

## YAMABE AND RIEMANN SOLITONS ON LORENTZIAN PARA-SASAKIAN MANIFOLDS

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**ABSTRACT.** In the present paper, we aim to study Yamabe soliton and Riemann soliton on Lorentzian para-Sasakian manifold. First, we proved, if the scalar curvature of an  $\eta$ -Einstein Lorentzian para-Sasakian manifold  $M$  is constant, then either  $\tau = n(n-1)$  or,  $\tau = n-1$ . Also we constructed an example to justify this. Next, it is proved that, if a three dimensional Lorentzian para-Sasakian manifold admits a Yamabe soliton for  $V$  is an infinitesimal contact transformation and  $tr \varphi$  is constant, then the soliton is expanding. Also we proved that, suppose a 3-dimensional Lorentzian para-Sasakian manifold admits a Yamabe soliton, if  $tr \varphi$  is constant and scalar curvature  $\tau$  is harmonic (i.e.,  $\Delta\tau = 0$ ), then the soliton constant  $\lambda$  is always greater than zero with either  $\tau = 2$ , or  $\tau = 6$ , or  $\lambda = 6$ . Finally, we proved that, if an  $\eta$ -Einstein Lorentzian para-Sasakian manifold  $M$  represents a Riemann soliton for the potential vector field  $V$  has constant divergence then either,  $M$  is of constant curvature 1 or,  $V$  is a strict infinitesimal contact transformation.

### 1. Introduction

It is well known that, the notion of Yamabe flow was first introduced by Richard Hamilton at the same time as Ricci flow [11]. A Yamabe flow is defined as a tool for constructing metrics of constant scalar curvature. On a smooth pseudo Riemannian manifold, Yamabe flow is defined as the evaluation of the metric  $g_0$  in time  $t$  to  $g = g(t)$  through the equation

$$(1.1) \quad \frac{\partial}{\partial t} g(t) = -\tau g, \quad g(0) = g_0,$$

where  $\tau$  is the scalar curvature of the metric  $g(t)$ . If a pseudo-Riemannian manifold  $M$  holds the relation

$$(1.2) \quad \mathcal{L}_V g = 2(\tau - \lambda)g$$

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for a vector field  $V$  on  $M$  and a constant  $\lambda$ , then  $M$  is said to have Yamabe soliton. Like the Ricci soliton [17, 18], the Yamabe soliton is said to be shrinking, steady or expanding according as  $\lambda < 0$ ,  $\lambda = 0$ , or  $\lambda > 0$ , respectively.

In the past two decades, many authors have studied Yamabe soliton on various types of manifolds [1, 5, 7, 25, 27]. Recently, Venkatesha et al., studied Yamabe soliton on three dimensional contact manifolds [24] and Ghosh studied Yamabe soliton on Kenmotsu manifold [10].

The notion of Ricci flow is generalized to the concept of Riemann flow (see [21], [22]). As an analogous to the Ricci flow, a Riemann flow has been introduced by Hiriča and Udrište [12] as a natural extension of the Ricci flow to a non-linear PDE and the metric  $g$  as a solution of the PDE. A Riemann soliton is defined as a self similar solution to the Riemann flow and is defined as

$$(1.3) \quad \frac{\partial}{\partial t} G(t) = -2R(g(t)), \quad t \in [0, I],$$

where  $R$  denotes the Riemannian curvature tensor associated with metric  $g$ ,  $G = g \otimes g$  and  $\otimes$  is Kulkarni-Nomizu product. If  $C$  and  $D$  are two  $(0, 2)$ -tensors, then  $C \otimes D$  is given by

$$(1.4) \quad \begin{aligned} (C \otimes D)(W, X, Y, Z) &= C(W, Z)D(X, Y) + C(X, Y)D(W, Z) \\ &\quad - C(W, Y)D(X, Z) - C(X, Z)D(W, Y). \end{aligned}$$

A pseudo-Riemannian manifold  $M$  is said to admit a Riemann soliton  $(g, V)$ , if there exist a vector field  $V$  and a constant  $\lambda$  on  $M$  such that

$$(1.5) \quad R + \frac{1}{2} \{ \lambda g \otimes g + g \otimes \mathcal{L}_V g \} = 0,$$

where  $\mathcal{L}_V$  is the Lie-derivative along  $V$ . In (1.5), if  $V = Df$ , where  $f$  is some smooth function and  $D$  represents the gradient operator of  $g$ , then the soliton is called a gradient Riemann soliton and is given by

$$(1.6) \quad 2R + \lambda g \otimes g + 2g \otimes \nabla^2 f = 0.$$

By Kulkarni-Nomizu product defined in (1.4) the soliton equation (1.5) becomes

$$(1.7) \quad \begin{aligned} &2R(W, X, Y, Z) + 2\lambda \{ g(X, Y)g(Z, W) - g(Y, W)g(X, Z) \} \\ &+ \{ g(W, Z)(\mathcal{L}_V g)(X, Y) + g(X, Y)(\mathcal{L}_V g)(W, Z) \\ &- g(W, Y)(\mathcal{L}_V g)(X, Z) - g(X, Z)(\mathcal{L}_V g)(W, Y) \} = 0 \end{aligned}$$

for all  $W, X, Y, Z \in \mathcal{X}(M)$ .

Moreover, contraction of the above expression over  $W, Z$  gives

$$(1.8) \quad \begin{aligned} &2S(X, Y) + 2(n-1)\lambda g(X, Y) + (n-2)(\mathcal{L}_V g)(X, Y) \\ &+ 2(\operatorname{div} V)g(X, Y) = 0. \end{aligned}$$

Similar to the Yamabe soliton, the Riemann soliton is steady, shrinking or expanding according as  $\lambda = 0$ ,  $\lambda < 0$  or  $\lambda > 0$ , respectively. In [8], [23], Naik et al., studied geometric properties of Riemann soliton in contact manifolds and in almost Kenmotsu manifolds. Further, in [4], we have studied Riemann soliton

on non-Sasakian  $(\kappa, \mu)$ -contact manifolds. In [6], De et al., studied an almost Riemann soliton in a non-cosymplectic normal almost contact metric manifold. Further, Blaga et al., considered Riemann soliton in  $(\alpha, \beta)$ -contact manifolds and gave some important geometric aspects [2]. This literature survey motivates us to study Yamabe and Riemann soliton on Lorentzian para-Sasakian manifolds.

