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SHARP COEFFICIENT INEQUALITIES FOR CERTAIN SUBCLASSES OF BI-UNIVALENT BAZILEVIČ FUNCTIONS

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Abstract. In the present paper, we introduce the subclasses $\mathfrak{B}_{1\Sigma}(\mu)$, $\mathrm{B}_{1\Sigma}(\mu,\gamma)$ and $U_\Sigma(\mu,\gamma)$ of bi-univalent Bazilevič functions which are defined in the open unit disk $\mathbb{D}$. Further, we obtain sharp estimates on initial coefficients a_2, a_3, a_4 and also sharp estimate on the Fekete-Szegö functional $a_3 - ka_2^2$ for the functions belong to these subclasses.

1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \mathbb{D}, n \in \mathbb{N}, a_n \in \mathbb{C}), \tag{1}$$

which are defined in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and satisfies the standard normalization conditions $[f(z) = 0, f'(z) = 1]_{z=0}$. Also, let $\mathcal{S}$ represent the subclass of $\mathcal{A}$, that contain functions of the form (1) which are univalent in $\mathbb{D}$. The Koebe function $z/(1-z)^2 = z + 2z^2 + 3z^3 + \cdots$ is the most important member of the class $\mathcal{S}$. Further, let $\mathcal{S}^*$ denote the subclass of $\mathcal{S}$, that contain functions which are star-like in $\mathbb{D}$. Whereas, $f \in \mathcal{A}$ given by (1) is known as a star-like function if it maps the open unit disk $\mathbb{D}$ to a star-like domain with respect to the origin. In addition, if $f \in \mathcal{S}^*$, then we have:

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad (z \in \mathbb{D}).$$

It is well known that if $f \in \mathcal{S}^*$, then $|a_n| \le n$ for every $n > 1$, $(n \in \mathbb{N})$ and the result is sharp for the Koebe function.

According to the Koebe one-quarter theorem (see [7]), the image of $\mathbb{D}$ under every $f \in \mathcal{S}$ contains a disk of radius one-quarter centered at origin. Thus, every $f \in \mathcal{S}$ has an inverse $f^{-1} : f(\mathbb{D}) \to \mathbb{D}$ that satisfies $f^{-1}(f(z)) = z$; $(|z| < 1)$ and $f(f^{-1}(w)) = w$; $(|w| < r_0(f), r_0(f) \ge 1/4)$. Moreover, for $f \in \mathcal{S}$, an

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analytic, univalent continuation of the inverse function $f^{-1} \equiv g$ to $\mathbb{D}$ is of the form:

$$g(w) = w + (-a_2)\, w^2 + (2a_2^2 - a_3)w^3 + (5a_2a_3 - 5a_2^3 - a_4)w^4 + \cdots . \tag{2}$$

A function $f \in \mathcal{S}$ given by (1), is said to be bi-univalent if $f^{-1} \in \mathcal{S}$ and the class of all such functions denoted by Σ is said to be the bi-univalent function class. The functions $z/\,(1-z)$, $-\log(1-z)$ and $(1/2)\log[(1+z)\,/\,(1-z)]$ are the members of the class Σ. However, the functions $z - \left(z^2/2\right)$, $z/\left(1-z^2\right)$ and also the Koebe function are not the members of Σ.

Lewin [12] introduced the concept of class Σ and proved that $|a_2| < 1.51$ for functions in it. After which, Brannan and Clunie [5] proved that $|a_2|_{f\in\Sigma} \leq \sqrt{2}$. Later, Netanyahu [15] showed that $\max |a_2|_{f\in\Sigma} = 4/3$, whereas Styer and Wright [24] showed the existence of $f \in \Sigma$ for which $|a_2| > 4/3$. Further, Tan [25] proved that $|a_2| \leq 1.485$ for functions in Σ. After invention of the class Σ, many researchers have been working to find out the connection between the coefficient bounds and geometrical properties of the functions in it.

Indeed, Lewin [12], Brannan and Taha [6], Srivastava et al. [23] etc. provided a solid base for the study of bi-univalent functions. After which, many researchers viz. [9,10,18,20,22] (also see the references therein) introduced several subclasses of Σ and found estimates on initial coefficients for functions in them. However, still the problem of sharp coefficient bound for $|a_n|$, $(n = 3, 4, 5, \ldots)$ is open.

