac{\left\|u_k^- + u_k^0\right\|}{\|u_k\|} \to 0 \tag{4.8}$$

as $k \to \infty$. By (F_8) and (2.4), one sees that

$$\begin{aligned}\Phi'(u_k)u_k^+ \ge\ & \left\|u_k^+\right\|^2 - \int_{\mathbb{R}} |f(x,u_k)| \left|u_k^+\right| dx \\ \ge\ & \left\|u_k^+\right\|^2 - a\eta_\infty \left\|u_k^+\right\| r^{\alpha-\beta} \int_{\{t\in\mathbb{R}:|u_k(x)|\ge r\}} |u_k|^\beta\, dx \\ & - ar^\alpha \eta_1 \left\|u_k^+\right\| - a\eta_1 \left\|u_k^+\right\|\end{aligned}$$

which, by (4.3), implies

$$\frac{\left\|u_k^+\right\|}{\|u_k\|} \to 0 \tag{4.9}$$

as $k \to \infty$. Hence by (4.8) and (4.9), we obtain

$$1 = \frac{\|u_k\|}{\|u_k\|} \le \frac{\left\|u_k^- + u_k^0\right\| + \left\|u_k^+\right\|}{\|u_k\|} \to 0$$

as $k \to \infty$, which is a contradiction. Hence (u_k) must be bounded. Moreover we have

$$\left\|u_k^+ - u^+\right\|^2 = (\Phi'(u_k) - \Phi'(u))(u_k^+ - u^+) + (\varphi'(u_k) - \varphi'(u))(u_k^+ - u^+).$$

Going to a subsequence if necessary, we may assume, by using Lemma 2.2, that $u_k \rightharpoonup u$ weakly in E and

$$u_k \to u \text{ in both } L^2(\mathbb{R}) \text{ and } L^\infty(\mathbb{R}) \text{ as } k \to \infty. \tag{4.10}$$

Since φ' is continuous, we deduce that $\varphi'(u_k) \to \varphi'(u)$ and therefore $u_k^+ \to u^+$ in E. From (4.10) and the equivalence of the norms on the finite dimensional subspace $E^- \oplus E^0$ we obtain that $u_k^0 \to u^0$ and $u_k^- \to u^-$ in E as $k \to \infty$. Hence (u_k) has a convergent subsequence, which shows that the (PS) condition holds. The proof of Lemma 4.6 is achieved. □

Lemma 4.7. *Assume that* $(\mathcal{A}_\sigma)$, (F_6) *and* (F_7) *are satisfied. Then there are constants* $\rho > 0$ *and* $\delta > 0$ *such that*

$$\Phi_{|S} \ge \delta,$$

where

$$S = \left\{u \in E^+ : \|u\| = \rho\right\}.$$

Proof. By (F_7), for all $\epsilon > 0$, there exists $\nu > 0$ such that

$$|f(x,u)| \leq \epsilon |u|, \ \forall x \in \mathbb{R}, \ \forall |u| \leq \nu,$$

which with (F_6) and the Mean Value Theorem gives

$$F(x,u) = \int_0^1 f(x,su)uds \leq \frac{\epsilon}{2}|u|^2, \ \forall x \in \mathbb{R}, \ \forall |u| \leq \nu.$$

Choose $\epsilon = (2\eta_2^2)^{-1}$ and take $\rho = \frac{\nu}{\eta_\infty}$, $\delta = \frac{\rho^2}{4}$. By (2.4), we get

$$\Phi(u) \geq \frac{1}{2}\|u\|^2 - \frac{\epsilon}{2}\int_{\mathbb{R}} |u(x)|^2 dx = \frac{1}{4}\|u\|^2 = \frac{\rho^2}{4} = \delta$$

for all $u \in S$. The proof of Lemma 4.7 is completed. □

Proof of Theorem 4.1

Lemma 4.8. *Assume that* $(\mathcal{A}_\sigma)$, (F_5), (F_6) *and* (F_9) *are satisfied. Let* $e \in E^+$ *with* $\|e\| = 1$. *Then there exist* $r_1, r_2 > 0$ *such that*

$$\Phi(u) \leq 0, \ \forall u \in \partial\Lambda$$

where

$$\Lambda = \{se : 0 \leq s \leq r_1\} \oplus \left\{u \in E^- \oplus E^0 : \|u\| \leq r_2\right\}.$$