The structure of this paper is as follows: After the accumulation of some basic results and formulas in Section 2, we show some non-existence curvature conditions on Lorentzian para-Sasakian manifold  $M$ . Also, we show that, if  $M$  is an  $\eta$ -Einstein and  $\tau$  is constant on  $M$ , then either  $\tau = n(n-1)$ , or  $\tau = n-1$ . Example has been constructed to justify this. In Section 3, we consider studying the Yamabe soliton and we establish a result that, if a three dimensional Lorentzian para-Sasakian metric  $g$  represents a Yamabe soliton for an infinitesimal contact transformation  $V$  with constant  $tr \varphi$ , then  $\lambda > 0$ . Further, we prove that, if a three dimensional Lorentzian para-Sasakian manifold with constant  $tr \varphi$  and  $\Delta\tau = 0$  admits a Yamabe soliton, then the soliton is expanding. Section 4, is devoted to study Riemann soliton on  $M$  under certain conditions, such as, (1)  $M$  is an  $\eta$ -Einstein and  $div V$  is constant, (2) for  $V = \xi$ , (3)  $V = Df$  and  $div V$  is constant.

## 2. Preliminaries

The Lorentzian para-Sasakian structure on a differentiable manifold  $M$  was first introduced by K. Matsumoto in 1989 and is defined as follows [13]:

An  $n$ -dimensional smooth manifold  $M$  together with 1-form  $\eta$ , a  $(1, 1)$  tensor  $\varphi$ , a unit vector field  $\xi$  and a Lorentzian metric  $g$  is said to have a Lorentzian para-Sasakian structure if it holds the following conditions:

$$(2.1) \quad \varphi\xi = 0, \quad \eta(\xi) = -1, \quad \varphi^2 X = X + \eta(X)\xi,$$

$$(2.2) \quad g(\varphi X, \varphi Y) = g(X, Y) + \eta(X)\eta(Y),$$

$$(2.3) \quad (\nabla_X \varphi)Y = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi,$$

$$(2.4) \quad \nabla_X \xi = \varphi X.$$

From the definition, it is known that

$$g(X, \xi) = \eta(X)$$

for all  $X$  belongs to  $\mathcal{X}(M)$ . And so the vector field  $\xi$  is time like, i.e.,

$$g(\xi, \xi) = -1$$

and  $\varphi$  is symmetric with respect to the metric  $g$ . Moreover, the geometric aspects of the Reeb vector field  $\xi$  have been exclusively studied by Wang in [26]. A smooth connected manifold  $M$  together with a Lorentzian para-Sasakian structure is said to be a Lorentzian para-Sasakian manifold. In recent years, the Lorentzian para-Sasakian manifold has been studied by many authors, [14–

16, 19, 20]. So we have the following expressions

$$(2.5) \quad R(X, Y)\xi = \eta(Y)X - \eta(X)Y,$$

$$(2.6) \quad R(\xi, Y)Z = g(Y, Z)\xi + \eta(Z)Y + 2\eta(Y)\eta(Z)\xi,$$

$$(2.7) \quad Q\xi = (n-1)\xi.$$

Moreover, the Reeb vector field  $\xi$  is never a Killing, i.e.,

$$(2.8) \quad (\mathcal{L}_\xi g)(Y, Z) = 2g(Z, \varphi Y)$$

as  $\varphi$  is linear and the rank of  $\varphi$  is  $n-1$ , so  $\mathcal{L}_\xi g \neq 0$  for all vector fields on  $\mathcal{X}(M)$ . Since,  $\varphi$  is symmetric. Therefore, we have

$$\operatorname{div} \xi = \operatorname{tr} \varphi,$$

where  $\operatorname{div}$  and  $\operatorname{tr}$  stand for divergence and trace, respectively.

**Definition 2.1.** A pseudo-Riemannian manifold  $M$  is said to be an  $\eta$ -Einstein if the Ricci operator  $Q$  satisfies

$$(2.9) \quad g(QX, Y) = \alpha g(X, Y) + \beta(\eta \otimes \eta)(X, Y),$$

where  $\alpha, \beta$  are the smooth functions on  $M$ .

Moreover, from [3], the expression of  $Q$  for an  $\eta$ -Einstein Lorentzian para-Sasakian manifold is given by

$$(2.10) \quad QX = \left\{ \frac{\tau}{n-1} - 1 \right\} X + \left\{ \frac{\tau}{n-1} - n \right\} \eta(X)\xi.$$

If  $M$  is a three-dimensional Lorentzian para-Sasakian manifold, then the expression of  $Q$  is given as

$$(2.11) \quad QX = \left\{ \frac{\tau}{2} - 1 \right\} X + \left\{ \frac{\tau}{2} - 3 \right\} \eta(X)\xi.$$

**Definition 2.2.** On a pseudo-Riemannian manifold  $M$ , any vector field  $V$  is said to be an infinitesimal contact transformation if it satisfies

$$(2.12) \quad \mathcal{L}_V \eta = \sigma \eta,$$

where  $\sigma$  is the smooth function on  $M$ . If  $\sigma = 0$ , then  $V$  is called to be strict.