Ram Singh [21] introduced the class $\mathcal{B}_1(\mu)$ of Bazilevič functions, that consist of functions $f \in \mathcal{A}$ for which:

$$\Re\left\{\left[\frac{f(z)}{z}\right]^{\mu-1} f^{'}(z)\right\} > 0, \quad (z \in \mathbb{D}, \mu \geq 0)\,.$$

In fact, it is known (see [13]) that $\mathcal{B}_1(\mu) \subset \mathcal{S}$ and $\mathcal{B}_1(0) \equiv \mathcal{S}^*$. Moreover, the subclass $\mathcal{B}_1(1)$ satisfies the condition $\Re\{f^{'}(z)\} > 0$, $z \in \mathbb{D}$ and reduce to the close-to-convex class. Singh [21] and Ali et al. [3] have used two different ways to obtain sharp bounds for first three coefficients of the class $\mathcal{B}_1(\mu)$.

We need the following lemmas to prove our main results.

Lemma 1.1 ([7]). *Let $\mathcal{P}$ denote the class of all analytic functions in $\mathbb{D}$ with positive real part. For $z \in \mathbb{D}$ if*

$$P(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in \mathcal{P},$$

then $|c_n| \leq 2$ for each $n \in \mathbb{N}$.

Lemma 1.2 ([14]). *If the functions F_1 and F_2 defined by*

$$F_1 = 1 + \sum_{n=1}^{\infty} b_n z^n \text{ and } F_2 = 1 + \sum_{n=1}^{\infty} c_n z^n \text{ belong to the class } \mathcal{P},$$

then the function $F = 1 + \frac{1}{2}\sum_{n=1}^{\infty} b_n c_n z^n$ *also belong to* $\mathcal{P}$.

Lemma 1.3 ([14]). *Let* $h(z) = 1 + \alpha_1 z + \alpha_2 z^2 + \cdots$ *and* $t(z) = 1 + t_1(z) = 1 + \beta_1 z + \beta_2 z^2 + \cdots$ *be the functions of the class* $\mathcal{P}$ *and for* $m, k \in \mathbb{N}$ *set*

$$\eta_m = \frac{1}{2^m}\left[1 + \frac{1}{2}\sum_{k=1}^{m}\binom{m}{k}\alpha_k\right], \quad \eta_0 = 1. \tag{3}$$

If A_n *is defined by*

$$\sum_{n=1}^{\infty} A_n z^n = \sum_{n=1}^{\infty} (-1)^{n+1}\,\eta_{n-1} t_1^n(z), \tag{4}$$

then for each $n \in \mathbb{N}$,

$$|A_n| \le 2.$$

In this paper, we define the subclasses $\mathfrak{B}_{1\Sigma}(\mu)$, $\mathrm{B}_{1\Sigma}(\mu,\gamma)$ and $U_{\Sigma}(\mu,\gamma)$ of Σ that are associated with the Bazilevič functions (for more details about Bazilevič functions see [4, 26, 27]). Moreover, by using the method of Ram Singh [21] along with the equating coefficient trick of Srivastava et al. [23], we obtain sharp bounds for the coefficients a_2, a_3 and a_4 for the functions belong to these subclasses.

2. Coefficient estimates for the class $\mathfrak{B}_{1\Sigma}(\mu)$

Definition. A function $f(z) \in \Sigma$ of the form (1) is said to be in the class $\mathfrak{B}_{1\Sigma}(\mu)$; $(\mu > 0)$ if the following two conditions are fulfilled:

$$\Re\left\{\frac{zf'(z)f(z)^{\mu-1}}{z^\mu}\right\} > 0, \quad (z \in \mathbb{D})$$

and

$$\Re\left\{\frac{wg'(w)g(w)^{\mu-1}}{w^\mu}\right\} > 0, \quad (w \in \mathbb{D}),$$

where g is of the form (2), be an extension of f^{-1} to $\mathbb{D}$.

