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SOME RESULTS CONCERNED WITH HANKEL DETERMINANT FOR $\mathcal{N}(\alpha)$ CLASS

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Abstract. In this paper, we give some results an upper bound of Hankel determinant of $H_2(1)$ for the classes of $\mathcal{N}(\alpha)$. We get a sharp upper bound for $H_2(1) = c_3 - c_2^2$ for $\mathcal{N}(\alpha)$ by adding $z_1, z_2, \ldots, z_n$ zeros of $f(z)$ which are different than zero. Moreover, in a class of analytic functions on the unit disc, assuming the existence of angular limit on the boundary point, the estimations below of the modulus of angular derivative have been obtained. Finally, the sharpness of the inequalities obtained in the presented theorems are proved.

1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z) = z + c_2z^2 + c_3z^3 + \cdots$ which are analytic in $D = \{z : |z| < 1\}$. Also, $\mathcal{N}(\alpha)$ be the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which satisfy

$$\left| \left(\frac{z}{f(z)} \right)^2 f'(z) - \alpha \right| < 1, \tag{1.1}$$

where $\alpha \in \mathbb{C}$. There are many studies about this inequality [12, 13, 21].

The certain analytic functions which are in the class of $\mathcal{N}(\alpha)$ on the unit disc D are considered in this paper. The subject of the present paper is to discuss some properties of the function $f(z)$ which belong to the class of $\mathcal{N}(\alpha)$ by applying Schwarz lemma. Schwarz lemma is a highly popular topic in electrical engineering. As exemplary applications, the use of positive real functions and boundary analysis of these functions for circuit synthesis can be given. Moreover, it is also possible to utilize Schwarz lemma for the analysis of transfer functions in control engineering and to design multi-notch filter structures in signal processing [15, 16].

In this paper, we will give the sharp estimates for the Hankel determinant of the class of analytic function $f \in \mathcal{A}$ will satisfy the condition (1.1). Also, the

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relationship between the coefficients of the Hankel determinant and the angular derivative of the function f, which provides the class $\mathcal{N}(\alpha)$, will be examined. In this examine, the coefficients c_2, c_3 and c_4 will be used.

Let $f \in \mathcal{A}$. The q^{th} Hankel determinant of f for $n \geq 0$ and $q \geq 1$ is stated by Noonan and Thomas [11] as

$$H_q(n) = \begin{vmatrix} c_n & c_{n+1} & \cdots & c_{n+q-1} \\ c_{n+1} & c_{n+2} & \cdots & c_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ c_{n+q-1} & c_{n+q} & \cdots & c_{n+2q-2} \end{vmatrix}, \ c_1 = 1.$$

From the Hankel determinant for $n = 1$ and $q = 2$, we have

$$H_2(1) = \begin{vmatrix} c_1 & c_2 \\ c_2 & c_3 \end{vmatrix} = c_3 - c_2^2.$$

Here, the Hankel determinant $H_2(1) = c_3 - c_2^2$ is well-known as Fekete-Szegö functional [5]. In [11], authors have obtained the upper bounds of the Hankel determinant $\left|c_2c_4 - c_3^2\right|$. Also, in [18], author has obtained the upper bounds the Hankel determinant $A_n^{(k)}$. Moreover, in [20], authors have given bounds for the Second Hankel determinant for class $\mathcal{M}_\alpha$. We will get a sharp upper bound for $H_2(1) = c_3 - c_2^2$ for $\mathcal{N}(\alpha)$ by adding $z_1, z_2, \ldots, z_n$ zeros of $f(z)$ which are different than zero in our study.

Let $f(z) \in \mathcal{N}(\alpha)$ and consider the following function

$$p(z) = \left(\frac{z}{f(z)}\right)^2 f'(z) - \alpha = 1 - \alpha + \left(c_3 - c_2^2\right) z^2 + \left(2c_4 - 4c_2c_3 + 2c_2^3\right) z^3 + \cdots.$$

It is an analytic function in D and $p(0) = 1 - \alpha$. Consider the function

$$m(z) = \frac{p(z) - p(0)}{1 - \overline{p(0)}p(z)} \frac{1}{\prod\limits_{i=1}^{n} \frac{z - z_i}{1 - \overline{z_i}z}}.$$