From [9], we have:

**Lemma 2.3.** *On an  $n$ -dimensional pseudo-Riemannian manifold  $M$ , if there exists a vector field  $V$  such that  $\mathcal{L}_V g = 2\rho g$ , where  $\rho$  is a smooth function, then the following equations hold true on  $M$*

$$(2.13) \quad (\mathcal{L}_V S)(X, Y) = g(X, Y)(\Delta\rho) - (n-2)g(\nabla_X D\rho, Y),$$

$$(2.14) \quad \mathcal{L}_V \tau = -2\rho\tau + 2(n-1)\Delta\rho,$$

where  $\Delta\rho = -\operatorname{div} D\rho$ . If  $\rho = \tau - \lambda$ , then  $\Delta\rho = \Delta\tau = -\operatorname{div} D\tau$ .

From Yano [28], we deduce the following computational formulas

$$(2.15) \quad \begin{aligned} 2g((\mathcal{L}_V \nabla)(X, Y), Z) &= (\nabla_X \mathcal{L}_V g)(Y, Z) + (\nabla_Y \mathcal{L}_V g)(X, Z) \\ &\quad - (\nabla_Z \mathcal{L}_V g)(X, Y) \end{aligned}$$

and

$$(2.16) \quad (\mathcal{L}_V R)(X, Y)Z = (\nabla_X \mathcal{L}_V \nabla)(Y, Z) - (\nabla_Y \mathcal{L}_V \nabla)(X, Z).$$

**Proposition 2.4.** *A Lorentzian para-Sasakian manifold  $M$  for  $\dim M > 1$ , never has the following curvature conditions:*

- $\eta$ -recurrent Ricci tensor.
- cyclic  $\eta$ -recurrent Ricci tensor.

*Proof.* Let  $M$  be an  $n$ -dimensional Lorentzian para-Sasakian manifold and the dimension  $n > 1$ .

- If suppose the Ricci curvature tensor  $S$  on  $M$  satisfies  $(\nabla_X S)(Y, Z) = \eta(X)S(Y, Z)$  (i.e., Ricci tensor is  $\eta$ -recurrent) for all  $X, Y, Z \in \mathcal{X}(M)$ .

By taking  $X = Y = \xi$  in this expression and from (2.7), we obtain

$$(2.17) \quad (n-1)\eta(Z) = 0,$$

this shows that  $n = 1$ . Which is a contradiction.

Similarly,

- If  $S$  is cyclic  $\eta$ -recurrent on  $M$ , then

$$(2.18) \quad \begin{aligned} (\nabla_Y S)(X, Z) + (\nabla_Z S)(X, Y) + (\nabla_X S)(Y, Z) &= \eta(Y)S(X, Z) + \eta(Z)S(X, Y) \\ &\quad + \eta(X)S(Y, Z). \end{aligned}$$

In this, by taking  $Y = Z = \xi$ , we get

$$(2.19) \quad -3(n-1)\eta(X) = 0,$$

which leads to the contradiction as  $n > 1$ . Hence the result is proved.  $\square$

**Lemma 2.5.** *On a Lorentzian para-Sasakian manifold, the following condition holds true:*

$$(2.20) \quad (\nabla_\xi Q)Y = 2(\text{tr } \varphi)\varphi^2 Y - 2\varphi QY.$$

*Proof.* Taking covariant derivative of (2.8) along the direction of  $X$  and from (2.3) we deduce

$$(2.21) \quad (\nabla_X \mathcal{L}_\xi g)(Y, Z) = 2\{g(X, Y)\eta(Z) + \eta(Y)g(X, Z) + 2\eta(X)\eta(Y)\eta(Z)\}.$$

In view of (2.15) and (2.21), we find

$$(2.22) \quad (\mathcal{L}_\xi \nabla)(Y, Z) = 2g(\varphi Y, \varphi Z)\xi.$$

Now, in (2.22), with the help of (2.3) and (2.4), we infer

$$(2.23) \quad \begin{aligned} (\nabla_X \mathcal{L}_\xi \nabla)(Y, Z) &= 2g(\varphi Y, \varphi Z)\varphi X + 2\eta(Y)g(X, \varphi Z)\xi \\ &\quad + 2\eta(Z)g(X, \varphi Y)\xi. \end{aligned}$$

By virtue of this, we obtain

$$(2.24) \quad (\nabla_Y \mathcal{L}_\xi \nabla)(X, Z) = 2g(\varphi X, \varphi Z)\varphi Y + 2\eta(X)g(Y, \varphi Z)\xi \\ + 2\eta(Z)g(X, \varphi Y)\xi.$$

On substituting the foregoing relations in (2.16) and then contracting (2.16) over  $X$  with respect to an orthonormal basis, gives

$$(2.25) \quad (\mathcal{L}_\xi S)(Y, Z) = 2g(\varphi Y, \varphi Z)(tr \varphi).$$

On the other hand, computing the left hand side of (2.25) by using (2.4) leads to

$$(2.26) \quad (\mathcal{L}_\xi S)(Y, Z) = g((\nabla_\xi Q)Y, Z) + 2g(\varphi QY, Z).$$

Hence, by equating (2.25) with (2.26) we obtain (2.20). This finishes the proof.  $\square$

**Lemma 2.6.** *On an  $\eta$ -Einstein Lorentzian para-Sasakian manifold  $M$  we have*

$$(2.27) \quad \xi\tau = -2\left(\frac{\tau}{n-1} - n\right)(tr \varphi).$$

*Proof.* Since  $M$  is  $\eta$ -Einstein, covariant derivative of equation (2.10) leads to obtain

$$(2.28) \quad (\nabla_X Q)Y = \left(\frac{X\tau}{n-1}\right)Y + \left(\frac{X\tau}{n-1}\right)\eta(Y)\xi + \left(\frac{\tau}{n-1} - n\right)\{g(X, \varphi Y)\xi \\ + \eta(Y)\varphi X\}.$$

Hence, fetching  $Y = \xi$  in the above relation and then taking contraction over  $X$  gives the condition (2.27).  $\square$