Proof. Let $e \in E^+$ with $\|e\| = 1$ and $F = span\{e\} \oplus E^- \oplus E^0$. By the proof of Lemma 3.4, there exists a constant $\epsilon_0 > 0$ such that

$$meas(\{x \in \mathbb{R} : |u(x)| \geq \epsilon_0 \|u\|\}) \geq \epsilon_0, \ \forall u \in F \setminus \{0\}. \tag{4.11}$$

For $u = u^- + u^0 + u^+ \in F$, let

$$\Omega_u = \{x \in \mathbb{R} : |u(x)| \geq \epsilon_0 \|u\|\}.$$

By (F_5), for $d = \frac{1}{2\epsilon_0^3} > 0$, there exists $R_1 > 0$ such that

$$F(x,u) \geq d|u|^2, \ \forall x \in \mathbb{R}, \ \forall |u| \geq R_1.$$

Hence one has

$$F(x,u(x)) \geq d|u(x)|^2 \geq d\epsilon_0^2 \|u\|^2 \tag{4.12}$$

for all $u \in F$ with $\|u\| \geq \frac{R_1}{\epsilon_0}$ and $x \in \Omega_u$. It follows from (F_9), (4.11) and (4.12) that

$$\begin{aligned} \Phi(u) &\leq \frac{1}{2}\|u^+\|^2 - \int_{\Omega_u} F(x,u(x))dx \\ &\leq \frac{1}{2}\|u^+\|^2 - d\epsilon_0^2\|u\|^2 meas(\Omega_u) \\ &\leq \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u\|^2 \leq 0 \end{aligned} \tag{4.13}$$

for all $u \in F$ with $\|u\| \geq \frac{R_1}{\epsilon_0}$. Let $r_1 > 0$ and denote

$$\Lambda = \{se : 0 \leq s \leq r_1\} \oplus \left\{u \in E^- \oplus E^0 : \|u\| \leq r_1\right\}.$$

Then we have

$$\partial\Lambda = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3,$$

where

$$\Lambda_1 = \left\{u \in E^- \oplus E^0 : \|u\| \leq r_1\right\},$$

$$\Lambda_2 = r_1 e + \left\{u \in E^- \oplus E^0 : \|u\| \leq r_1\right\},$$

$$\Lambda_3 = \{se : 0 \leq s \leq r_1\} \oplus \left\{u \in E^- \oplus E^0 : \|u\| = r_1\right\}.$$

By (4.13), one has

$$\Phi(u) \leq 0, \ \forall u \in \Lambda_2 \cup \Lambda_3$$

for all $r_1 \geq \frac{R}{\epsilon_0}$. From (F_6), we have

$$\Phi(u) \leq 0, \ \forall u \in E^- \oplus E^0,$$

which implies that

$$\Phi(u) \leq 0, \ \forall u \in \Lambda_1.$$

Hence we have

$$\Phi(u) \leq 0, \ \forall u \in \partial\Lambda,$$

for all $r_1 > \max\left\{\rho, \frac{R_1}{\epsilon_0}\right\}$, where ρ is defined in Lemma 4.7, which completes the proof of Lemma 4.8. □

By Lemma 4.4, Φ has a critical point u satisfying $\Phi(u) \geq \delta > 0$ where δ is given by Lemma 4.8. Since $\Phi(0) = 0$, then u is nontrivial and $(\mathcal{F})$ possesses a nontrivial homoclinic solution. The proof of Theorem 4.1 is achieved.