Theorem 2.1. *Let* $f(z) \in \mathfrak{B}_{1\Sigma}(\mu), (\mu > 0)$ *be given by* (1). *Then we have the following sharp estimates*:

$$|a_2| \le \begin{cases} \sqrt{\frac{2(3+\mu)}{(2+\mu)(1+\mu)^2}}, & 0 < \mu \le 1, \\ \sqrt{\frac{2}{2+\mu}}, & \mu \ge 1, \end{cases} \tag{5}$$

$$|a_3| \le \begin{cases} \frac{2(3+\mu)}{(2+\mu)(1+\mu)^2}, & 0 < \mu \le 1, \\ \frac{2}{2+\mu}, & \mu \ge 1, \end{cases} \tag{6}$$

$$|a_4| \le \begin{cases} \frac{2}{3+\mu} + \frac{4(1-\mu)\left(5+3\mu+\mu^2\right)}{3(2+\mu)(1+\mu)^3}, & 0 < \mu \le 1, \\ \frac{2}{3+\mu}, & \mu \ge 1. \end{cases} \tag{7}$$

Proof. Since $f \in \mathfrak{B}_{1\Sigma}(\mu)$, by definition we have

$$\frac{z^{1-\mu} f'(z)}{f(z)^{1-\mu}} = P(z), \tag{8}$$

$$\frac{w^{1-\mu} g'(w)}{g(w)^{1-\mu}} = Q(w), \tag{9}$$

for some $P(z), Q(w) \in \mathcal{P}$. On setting

$$P(z) = 1 + c_1 z + c_2 z^2 + \cdots + c_n z^n + \cdots , \tag{10}$$

$$Q(w) = 1 + d_1 w + d_2 w^2 + \cdots + d_n w^n + \cdots \tag{11}$$

and then comparing the coefficients in (8) and (9) we obtain

$$(1+\mu)\, a_2 = c_1, \tag{12}$$

$$(2+\mu)\, a_3 = c_2 + (1-\mu)\, c_1 a_2 - \frac{\mu}{2}(1-\mu)\, a_2^2, \tag{13}$$

$$\begin{aligned}(3+\mu)\, a_4 = c_3 + (1-\mu)\, c_2 a_2 + \left[(1-\mu)\, a_3 - \frac{\mu}{2}(1-\mu)\, a_2^2\right] c_1 \\ - \mu(1-\mu)\, a_2 a_3 + \frac{\mu(1-\mu)(1+\mu)}{6} a_2^3,\end{aligned} \tag{14}$$

and

$$-(1+\mu)\, a_2 = d_1, \tag{15}$$

$$(2+\mu)\left(2a_2^2 - a_3\right) = d_2 - (1-\mu)\, d_1 a_2 - \frac{\mu}{2}(1-\mu)\, a_2^2, \tag{16}$$

$$\begin{aligned}-(3+\mu)\left(5a_2^3 - 5a_2 a_3 + a_4\right) = d_3 &- (1-\mu)\, d_2 a_2 \\ &- \left[(1-\mu)\, a_3 + \frac{\mu}{2}(1-\mu)\, a_2^2\right] d_1 \\ &- 2(1-\mu)\, a_2 a_3 \\ &- \left[2(1-\mu) + \frac{\mu(1-\mu)(1+\mu)}{6}\right] a_2^3.\end{aligned} \tag{17}$$

Equation (12) and (15) together yields

$$c_1 = -d_1 \quad \text{and} \quad |a_2| \le \frac{2}{1+\mu}.$$

Adding equations (13) and (16) we get

$$a_2^2 = \frac{(c_2 + d_2)}{2(2+\mu)} + \frac{(1-\mu)}{2(1+\mu)^2} c_1^2, \tag{18}$$

whereas, subtracting (16) from (13) and then using (18), we get

$$a_3 = \frac{c_2}{(2+\mu)} + \frac{(1-\mu)}{2(1+\mu)^2} c_1^2. \tag{19}$$

Application of the fact $|c_n| \leq 2, |d_n| \leq 2$; $n := 1, 2, \ldots$ in (18) and (19) along with $0 < \mu \leq 1$ proves the first part of the inequalities (5) and (6).

To prove the second part, we use

$$c_2 = \frac{1}{2}c_1^2 + \delta\left(2 - \frac{1}{2}|c_1|^2\right), \quad |\delta| \leq 1, \tag{20}$$

which is a consequence of the Carathéodory-Toeplitz inequality:

$$\left|c_2 - \frac{1}{2}c_1^2\right| \leq 2 - \frac{1}{2}|c_1|^2.$$

Performing elementary calculations along with the equality (20), we obtain the second part of the inequalities (5) and (6).