**Theorem 2.7.** *Let  $\tau$  be the scalar curvature of an  $n$ -dimensional  $\eta$ -Einstein Lorentzian para-Sasakian manifold  $M$ . If  $\tau$  is constant, then either  $\tau = n(n-1)$  with  $(tr \varphi) = \pm(n-1)$ , or  $\tau = (n-1)$  with  $(tr \varphi) = 0$ .*

*Proof.* Suppose  $\tau$  is constant on  $M$ , then  $\xi\tau = 0$  and from (2.27), we get

$$(2.29) \quad (\tau - n(n-1))(tr \varphi) = 0.$$

From (2.28) we get  $(\nabla_\xi Q)X = 0$ , which in (2.20) for  $Y = \varphi Y$  implies

$$(2.30) \quad (tr \varphi)\varphi Y - Q\varphi^2 Y = 0.$$

Contracting this over  $Y$  and with the help of (2.10), we find

$$(2.31) \quad (tr \varphi)^2 - \tau + (n-1) = 0.$$

On solving (2.31) by using (2.29) we obtain, either  $\tau = (n-1)$  with  $(tr \varphi) = 0$ , or  $\tau = n(n-1)$  with  $(tr \varphi) = \pm(n-1)$ . Hence the result is proved.  $\square$

From the above theorem, we can also state that:

**Theorem 2.8.** *Let  $M$  be an  $n$ -dimensional Lorentzian para-Sasakian manifold and the scalar curvature  $\tau$  is constant on  $M$ . If  $\tau$  is neither  $n(n-1)$  nor  $(n-1)$ , then  $M$  never be an  $\eta$ -Einstein manifold.*

**Example 2.9.** Here we construct the 5-dimensional Lorentzian para-Sasakian manifold  $M$ . We consider  $M = \{(u, v, w, x, y) \in \mathbb{R}^5\}$ , where  $(u, v, w, x, y)$  are the standard coordinates in  $\mathbb{R}^5$ .

Let  $\{v_1, v_2, v_3, v_4, v_5\}$  be the basis for  $M$  and the Lorentzian metric  $g$  is defined as the

$$(2.32) \quad g(v_i, v_j) = \begin{cases} 0 & \text{for } i \neq j, \\ 1 & \text{for } i = j \text{ and } i \neq 3, \\ -1 & \text{for } i = j = 3. \end{cases}$$

Let  $\nabla$  be the Levi-Civita connection corresponding to  $g$  and we have

$$\begin{aligned} [v_1, v_2] &= 0, & [v_1, v_3] &= -v_1, & [v_1, v_4] &= 0, \\ [v_1, v_5] &= v_1, & [v_2, v_3] &= -v_2, & [v_2, v_4] &= v_2, \\ [v_2, v_5] &= v_2, & [v_3, v_4] &= v_4, & [v_3, v_5] &= v_5, & [v_4, v_5] &= -v_5. \end{aligned}$$

Let the  $(1, 1)$  tensor field  $\varphi$  is defined by

$$(2.33) \quad \varphi v_1 = -v_1, \quad \varphi v_2 = -v_2, \quad \varphi v_3 = 0, \quad \varphi v_4 = -v_4, \quad \varphi v_5 = -v_5.$$

Let  $\eta$  be the 1-form defined by  $\eta(X) = g(X, v_3)$  for any vector field  $X$  on  $\mathcal{X}(M)$ . Then, by the linearity of  $\varphi$  and  $g$ , we find

$$(2.34) \quad \eta(v_3) = -1,$$

$$(2.35) \quad \varphi^2 = I + \eta \otimes \xi,$$

$$(2.36) \quad g(\varphi \cdot, \varphi \cdot) = (g + \eta \otimes \eta)(\cdot, \cdot).$$

By the Koszul's formula, we find

$$\begin{aligned} \nabla_{v_1} v_1 &= -v_3 - v_5, & \nabla_{v_1} v_2 &= 0, & \nabla_{v_1} v_3 &= -v_1, & \nabla_{v_1} v_4 &= 0, & \nabla_{v_1} v_5 &= v_1, \\ \nabla_{v_2} v_1 &= 0, & \nabla_{v_2} v_2 &= -v_3 - v_4 - v_5, & \nabla_{v_2} v_3 &= -v_2, & \nabla_{v_2} v_4 &= v_2, & \nabla_{v_2} v_5 &= v_2, \\ \nabla_{v_3} v_1 &= 0, & \nabla_{v_3} v_2 &= 0, & \nabla_{v_3} v_3 &= 0, & \nabla_{v_3} v_4 &= 0, & \nabla_{v_3} v_5 &= 0, \\ \nabla_{v_4} v_1 &= 0, & \nabla_{v_4} v_2 &= 0, & \nabla_{v_4} v_3 &= -v_4, & \nabla_{v_4} v_4 &= -v_3, & \nabla_{v_4} v_5 &= 0, \\ \nabla_{v_5} v_1 &= 0, & \nabla_{v_5} v_2 &= 0, & \nabla_{v_5} v_3 &= -v_5, & \nabla_{v_5} v_4 &= v_5, & \nabla_{v_5} v_5 &= -v_3 - v_4. \end{aligned}$$

Hence, we can conclude that  $(\varphi, v_3, \eta, g)$  defines a Lorentzian para-Sasakian structure on  $M$  and so  $M$  is a Lorentzian para-Sasakian manifold. Let  $R$  be the Riemannian curvature and  $S$  is the Ricci tensor and by the above relations, we evaluated the following conditions

$$\begin{aligned} R(v_1, v_2)v_2 &= 0, & R(v_1, v_3)v_3 &= -v_1, & R(v_1, v_4)v_4 &= v_1, & R(v_1, v_5)v_5 &= 0, \\ R(v_2, v_3)v_3 &= -v_2, & R(v_2, v_4)v_4 &= 0, & R(v_2, v_5)v_5 &= -v_2, & R(v_3, v_4)v_4 &= v_3, \\ R(v_3, v_5)v_5 &= v_3 + v_4, & R(v_4, v_5)v_5 &= 0. \end{aligned}$$

And from the above relations, we obtain

$$\begin{aligned} S(v_1, v_1) &= 2, & S(v_2, v_2) &= 0, & S(v_3, v_3) &= -4, \\ S(v_4, v_4) &= 2, & S(v_5, v_5) &= 0. \end{aligned}$$

Since,  $M$  is 5-dimensional and the scalar curvature is 8. Moreover,  $S(v_1, v_1) \neq S(v_2, v_2)$  shows that  $M$  is never an  $\eta$ -Einstein. Hence this verifies Theorem 2.8.