Proof of Theorem 4.2

We have $\Phi(0) = 0$ and since $F(x,u)$ is even with respect to the second variable, then Φ is even. The assumptions (Φ_1), (Φ_2) are proved above. Let us prove $(\Phi_3^{'})$. Let $\tilde{E} \subset E$ be a finite dimensional subspace of E, there exists $m \geq 1$ such that $\tilde{E} \subset E^- \oplus E^0 \oplus span\{w_1, \ldots, w_m\} = X^m$, where $w_k = e_{\bar{k}+k}$, $k \geq 1$. Replacing the subspace $F = span\{e\} \oplus E^- \oplus E^0$, introduced in the proof of Lemma 4.8, by the subspace X^m and following the same steps, we obtain $R_m > 0$ such that

$$\Phi(u) \leq \frac{1}{2}\left\|u^+\right\|^2 - \frac{1}{2}\|u\|^2 \leq 0, \ \forall u \in X^m, \ \|u\| \geq R_m.$$

Hence $(\Phi_3^{'})$ is verified. Therefore, by Lemma 4.5, Φ possesses an unbounded sequence of critical points. Hence $(\mathcal{F})$ possesses infinitely may homoclinic solutions. The proof of Theorem 4.2 is completed.

5. Local conditions

In this Section, we are interested in a general case where the potential $F(x,u)$ satisfies only locally conditions near the origin with respect to u and do not satisfy any additional hypotheses at infinity. More precisely, we present the following assumptions:

(F_{10}) There exist constants $r, c > 0$ and $\nu \in]0,1[$ such that

$$|f(x,u)| \leq c\,|u|^{\nu}\,,\ \forall x \in \mathbb{R},\ |u| \leq r;$$

(F_{11}) There exists $\rho \in]0,r]$ such that

$$F(x,-u) = F(x,u),\ \text{and}\ F(x,u) \geq 0,\ \forall x \in \mathbb{R}\ \ |u| \leq \rho;$$

(F_{12}) $\lim_{|u|\to 0} \frac{|F(x,u)|}{|u|^2} = +\infty,$ uniformly for all $x \in \mathbb{R}$.

Our main result in this Section reads as follows.

Theorem 5.1. *Assume that $(\mathcal{A}_\sigma)$ and (F_{10})-(F_{12}) are satisfied. Then $(\mathcal{F})$ possesses infinitely many nontrivial homoclinic orbits (u_k) such that*

$$\Phi(u_k) = \frac{1}{2}\int_{\mathbb{R}}[u_k^{''}(x)^2 - \omega u_k^{'}(x)^2 + a(x)u_k(x)^2]dx - \int_{\mathbb{R}} F(t,u_k(x))dx \to 0$$

as $k \to \infty$.

In the following, we give some examples which satisfy our assumptions.

Example 5.2 (The subquadratic case at infinity)**.** Let

$$F(x,u) = h(x)\,|u|^{\theta}\ln(1+|u|^2),$$

where $(x,u) \in \mathbb{R}^2$, $\theta \in]1,2[$ and $h \in C(\mathbb{R},\mathbb{R})$ with $0 < \inf_{x\in\mathbb{R}} h(x) \leq \sup_{x\in\mathbb{R}} h(x) < \infty$. It is easy to see that $F(x,u)$ satisfies the conditions (F_{10})-(F_{12}) and the subquadratic condition at infinity, i.e.,

$$\lim_{|u|\to\infty} \frac{F(x,u)}{|u|^2} = 0.$$

Example 5.3 (The superquadratic case at infinity)**.** Let

$$F(x,u) = h(x)(|u|^{\nu} + |u|^{\theta}\ln(1+|u|^2))$$

where $(x,u) \in \mathbb{R}\times\mathbb{R}$, $\theta \in]1,2[$, $\nu \in]2,\infty[$ and $h \in C(\mathbb{R},\mathbb{R})$ with $0 < \inf_{x\in\mathbb{R}} h(x) \leq \sup_{x\in\mathbb{R}} h(x) < \infty$. It is easy to see that $F(x,u)$ satisfy the conditions (F_{10})-(F_{12}) and the superquadratic condition at infinity, i.e.,