Observe that the first parts of the inequalities for a_2 and a_3 are sharp for the functions f_1 and g_1 defined by:

$$\frac{z^{1-\mu} f_1'(z)}{f_1(z)^{1-\mu}} = \frac{1+z}{1-z} \quad \text{and} \quad \frac{w^{1-\mu} g_1'(w)}{g_1(w)^{1-\mu}} = \frac{1-w}{1+w},$$

whereas the second parts are sharp for f_2 and g_2 that satisfies:

$$\frac{z^{1-\mu} f_2'(z)}{f_2(z)^{1-\mu}} = \frac{1+z^2}{1-z^2} \quad \text{and} \quad \frac{w^{1-\mu} g_2'(w)}{g_2(w)^{1-\mu}} = \frac{1+w^2}{1-w^2}.$$

Next, for proof of the third inequality we compile the outputs of addition and subtraction of equations (14) and (17), which yields

$$\begin{aligned}(3+\mu)\, a_4 = {} & c_3 + (1-\mu)\, c_2 a_2 + \left[(1-\mu)\, a_3 - \frac{\mu}{2}\,(1-\mu)\, a_2^2\right] c_1 \\ & - \mu\,(1-\mu)\, a_2 a_3 + \frac{\mu\,(1-\mu)\,(1+\mu)}{6} a_2^3,\end{aligned}$$

which, by eliminating a_2 and a_3 produce

$$(3+\mu)\, a_4 = c_3 + \frac{(1-\mu)\,(3+\mu)}{(1+\mu)} \left[\frac{c_1 c_2}{2+\mu} + \frac{(1-2\mu)}{6\,(1+\mu)^2} c_1^3\right]. \tag{21}$$

From this, the first part of the inequality (7) is trivial for $0 < \mu \leq 1/2$. Next, for $1/2 < \mu \leq 1$ we use the equality (20) in (21) to eliminate c_2, which on applying simple calculus implies that the expression in the square bracket of equation (21) attains its maximum when $c_1 = c_2 = 2$. Thus, for $1/2 < \mu \leq 1$, $|a_4|$ attain its maximum when $|c_n| = 2$, $(n := 1, 2, 3)$ which completes the first part of the inequality (7).

Finally, to obtain bound on a_4 for $\mu \geq 1$ we use Lemmas 1.2 and 1.3 given by Nehari and Netanyahu [14]. From the equation (4) we get

$$A_3 = \beta_3 - 2\eta_1 \beta_1 \beta_2 + \eta_2 {\beta_1}^3. \tag{22}$$

Lemma 1.2 along with $|A_n| \leq 2$ and $P(z) = 1 + c_1 z + c_2 z^2 + \cdots \in \mathcal{P}$ yields

$$\left|\frac{1}{2}\beta_3 c_3 - \frac{1}{2}\eta_1 \beta_1 \beta_2 c_1 c_2 + \frac{1}{8}\eta_2 {\beta_1}^3 {c_1}^3\right| \leq 2. \tag{23}$$

On comparing (21) and (23) with $\mu \geq 1$, we conclude that

$$(3+\mu)\left|a_4\right| = \left|c_3 + \frac{(1-\mu)(3+\mu)}{(1+\mu)}\left[\frac{c_1c_2}{2+\mu} + \frac{(1-2\mu)}{6(1+\mu)^2}c_1^3\right]\right| \leq 2,$$

if there exist the functions $h(z), t(z) \in \mathcal{P}$ given by

$h(z) = 1 + \alpha_1 z + \alpha_2 z^2 + \alpha_3 z^3 + \cdots$ and $t(z) = 1 + \beta_1 z + \beta_2 z^2 + \beta_3 z^3 + \cdots$

such that $\beta_3 = 2$,

$$\frac{1}{2}\eta_1\beta_1\beta_2 = \frac{(\mu-1)(3+\mu)}{(1+\mu)(2+\mu)} \quad \text{and} \quad \frac{1}{8}\eta_2{\beta_1}^3 = \frac{(\mu-1)(2\mu-1)(3+\mu)}{6(1+\mu)^3}, \tag{24}$$

where η_m, $(m \in \mathbb{N})$ is given by the equation (3), which implies that

$$\eta_1 = \frac{1}{2}\left(1 + \frac{1}{2}\alpha_1\right) \quad \text{and} \quad \eta_2 = \frac{1}{4}\left(1 + \alpha_1 + \frac{1}{2}\alpha_2\right). \tag{25}$$