**Example 2.10.** Let us consider a manifold  $M = \{(x, y, z) \in \mathbb{R}^3\}$  and the orthonormal basis  $\{u_1, u_2, u_3\}$  on  $M$ , with the Lorentzian metric  $g$  satisfying

$$\begin{aligned} g(u_i, u_j) &= 0 \quad \text{for } i \neq j, \\ g(u_1, u_1) &= g(u_2, u_2) = 1, \\ g(u_3, u_3) &= -1. \end{aligned}$$

Define 1-form  $\eta$  and the vector field  $\xi$  by

$$\eta(X) = g(X, u_3), \quad \xi = u_3.$$

Let  $\nabla$  be the Levi-Civita connection corresponding to  $g$  and is defined by

$$[u_1, u_2] = 0, \quad [u_1, u_3] = -u_1, \quad [u_2, u_3] = -u_2,$$

and the tensor field  $\varphi$  is defined by

$$\varphi u_1 = -u_1, \quad \varphi u_2 = -u_2, \quad \varphi u_3 = 0.$$

Use of Koszul's formula gives the following relations

$$\begin{aligned} \nabla_{u_1} u_1 &= -u_3, & \nabla_{u_1} u_2 &= 0, & \nabla_{u_1} u_3 &= -u_1, \\ \nabla_{u_2} u_1 &= 0, & \nabla_{u_2} u_2 &= -u_3, & \nabla_{u_2} u_3 &= -u_2, \\ \nabla_{u_3} u_1 &= 0, & \nabla_{u_3} u_2 &= 0, & \nabla_{u_3} u_3 &= 0. \end{aligned}$$

From the above relations, it is clear that  $(\nabla_X \varphi)Y = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi$  and  $\nabla_X \xi = \varphi X$  for any vector fields  $X, Y$ . Hence, the defined structure  $(\varphi, \xi = u_3, \eta, g)$  is a Lorentzian para-Sasakian structure on  $M$ . Then the corresponding Riemannian curvature tensor and Ricci tensor have been calculated as follows:

$$\begin{aligned} R(u_1, u_2)u_2 &= u_1, & R(u_1, u_3)u_3 &= -u_1, & R(u_2, u_1)u_1 &= u_2, \\ R(u_2, u_3)u_3 &= -u_2, & R(u_3, u_1)u_1 &= u_3, & R(u_3, u_2)u_2 &= u_3. \end{aligned}$$

and

$$\begin{aligned} S(u_1, u_1) &= S(u_2, u_2) = 2, & S(u_3, u_3) &= -2, \\ S(u_1, u_2) &= S(u_1, u_3) = S(u_2, u_3) = 0. \end{aligned}$$

Clearly, the constructed structure  $(\varphi, \xi, \eta, g)$ , for  $\xi = u_3$  is an Einstein Lorentzian para-Sasakian structure with  $\tau = 6$  and  $tr \varphi = -2$ . This verifies Theorem 2.7.



### 3. Yamabe soliton

**Theorem 3.1.** *If a Lorentzian para-Sasakian metric  $g$  represents a Yamabe soliton, then the scalar curvature  $\tau$  is constant if and only if  $V$  is Killing.*

*Proof.* Suppose  $M$  has a constant scalar curvature and  $g$  is a Yamabe soliton. Then by equation (1.2) we can deduce that,  $\nabla_X \mathcal{L}_V g = 0$ . And by using this in the computational formula (2.15), we obtain

$$(3.1) \quad (\mathcal{L}_V \nabla)(Y, Z) = 0,$$

this implies getting

$$(3.2) \quad (\nabla_X \mathcal{L}_V \nabla)(Y, Z) = 0.$$

As a result, the preceding condition in (2.16) produces

$$(3.3) \quad (\mathcal{L}_V R)(X, Y)Z = 0.$$

Substituting  $Y = Z = \xi$  in the previous relation and then tracing the resulting equation with the aid of (1.2), we find

$$(3.4) \quad \eta(\mathcal{L}_V \xi) = \tau - \lambda = 0.$$

Therefore, use of this in (1.2) proves that  $V$  is Killing.

Conversely, if the soliton vector field  $V$  is Killing, then from the expression (1.2), it is obvious that  $\tau = \lambda$ . Since  $\lambda$  is constant, which means  $\tau$  is also constant. This completes the proof.  $\square$

**Corollary 3.2.** *If  $g$  is a Lorentzian para-Sasakian metric, then  $g$  never satisfies Yamabe equation for  $V = \xi$ .*

*Proof.* If suppose a Lorentzian para-Sasakian metric  $g$  is a Yamabe soliton for  $V = \xi$ , then the equation (1.2), on  $(\xi, \xi)$  gives  $\tau - \lambda = 0$ . Later, this in (1.2) shows  $\xi$  is Killing. But, as we know, if  $\xi$  is Killing then by the condition (2.8)  $\varphi = 0$ , which is a contradiction. Therefore,  $V$  is never a Reeb vector field  $\xi$ .  $\square$

Here we justify the above theorem by the following example:

**Example 3.3.** In Example 2.9, if manifold  $M$  holds Yamabe soliton for  $V = \xi = v_3$ , then, by computing (1.2) on  $(v_3, v_3)$ , we acquire

$$(3.5) \quad (\mathcal{L}_{v_3} g)(v_3, v_3) = 2(\lambda - \tau) = 0,$$

this implies  $\tau = \lambda$ , at one more time, evaluating (1.2) on  $(v_2, v_2)$  gives

$$2g(\nabla_{v_2} v_3, v_2) = -2 = 0,$$

which is a contradiction. Therefore it verifies Corollary 3.2.