$$\lim_{|u|\to\infty} \frac{F(x,u)}{|u|^2} = +\infty.$$

Example 5.4 (The asymptotically quadratic case at infinity)**.** Let

$$F(x,u) = \frac{1}{2}S(x)u^2 + |u|^{\theta}\ln(1+|u|^2),$$

where $S:\mathbb{R}\to\mathbb{R}$ is a continuous bounded function and $\theta\in]1,2[$. It is clear that $F(x,u)$ is asymptotically quadratic at infinity with respect to u and satisfies the conditions (F_{10})-(F_{12}).

Proof of Theorem 5.1

Consider the continuously differentiable functional $\Phi:E\to\mathbb{R}$ introduced in Section 3 whose critical points on E are the homoclinic solutions of the equation $(\mathcal{F})$.

We shall use the following Variant Symmetric Mountain Pass Lemma due to Kajikiya [5] to prove our result. We will first recall the notion of genus.

Let E be a Banach space and let A be a subset of E. A is said to be symmetric if $u\in A$ implies $-u\in A$. For a closed symmetric set A which does not contain the origin, we define the genus $\gamma(A)$ of A by the smallest integer k for which there exists an odd continuous mapping from A to $\mathbb{R}^k\setminus\{0\}$. If such a k does not exist, we define $\gamma(A)=+\infty$. Moreover, we set $\gamma(\phi)=0$. Let

$$\Gamma_k=\{A\subset E: A \text{ is a closed symmetric subset, } 0\notin A,\ \gamma(A)\geq k\}.$$

The properties of genus used in the proof of our main result are summarized as follows.

Lemma 5.5 ([5])**.** *Let A and B be closed symmetric subsets of E that do not contain the origin. Then the following hold.*

a) *If $A\subset B$, then $\gamma(A)\leq\gamma(B)$.*

b) *The n-dimensional sphere S^n has a genus of $n+1$ by the Borsuk-Ulam theorem.*

Lemma 5.6 ([5])**.** *Let E be an infinite-dimensional Banach space and $\Phi\in C^1(E,\mathbb{R})$ satisfies the following.*

(Φ_1) *$\Phi(0)=0$, Φ is even and bounded from below and Φ satisfies the (PS)-condition;*

(Φ_2) *For each $k\in\mathbb{N}$, there exists $A_k\subset\Gamma_k$ such that*

$$\sup_{u\in A_k}\Phi(u)<0.$$

Then Φ possesses a sequence of critical points (u_k) such that

$$\Phi(u_k)\leq 0,\ u_k\neq 0,\ \forall k\in\mathbb{N} \text{ and } \lim_{k\to\infty}u_k=0.$$

Now, let $\theta\in C^1(\mathbb{R}^+,\mathbb{R}^+)$ satisfying

$$\begin{cases}\theta(s)=1 \text{ for } s\in[0,\frac{\rho}{2\eta_\infty}],\ \theta(s)=0 \text{ for } s\geq\frac{\rho}{\eta_\infty},\\ \theta'(s)<0 \text{ for } \frac{\rho}{2\eta_\infty}<s<\frac{\rho}{\eta_\infty},\end{cases}\tag{5.1}$$

where ρ is defined in (F_{11}). Consider the new functional ψ defined on E by

$$\psi(u)=\frac{1}{2}\|u\|^2-\theta(\|u\|)\Big(\|u^-\|^2+\frac{1}{2}\|u^0\|^2+\int_{\mathbb{R}}F(x,u)dx\Big).$$

Remark 5.7. It is clear that $\psi \in C^1(E,\mathbb{R})$, $\psi(u) = \Phi(u)$ for all $\|u\| \leq \frac{\rho}{2\eta_\infty}$ and thus critical points of ψ satisfying $\|u\| \leq \frac{\rho}{2\eta_\infty}$ are exactly critical points of Φ. Consequently, to prove our result, we will apply Lemma 5.6 to the functional ψ instead of Φ.