Also, on choosing $\beta_1 = \beta_2 = 2$, relations in (24) gives

$$\eta_1 = \frac{(\mu-1)(3+\mu)}{2(1+\mu)(2+\mu)} \quad \text{and} \quad \eta_2 = \frac{(\mu-1)(2\mu-1)(3+\mu)}{6(1+\mu)^3}. \tag{26}$$

Equating the values of η_1 in (25) and (26), we get

$$\alpha_1 = \frac{-2(5+\mu)}{(1+\mu)(2+\mu)}. \tag{27}$$

Since $|\alpha_1| \leq 2$ for all $\mu \geq 1$, this value is acceptable. Next, on equating the values of η_2 in (25) and (26) in light of (27), we obtain

$$\alpha_2 = \frac{2\left(\mu^4 + 5\mu^3 + 11\mu^2 - 19\mu + 36\right)}{3(2+\mu)(1+\mu)^3}, \tag{28}$$

which also satisfies that $|\alpha_2| \leq 2$ for all $\mu \geq 1$. Finally, to construct the functions $h(z)$ and $t(z)$, it is evident to have $t(z) = (1+z)/(1-z) \in \mathcal{P}$ and a suitable choice of $h(z)$ is the function

$$h(z) = \frac{L(1-z)}{(1+z)} + \frac{M\left(1 + Nz^2\right)}{(1 - Nz^2)},$$

where

$$L = \frac{(\mu+5)}{(\mu+1)(\mu+2)}, \quad M = \frac{(\mu-1)(\mu+3)}{(\mu+1)(\mu+2)}$$

and

$$N = \frac{\left(\mu^4 + 2\mu^3 - 10\mu^2 - 14\mu + 21\right)}{3(\mu+1)^2(\mu-1)(\mu+3)}.$$

Observe here that L, M, N are all positive, $L+M = 1$ and $N \leq 1$ for $\mu \geq 1$ and hence we have $h(z) \in \mathcal{P}$. Moreover, on expanding $h(z)$ in ascending powers of z shows that the coefficients of z and z^2 in this expansion are equal to α_1 and α_2 given by equations (27) and (28), respectively. Hence, we have

$$(3+\mu)\left|a_4\right| \leq 2 \quad \text{for} \quad \mu \geq 1,$$

which proves the second part of the inequality (7) and the result is sharp for the functions f_3 and g_3 defined by:

$$\frac{z^{1-\mu}f_3'(z)}{f_3(z)^{1-\mu}}=\frac{1+z^3}{1-z^3}\quad\text{and}\quad\frac{w^{1-\mu}g_3'(w)}{g_3(w)^{1-\mu}}=\frac{1+w^3}{1-w^3}.$$

□

On setting $\mu = 1$ in Theorem 2.1, we get the following corollary as an improvement in Theorem 2 given by Srivastava et al. [23].

Corollary 2.2. *Let $f(z) \in \mathcal{H}_\Sigma(0) \equiv \mathcal{H}_\Sigma$ be given by (1). Then we have the following sharp estimates*:

$$|a_2| \le \sqrt{\frac{2}{3}},\quad |a_3| \le \frac{2}{3},\quad |a_4| \le \frac{1}{2}.$$

Further, as a consequence of Theorem 2.1, we obtain the following result known as the Fekete-Szegö problem for the class $\mathfrak{B}_{1\Sigma}(\mu)$.

Theorem 2.3. *Let $f(z) \in \mathfrak{B}_{1\Sigma}(\mu), (\mu > 0)$ be given by (1). Then we have the following sharp estimate*:

$$|a_3 - ka_2^2| \le \begin{cases} \frac{2(3+\mu)}{(2+\mu)(1+\mu)^2}|1-k|, & 0<\mu\le 1,\\ \frac{2}{2+\mu}|1-k|, & \mu \ge 1,\end{cases}$$

where k is some real number.