**Theorem 3.4.** *Let  $g$  be a Lorentzian para-Sasakian metric and it admits Yamabe soliton for  $V$  is an infinitesimal contact transformation, if  $\tau$  is constant in the direction of  $\xi$  then  $V$  is Killing.*

*Proof.* From Definition 2.2 and from the equation (1.2) we can easily find that

$$(3.6) \quad \sigma = (\tau - \lambda),$$

and as we know  $\eta$  is closed on  $M$ , i.e.,  $d\eta = 0$ , therefore applying  $d$  on both sides of relation (2.12) provides

$$(3.7) \quad (d\sigma \wedge \eta)(X, Y) = 0.$$

In the above equation for  $X = \xi$  we get  $Y\sigma = -(\xi\sigma)\eta(Y)$ . So  $\sigma$  is constant if  $\xi\sigma$  is zero. Since  $\xi\tau = 0$ , then by (3.6), we have  $\xi\sigma = 0$ , which shows  $\sigma$  is constant on  $M$  and consequently  $\tau$  is also constant on  $M$ . Therefore, from Theorem 3.1 the proof is completed.  $\square$

**Theorem 3.5.** *Let  $M$  be a three-dimensional Lorentzian para-Sasakian manifold and admits a Yamabe soliton for the potential vector field  $V$ , where  $V$  is an infinitesimal contact transformation. If the trace of  $\varphi$  is constant, then the soliton is expanding.*

*Proof.* For a 3-dimensional Lorentzian para-Sasakian manifold the expression of Ricci tensor is given by

$$(3.8) \quad S = \left\{ \frac{\tau}{2} - 1 \right\} g + \left\{ \frac{\tau}{2} - 3 \right\} \eta \otimes \eta.$$

Taking the Lie-derivative of the above condition in the direction of  $V$  results in the following

$$(3.9) \quad \begin{aligned} (\mathcal{L}_V S)(Y, Z) &= \left( \frac{\mathcal{L}_V \tau}{2} \right) g(Y, Z) + \left\{ \frac{\tau}{2} - 1 \right\} (\mathcal{L}_V g)(Y, Z) + \left( \frac{\mathcal{L}_V \tau}{2} \right) \eta(Y)\eta(Z) \\ &+ \left\{ \frac{\tau}{2} - 3 \right\} (\mathcal{L}_V \eta \otimes \eta)(Y, Z). \end{aligned}$$

We can also have

$$(3.10) \quad \begin{aligned} g((\mathcal{L}_V Q)Y, Z) &= \left( \frac{\mathcal{L}_V \tau}{2} \right) g(Y, Z) + \left( \frac{\mathcal{L}_V \tau}{2} \right) \eta(Y)\eta(Z) + \left\{ \frac{\tau}{2} - 3 \right\} \{ \eta(Z)(\mathcal{L}_V \eta)Y \\ &+ g(\mathcal{L}_V \xi, Z)\eta(Y) \}. \end{aligned}$$

From equation (1.2), we derive

$$(3.11) \quad (\mathcal{L}_V S)(Y, Z) - g((\mathcal{L}_V Q)Y, Z) = 2(\tau - \lambda)S(Y, Z).$$

As from (1.2), we have  $\eta(\mathcal{L}_V \xi) = (\tau - \lambda)$ . Next, by putting  $Y = Z = \xi$  in equation (3.11) and with the help of (3.9) and (3.10) we find that

$$(3.12) \quad (\mathcal{L}_V S)(\xi, \xi) = -4(\tau - \lambda).$$

Since, from (2.13) we have

$$(3.13) \quad (\mathcal{L}_V S)(\xi, \xi) = -\Delta\tau - g(\nabla_\xi D\tau, \xi).$$

On equating (3.12) with (3.13), we obtain

$$(3.14) \quad 4(\tau - \lambda) = \Delta\tau + \xi(\xi\tau).$$

Since  $V$  is an infinitesimal contact transformation, thus, from the conditions (2.12) and (1.2), we have that  $X\sigma = X\tau = 0$  for all  $X$  orthogonal to  $\xi$ . Later, this implies getting

$$(3.15) \quad D\tau = -(\xi\tau)\xi.$$

Now differentiating this along  $Y$  provides

$$(3.16) \quad \nabla_Y D\tau = -\{Y(\xi\tau)\}\xi - (\xi\tau)\nabla_Y \xi.$$

Further, we proceed with the condition  $tr \varphi = constant$ . If the trace of  $\varphi$  is constant, then from (2.27) we obtain

$$(3.17) \quad \xi(\xi\tau) = -(\xi\tau)(tr \varphi) = (\tau - 6)(tr \varphi)^2.$$

In equation (2.27), the fact that  $g(X, D\tau) = 0$  for any  $X$  orthogonal to  $\xi$  enables us to find

$$(3.18) \quad X(\xi\tau) = -(X\tau)(tr \varphi) = 0,$$

for all  $X$  perpendicular to  $\xi$ .