Lemma 5.8. *Assume that* $(\mathcal{A}_\sigma)$, (F_{10}) *and* (F_{11}) *are satisfied. Then* ψ *satisfies the* (PS)*-condition.*

Proof. Let (u_n) be a Palais-Smale sequence, that is $(\psi(u_n))$ is bounded and $\psi'(u_n) \to 0$ as $n \to \infty$. If $u \in E$ with $\|u\| \geq \frac{\rho}{\eta_\infty}$, then by the definition of θ and ψ, we have

$$\psi(u) = \frac{1}{2}\|u\|^2,$$

which implies that

$$\psi(u) \to +\infty \text{ as } \|u\| \to \infty. \tag{5.2}$$

Since $(\psi(u_n))$ is bounded, then (5.2) implies that (u_n) is bounded. Thus, passing to a subsequence if necessary, we can assume by Lemma 2.2 that $u_n \rightharpoonup u = u^- + u^0 + u^+$, $u_n^+ \rightharpoonup u^+$ and $u_n^+ \to u^+$ in $L^1(\mathbb{R})$. On the other hand, if $\|u_n\| \geq \frac{\rho}{\eta_\infty}$, we have $\psi'(u_n)u_n = \|u_n\|^2 \geq \frac{\rho^2}{\eta_\infty^2}$ contradicting the fact that $\psi'(u_n) \to 0$ as $n \to \infty$. Hence, we can assume that $\|u_n\| \leq \frac{\rho}{\eta_\infty}$ for all $n \in \mathbb{N}$, which with (2.4) implies that

$$|u_n(x)| \leq \|u_n\|_{L^\infty} \leq \eta_\infty \|u_n\| \leq \rho.$$

This, jointly with (F_{10}) implies that

$$\begin{aligned} \left|\int_{\mathbb{R}} f(x,u_n)(u_n^+ - u^+)dx\right| &\leq \int_{\mathbb{R}} |f(x,u_n)|\,|u_n^+ - u^+|\,dx \\ &\leq c\rho^\nu \int_{\mathbb{R}} |u_n^+ - u^+|\,dx \to 0 \end{aligned} \tag{5.3}$$

as $n \to \infty$.

Now, we have

$$\begin{aligned} &\psi'(u_n)(u_n^+ - u^+) \\ &= \langle u_n, u_n^+ - u^+\rangle - \theta'(\|u_n\|)\langle \frac{u_n}{\|u_n\|}, u_n^+ - u^+\rangle \Big(\frac{1}{2}\|u_n^0\|^2 + \|u_n^-\|^2 + \int_{\mathbb{R}} W(t,u_n)dx\Big) \\ &\quad - \theta(\|u_n\|)\int_{\mathbb{R}} f(x,u_n)(u_n^+ - u^+)dx, \end{aligned}$$

which with (5.3) and the fact that $\psi'(u_n) \to 0$ implies

$$\Big[1 - \frac{\theta'(\|u_n\|)}{\|u_n\|}\Big(\frac{1}{2}\|u_n^0\|^2 + \|u_n^-\|^2 + \int_{\mathbb{R}} W(t,u_n)dx\Big)\Big]\langle u_n, u_n^+ - u^+\rangle \to 0 \tag{5.4}$$

as $n \to \infty$. Since $\|u_n\| \leq \frac{\rho}{\eta_\infty}$ then $|u_n(x)| \leq \rho$ and $F(x, u_n(x)) \geq 0$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$ by (F_{11}). Hence, the definition of θ implies

$$1 - \frac{\theta'(\|u_n\|)}{\|u_n\|}\Big(\frac{1}{2}\left\|u_n^0\right\|^2 + \left\|u_n^-\right\|^2 + \int_{\mathbb{R}} F(x, u_n)dx\Big) \geq 1, \ \forall n \in \mathbb{N}.$$

It follows from (5.4) that

$$\langle u_n, u_n^+ - u^+\rangle \to 0 \text{ as } n \to \infty.$$

By virtue of $u_n^+ \rightharpoonup u^+$, we have $\|u_n^+\| \to \|u^+\|$ and then $u_n^+ \to u^+$.