Proof. Using equations (18) and (19), we get

$$\begin{aligned} a_3 - ka_2^2 &= \left(\frac{c_2}{(2+\mu)}+\frac{(1-\mu)}{2(1+\mu)^2}c_1^2\right)-k\left(\frac{(c_2+d_2)}{2(2+\mu)}+\frac{(1-\mu)}{2(1+\mu)^2}c_1^2\right)\\ &= \frac{2c_2-k(c_2+d_2)}{2(2+\mu)}+\frac{(1-k)(1-\mu)}{2(1+\mu)^2}c_1^2. \end{aligned}$$

Now, using the equality (20) along with the fact $c_1 = -d_1$ yields

$$a_3 - ka_2^2 = (1-k)\left[\frac{c_1^2+\delta\left(4-|c_1|^2\right)}{2(2+\mu)}+\frac{(1-\mu)}{2(1+\mu)^2}c_1^2\right],\quad |\delta|\le 1,$$

which, in light of Lemma 1.1 gives

$$\left|a_3 - ka_2^2\right| \le |1-k|\left[\frac{2}{(2+\mu)}+\frac{2(1-\mu)}{(1+\mu)^2}\right].$$

This proves the required inequality according to the restrictions on μ. □

Remark 2.4. Clearly, for $k = 0$ Theorem 2.3 gives the sharp bound for $|a_3|$ and for $k = 1$ it shows that $\left|a_3 - a_2^2\right| = 0$.

3. Coefficient estimates for the class $\mathbf{B_{1\Sigma}(\mu,\gamma)}$

We have used the generalization of the univalance criterion appeared in the paper of Aksentév [1] (also see [2] and the result by Ozaki and Nunokawa [17]). According to it, for $f(z) \in \mathcal{A}$, if

$$\left|\frac{z^{1-\mu} f^{'}(z)}{f(z)^{1-\mu}} - 1\right| < 1, \quad (z \in \mathbb{D}, 0 < \mu \leq 1),$$

then $f(z)$ is univalent in $\mathbb{D}$ and hence $f(z) \in \mathcal{S}$. Also, let $T(\mu,\gamma)$ denote the class of functions $f(z) \in \mathcal{A}$ such that

$$\left|\frac{z^{1-\mu} f^{'}(z)}{f(z)^{1-\mu}} - 1\right| < \gamma, \quad (z \in \mathbb{D}, 0 < \gamma \leq 1),$$

where $T(\mu,1) = T(\mu)$. Clearly, $T(\mu,\gamma) \subset T(\mu) \subset \mathcal{S}$. Moreover, for $f(z) \in T(\mu,\gamma)$ (see Kuroki et al. [11]), we have

$$\Re\left(\frac{z^{1-\mu} f^{'}(z)}{f(z)^{1-\mu}}\right) > 1 - \gamma, \quad (z \in \mathbb{D}).$$

Further, Ponnusamy [19] shown that for $f \in \mathcal{A}$,

$$\left|f^{'}(z)\left(\frac{f(z)}{z}\right)^{\mu-1} - 1\right| < \gamma, \quad (z \in \mathbb{D}, \mu > 0, 0 < \gamma < 1)$$

is a condition of star-likeness in $\mathbb{D}$.

Definition. A function $f(z) \in \Sigma$ of the form (1) is said to be in the class $\mathrm{B}_{1\Sigma}(\mu,\gamma)$; $(0 < \mu \leq 1,\ 0 < \gamma \leq 1)$ if the following two conditions are fulfilled:

$$\left|\frac{z^{1-\mu} f^{'}(z)}{f(z)^{1-\mu}} - 1\right| < \gamma, \quad (z \in \mathbb{D})$$

and

$$\left|\frac{w^{1-\mu} g^{'}(w)}{g(w)^{1-\mu}} - 1\right| < \gamma, \quad (w \in \mathbb{D}),$$

where g is of the form (2), be an extension of f^{-1} to $\mathbb{D}$.

Theorem 3.1. *Let* $f(z) \in \mathrm{B}_{1\Sigma}(\mu,\gamma), (0 < \mu \leq 1, 0 < \gamma \leq 1)$ *be given by* (1). *Then we have the following sharp estimates*:

$$|a_2| \leq \sqrt{\frac{2\gamma}{2+\mu} + \frac{2(1-\mu)\gamma^2}{(1+\mu)^2}},$$

$$|a_3| \leq \frac{2\gamma}{2+\mu} + \frac{2(1-\mu)\gamma^2}{(1+\mu)^2},$$

$$|a_4| \leq \frac{2\gamma}{3+\mu} + \frac{4\gamma^2(1-\mu)\left[3(1+\mu)^2 + \gamma(1-2\mu)(2+\mu)\right]}{3(2+\mu)(1+\mu)^3}$$

and for some real number k,

$$\left|a_3 - ka_2^2\right| \leq |1-k| \left[\frac{2\gamma}{2+\mu} + \frac{2(1-\mu)\gamma^2}{(1+\mu)^2}\right].$$