Next, tracing (3.16) over  $Y$  and then using above relation yields

$$(3.19) \quad -\Delta\tau = -\{\xi(\xi\tau)\} - (\xi\tau)(tr \varphi).$$

On substituting (3.17) and (3.19) in (3.14) we get

$$(3.20) \quad -4(\tau - \lambda) = -2(\tau - 6)(tr \varphi)^2 + (\tau - 6)(tr \varphi)^2,$$

differentiating (3.20) along  $\xi$  and using (2.27), we have

$$(3.21) \quad (\tau - 6)\{4(tr \varphi) - (tr \varphi)^3\} = 0.$$

Note that the trace of  $\varphi$  is constant. Therefore, from the above equation, there are three cases that arise: either  $\tau = 6$ , or  $(tr \varphi) = 0$ , or  $(tr \varphi)^2 = 4$ . *First case* itself proves the result. Next, let us deal with *second case*, i.e.,  $(tr \varphi) = 0$ , which in (2.27) finds  $\xi\tau = 0$  and from (2.28) for  $n = 3$  gives  $(\nabla_\xi Q)Y = 0$ , use of this in (2.20) enables us to find  $\tau = 2$ . Finally, if  $(tr \varphi)^2 = 4$ , which in (3.20) finds  $\lambda = 6$ . Hence, by Theorem 3.1 the proof is completed.  $\square$

**Theorem 3.6.** *Let  $M$  be a Lorentzian para-Sasakian manifold of dimension three and admits a Yamabe soliton  $(g, V, \lambda)$ . If  $tr \varphi$  is constant and the scalar curvature  $\tau$  is harmonic, i.e.,  $\Delta\tau = 0$ , then the soliton is expanding with either  $V$  is Killing, or  $\lambda = 6$ .*

*Proof.* Suppose a three-dimensional Lorentzian para-Sasakian manifold  $M$  admits a Yamabe soliton. If  $tr \varphi$  is constant and  $\Delta\tau = 0$ , then from (2.27) we have

$$(3.22) \quad \xi(\xi\tau) = (\tau - 6)(tr \varphi)^2.$$

Use of foregoing condition in (3.14) and the harmonic scalar curvature condition provides

$$(3.23) \quad 4(\tau - \lambda) - (\tau - 6)(tr \varphi)^2 = 0.$$

Taking covariant derivative of preceding relation along  $\xi$  and from (2.27), we yields

$$(3.24) \quad (\tau - 6)(tr \varphi)\{4 - (tr \varphi)^2\} = 0.$$

Hence, from the above equation we conclude that either  $\tau = 6$ , or  $\tau = 2$ , or  $\lambda = 6$ . This finishes the proof.  $\square$

#### 4. Riemann soliton

**Theorem 4.1.** *Let  $M$  ( $\dim M = n > 2$ ) be an  $\eta$ -Einstein Lorentzian para-Sasakian manifold and represents a Riemann soliton for  $V$  has a constant divergence. Then either  $V$  is strict infinitesimal contact transformation or  $M$  is of constant curvature 1.*

*Proof.* By the hypothesis,  $divV$  is constant. Therefore, the contraction of equation (1.8) gives an expression for  $\tau$  and shows  $\tau$  is constant on  $M$ . Taking the covariant derivative of equation (2.10) leads to obtaining

$$(4.1) \quad g((\nabla_X Q)Y, Z) = \left(\frac{\tau}{n-1} - n\right) \{\eta(Z)g(\varphi X, Y) + \eta(Y)g(\varphi X, Z)\}.$$

In view of the above condition and from (1.8), we derive

$$(\nabla_X \mathcal{L}_V g)(Y, Z) = \frac{-2}{n-2} \left(\frac{\tau}{n-1} - n\right) \{\eta(Z)g(\varphi X, Y) + \eta(Y)g(\varphi X, Z)\}.$$

Use of foregoing relation in the computational formula (2.15) yields

$$(\mathcal{L}_V \nabla)(X, Y) = \frac{-2}{n-2} \left(\frac{\tau}{n-1} - n\right) g(X, \varphi Y)\xi.$$

By the help of above condition and equation (2.3), we obtain

$$\begin{aligned} (\nabla_X \mathcal{L}_V \nabla)(Y, Z) &= \frac{-2}{n-2} \left(\frac{\tau}{n-1} - n\right) \{g(\varphi X, \varphi Y)\eta(Z)\xi \\ &\quad + g(\varphi X, \varphi Z)\eta(Y)\xi + g(Y, \varphi Z)\varphi X\}. \end{aligned}$$

With the help of previous equation, the right side of the relation (2.16) is computed as

$$\begin{aligned} (\mathcal{L}_V R)(X, Y)Z &= \frac{-2}{n-2} \left(\frac{\tau}{n-1} - n\right) \{g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi \\ &\quad + g(Y, \varphi Z)\varphi X - g(X, \varphi Z)\varphi Y\}. \end{aligned}$$

Tracing this over  $X$  implies

$$(4.2) \quad (\mathcal{L}_V S)(Y, Z) = \frac{-2}{n-2} \left(\frac{\tau}{n-1} - n\right) \{(tr \varphi)g(Y, \varphi Z)\}.$$

In equation (4.2), by placing  $Z = \xi$  and from (2.7), we obtain

$$(4.3) \quad (n-1)(\mathcal{L}_V \eta)Y = g(QY, \mathcal{L}_V \xi).$$

In order to find  $g(QY, \mathcal{L}_V \xi)$ , we go through an  $\eta$ -Einstein condition. By taking an inner product of (2.10) with  $\mathcal{L}_V \xi$  we find the following:

$$(4.4) \quad g(QX, \mathcal{L}_V \xi) = \left( \frac{\tau}{n-1} - 1 \right) g(X, \mathcal{L}_V \xi) + \left( \frac{\tau}{n-1} - n \right) \eta(X) \eta(\mathcal{L}_V \xi).$$

In (1.8), for  $Y = \xi$  and the expansion of  $\mathcal{L}_V g$  provides

$$(4.5) \quad (n-2)g(X, \mathcal{L}_V \xi) = \{2(n-1)(1+\lambda) + 2(\operatorname{div} V)\} \eta(X) + (n-2)(\mathcal{L}_V \eta)X.$$