Noting that E^- and E^0 are finite dimensional subspaces, so we have $u_n^- \to u^-$ and $u_n^0 \to u^0$. Therefore $u_n \to u$ in E and ψ satisfies the (PS)-condition. The proof of Lemma 5.8 is completed. □

Now, the definitions of ψ and θ imply that $\psi(u) = \frac{1}{2}\|u\|^2 = \psi(-u)$ for all $\|u\| \geq \frac{\rho}{\eta_\infty}$. If $\|u\| \leq \frac{\rho}{\eta_\infty}$, we have as above $|u(x)| \leq \rho$ for all $x \in \mathbb{R}$, which together with (F_{11}) implies $F(x, -u(x)) = F(x, u(x))$ for all $x \in \mathbb{R}$ and $\psi(-u) = \psi(u)$. Thus ψ is even in E.

We claim that ψ is bounded from below. If not, there exists a sequence (u_n) such that

$$\psi(u_n) \to -\infty \text{ as } n \to \infty. \tag{5.5}$$

By (F_{10}), (F_{11}) and the definitions of ψ and θ, it is easy to verify that ψ maps bounded sets into bounded sets. It follows from (5.5) that $\|u_n\| \to \infty$ as $n \to \infty$. Thus (5.2) implies that $\psi(u_n) \to +\infty$ as $n \to \infty$, which contradicts (5.5). Hence the condition (Φ_1) of Lemma 5.6 is verified.

Finally, we show that ψ satisfies condition (Φ_2) of Lemma 5.6. For any positive integer k, let

$$E_k = \oplus_{m=1}^k X_m, \ X_m = \mathbb{R}e_m,$$

where the sequence (e_m) is defined in Section 2. Since E_k is finite dimensional, there exists a positive constant β_k such that

$$\|u\| \leq \beta_k \|u\|_{L^2}, \ \forall u \in E_k. \tag{5.6}$$

By (F_{12}), there exists a constant $R > 0$ such that

$$F(x, u) \geq \beta_k^2 |u|^2, \ \forall x \in \mathbb{R}, \ |u| \leq R. \tag{5.7}$$

Let $u \in E$ such that $\|u\| \leq \frac{R}{\eta_\infty}$, we know that $|u(x)| \leq R$ for all $x \in \mathbb{R}$, thus by (5.7) we get

$$F(x, u(x)) \geq \beta_k^2 |u(x)|^2, \ \forall x \in \mathbb{R}. \tag{5.8}$$

Therefore, by (5.6) and (5.8), for all $u \in E_k$ with $0 < \|u\| = r_k \leq \min\left\{\frac{\rho}{2\eta_\infty}, \frac{R}{\eta_\infty}\right\}$, we have

$$\psi(u) \leq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} \beta_k^2 |u(x)|^2 dx - \frac{1}{2}\|u\|^2 = -\frac{1}{2}r_k^2,$$

which implies

$$\{u \in E_k \setminus \{0\} : \|u\| = r_k\} \subset A_k, \tag{5.9}$$

where

$$A_k = \left\{ u \in E : \psi(u) \leq -\frac{1}{2}\eta_k^2 \right\}.$$

Thus Lemma 5.5 and (5.9) imply

$$\gamma(A_k) \geq \gamma\Big(\{u \in E_k \setminus \{0\} : \|u\| = r_k\} \Big) \geq k$$

hence, by the definition of Γ_k, we have $A_k \subset \Gamma_k$. Moreover, the definition of A_k implies

$$\sup_{u \in A_k} \psi(u) \leq -\frac{1}{2}\eta_k^2 < 0.$$

All the conditions of Lemma 5.6 hold and the proof of Theorem 5.1 is finished by this Lemma.

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