Proof. Given $f \in \mathrm{B}_{1\Sigma}(\mu, \gamma)$. Thus, we have

$$\Re\left\{\frac{z^{1-\mu} f'(z)}{f(z)^{1-\mu}}\right\} > 1 - \gamma$$

and

$$\Re\left\{\frac{w^{1-\mu} g'(w)}{g(w)^{1-\mu}}\right\} > 1 - \gamma.$$

Hence we can write

$$\frac{z^{1-\mu} f'(z)}{f(z)^{1-\mu}} = (1-\gamma) + \gamma P(z) \tag{29}$$

and

$$\frac{w^{1-\mu} g'(w)}{g(w)^{1-\mu}} = (1-\gamma) + \gamma Q(w), \tag{30}$$

where $P(z), Q(w) \in \mathcal{P}$ are of the form (10) and (11), respectively.

On equating the coefficients in (29) and (30), we obtain the similar equations as the equations (12) to (17) with the replacement of c_1, c_2, c_3 by $c_1\gamma, c_2\gamma, c_3\gamma$ respectively. Hence, we can construct the further proof similarly as the proof of Theorem 2.1. □

Observe that the classes $\mathfrak{B}_{1\Sigma}(\mu); (0 < \mu \leq 1)$ and $\mathrm{B}_{1\Sigma}(\mu, \gamma); (\gamma = 1)$ are agree with all their corresponding estimates of a_2, a_3 and a_4.

Now we consider the class $\mathcal{U}(\mu, \gamma)$, introduced by Obradović [16] that consists of functions $f \in \mathcal{A}$ which satisfy the condition:

$$\left|\left(\frac{z}{f(z)}\right)^{1+\mu} f'(z) - 1\right| < \gamma, \quad (z \in \mathbb{U}, 0 < \mu < 1, 0 < \gamma < 1).$$

The univalency problem for this class $\mathcal{U}(\mu, \gamma)$ with μ as a complex number has been studied by Fournier and Ponnusamy [8]. Observe that, for $\mu < 0$ this class correlates to the class $\mathrm{B}_{1\Sigma}(\mu, \gamma)$.

Definition. A function $f(z) \in \Sigma$ of the form (1) is said to be in the class $U_\Sigma(\mu, \gamma)$; $(0 < \mu \leq 1,\ 0 < \gamma \leq 1)$ if the following two conditions are fulfilled:

$$\left|\left(\frac{z}{f(z)}\right)^{1+\mu} f'(z) - 1\right| < \gamma, \quad (z \in \mathbb{D})$$

and

$$\left|\left(\frac{w}{g(w)}\right)^{1+\mu} g^{'}(w)-1\right|<\gamma, \quad (w\in\mathbb{D}),$$

where g is of the form (2), be an extension of f^{-1} to $\mathbb{D}$.

The following theorem, which we state without proof here, is a consequence of Theorem 3.1.

Theorem 3.2. *Let $f(z)\in U_{\Sigma}(\mu,\gamma),(0<\mu\leq 1,0<\gamma\leq 1)$ be given by* (1). *Then we have the following sharp estimates*:

$$|a_2|\leq\sqrt{\frac{2\gamma}{2-\mu}+\frac{2\left(1+\mu\right)\gamma^2}{\left(1-\mu\right)^2}},$$

$$|a_3|\leq\frac{2\gamma}{2-\mu}+\frac{2\left(1+\mu\right)\gamma^2}{\left(1-\mu\right)^2},$$

$$|a_4|\leq\frac{2\gamma}{3-\mu}+\frac{4\gamma^2\left(1+\mu\right)\left[3\left(1-\mu\right)^2+\gamma\left(1+2\mu\right)\left(2-\mu\right)\right]}{3\left(2-\mu\right)\left(1-\mu\right)^3}$$

and for some real number k,

$$\left|a_3-ka_2^2\right|\leq|1-k|\left[\frac{2\gamma}{2-\mu}+\frac{2\left(1+\mu\right)\gamma^2}{\left(1-\mu\right)^2}\right].$$

Remark 3.3. All the above results of the class $U_{\Sigma}(\mu,\gamma)$ with $-1\leq\mu<0$ are agree with the class $\mathrm{B}_{1\Sigma}(\mu,\gamma)$.

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