For  $n > 2$ , by taking  $Y = \xi$  in (4.3) and by the fact that  $Q\xi = (n-1)\xi$  we obtain the value  $\eta(\mathcal{L}_V \xi) = 0$ . Finally, substituting (4.5) in (4.4) (minding that  $n > 2$ ) and then the use of the resulting equation in (4.3) gives

$$(4.6) \quad \left( n - \frac{\tau}{n-1} \right) (\mathcal{L}_V \eta)X = \left( \frac{\tau}{n-1} - 1 \right) \left( \frac{2(n-1)(1+\lambda) + 2(\operatorname{div} V)}{n-2} \right) \eta(X).$$

For an  $\eta$ -Einstein Lorentzian para-Sasakian manifold with constant  $\tau$ , we have from Theorem 2.7 that either  $\tau = n-1$  or  $\tau = n(n-1)$ . Therefore, if  $\tau = n-1$ , then the preceding equation shows that  $V$  is a strictly infinitesimal contact transformation. This completes the either part of the theorem. Next, if suppose  $\tau = n(n-1)$ , then from (4.6) we infer

$$(4.7) \quad (n-1)(1+\lambda) + \operatorname{div} V = 0.$$

Moreover, contraction of (1.8) leads to achieve

$$(4.8) \quad n + n\lambda + 2(\operatorname{div} V) = 0.$$

On solving (4.7) and (4.8), we obtain  $\lambda = -1$  and  $\operatorname{div} V = 0$ . Making use of the resulting equations and  $QX = (n-1)X$  in (1.8) provides  $\mathcal{L}_V g = 0$ , i.e.,  $V$  is Killing. Thus, from (1.7), we conclude that, manifold  $M$  is of constant curvature 1.  $\square$

**Theorem 4.2.** *If  $(\varphi, \xi, \eta, g)$  is a Lorentzian para Sasakian structure on an  $n$ -dimensional manifold  $M$ , then for  $n > 2$ ,  $g$  never a Riemann soliton  $(g, \xi)$ .*

*Proof.* If suppose a Lorentzian para-Sasakian metric  $g$  is a Riemann soliton for  $V = \xi$ , then from (1.8) we have

$$(4.9) \quad 2S(X, Y) + \{2(n-1)\lambda + 2(\operatorname{tr} \varphi)\}g(X, Y) + 2(n-2)g(\varphi X, Y) = 0.$$

Choosing  $X = Y = \xi$  in the foregoing relation we get

$$(4.10) \quad \operatorname{tr} \varphi = -(n-1)(1+\lambda).$$

Contracting (4.9) over  $X, Y$ , and from the above condition we find

$$(4.11) \quad \tau = -\lambda n(n-1) + 2(n-1)(n-1)(1+\lambda).$$

Since  $\lambda$  is constant, which implies  $\tau$  is constant on  $M$  and from (4.9), we deduce

$$(4.12) \quad (\nabla_X Q)Y = -(n-2)(\nabla_X \varphi)Y.$$

In the above relation putting  $Y = \xi$  and then contracting over  $X$  finds  $(n - 2)(n - 1) = 0$ . But this is a contradiction to our assumption that  $n > 2$ . This completes the proof.  $\square$

**Example 4.3.** In Example 2.10, if  $g$  represents a Riemann soliton  $(g, \xi)$ , then in equation (1.7) for  $W = Z = u_1$  and  $X = Y = u_2$ , we have

$$(4.13) \quad 2 + 2\lambda + (\mathcal{L}_{u_3}g)(u_2, u_2) + (\mathcal{L}_{u_3}g)(u_1, u_1) = 0,$$

which finds  $\lambda = -1$ . Again, in (1.7) for  $W = Z = u_2$  and  $X = Y = u_3$  we get

$$(4.14) \quad -2 + 2 + 2g(\nabla_{u_2}u_3, u_2) = 0.$$

Since,  $g(\nabla_{u_2}u_3, u_2) = -1$ , use of this in the preceding relation leads to a contradiction. Hence,  $g$  never admits a Riemann soliton for  $V$  being a Reeb vector field  $\xi$ .

**Theorem 4.4.** *If a Lorentzian para-Sasakian metric  $g$  supports a Riemann soliton for  $V = Df$  with divergence of  $V$  (i.e.,  $\text{div}Df = -\Delta f$ ) constant, then  $M$  is of constant curvature 1 and the scalar curvature  $\tau = n(n - 1)$ .*

*Proof.* If the vector  $V$  in (1.7) is a gradient of a smooth function  $f$ , then the relation (1.8) reduces to

$$(4.15) \quad QW + \lambda(n - 1)W - (\Delta f)W + (n - 2)\nabla_W Df = 0.$$

If  $\Delta f$  is constant, then the contraction of (4.15) shows that the scalar curvature  $\tau$  constant. Further, from equation (4.15), we derive the following relation

$$(4.16) \quad (\nabla_X Q)W = -(n - 2)\{\nabla_X \nabla_W Df + \nabla_{\nabla_X W} Df\}.$$

So, from this and equation (4.15), we find

$$(4.17) \quad (n - 2)R(X, W)Df = -(\nabla_X Q)W + (\nabla_W Q)X.$$

For  $n \geq 3$ , in the above expression setting  $X = \xi$  and then taking the scalar product of the resulting condition with  $\xi$  gives  $g(R(\xi, W)Df, \xi) = 0$ . Next, contraction of (4.17) over  $X$  with respect to an orthonormal basis provides  $(n - 2)QDf = 0$ . This implies  $f$  is constant along  $\xi$ . Further, the use of equation (2.5) in  $g(R(\xi, W)\xi, Df) = 0$  shows  $Wf = 0$ , i.e.,  $f$  is constant. Hence, the equation (1.7) turns to

$$(4.18) \quad R(X, Y)Z = -\lambda\{g(Y, Z)X - g(X, Z)Y\}.$$

Replacing  $Y$  and  $Z$  by  $\xi$  and  $X$  by  $\varphi X$  in (4.18) and by the virtue of (2.5), we get the value of  $\lambda$  as  $-1$ . Hence the theorem is proved.  $\square$

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