Here, $m(z)$ is an analytic function in D, $m(0) = 0$ and $|m(z)| < 1$ for $z \in D$. Therefore, the function $m(z)$ satisfies the condition of Schwarz lemma [6]. From the Schwarz lemma, we obtain

$$\begin{aligned} m(z) &= \frac{p(z) - p(0)}{1 - \overline{p(0)}p(z)} \frac{1}{\prod\limits_{i=1}^{n} \frac{z - z_i}{1 - \overline{z_i}z}} \\ &= \frac{1 - \alpha + \left(c_3 - c_2^2\right) z^2 + \left(2c_4 - 4c_2c_3 + 2c_2^3\right) z^3 + \cdots - (1 - \alpha)}{\left[1 - (1 - \overline{\alpha})\left(1 - \alpha + \left(c_3 - c_2^2\right) z^2 + \left(2c_4 - 4c_2c_3 + 2c_2^3\right) z^3 + \cdots\right)\right] \prod\limits_{i=1}^{n} \frac{z - z_i}{1 - \overline{z_i}z}} \\ &= \frac{\left(c_3 - c_2^2\right) z^2 + \left(2c_4 - 4c_2c_3 + 2c_2^3\right) z^3 + \cdots}{\left[1 - (1 - \overline{\alpha})\left(1 - \alpha + \left(c_3 - c_2^2\right) z^2 + \left(2c_4 - 4c_2c_3 + 2c_2^3\right) z^3 + \cdots\right)\right] \prod\limits_{i=1}^{n} \frac{z - z_i}{1 - \overline{z_i}z}}, \end{aligned}$$

$$\frac{m(z)}{z^2}=\frac{\left(c_3-c_2^2\right)+\left(2c_4-4c_2c_3+2c_2^3\right)z+\cdots}{\left[1-(1-\overline{\alpha})\left(1-\alpha+\left(c_3-c_2^2\right)z^2+\left(2c_4-4c_2c_3+2c_2^3\right)z^3+\cdots\right)\right]\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}},$$

$$\frac{\left|c_3-c_2^2\right|}{\left(1-|1-\alpha|^2\right)\prod_{i=1}^{n}|z_i|}=\frac{|H_2(1)|}{\left(1-|1-\alpha|^2\right)\prod_{i=1}^{n}|z_i|}\leq 1$$

and

$$|H_2(1)|\leq\left(1-|1-\alpha|^2\right)\prod_{i=1}^{n}|z_i|.$$

Now, let us show the sharpness of this inequality. Let

$$\frac{p(z)-p(0)}{1-\overline{p(0)}p(z)}=z^2\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}$$

and

$$p(z)=\frac{z^2\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}+p(0)}{1+\overline{p(0)}z^2\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}}.$$

From the definition of $p(z)$, we take

$$\left(\frac{z}{f(z)}\right)^2f'(z)-\alpha=\frac{z^2\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}+p(0)}{1+\overline{p(0)}z^2\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}},$$

$$1-\alpha+\left(c_3-c_2^2\right)z^2+\left(2c_4-4c_2c_3+2c_2^3\right)z^3+\cdots=\frac{z^2\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}+p(0)}{1+\overline{p(0)}z^2\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}},$$

$$\begin{aligned}\left(c_3-c_2^2\right)z^2+\left(2c_4-4c_2c_3+2c_2^3\right)z^3+\cdots&=\frac{z^2\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}+p(0)}{1+\overline{p(0)}z^2\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}}+\alpha-1\\&=\frac{\left(1-|1-\alpha|^2\right)z^2\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}}{1+(1-\overline{\alpha})z^2\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}}\end{aligned}$$

and

$$\left(c_3-c_2^2\right)+\left(2c_4-4c_2c_3+2c_2^3\right)z+\cdots=\frac{\left(1-|1-\alpha|^2\right)\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}}{1+(1-\overline{\alpha})z^2\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}}.$$

Passing to limit $(z \to 0)$ in the last equality yields

$$\left|c_3 - c_2^2\right| = |H_2(1)| = \left(1 - |1-\alpha|^2\right) \prod_{i=1}^{n} |z_i| .$$

We thus obtain the following lemma.

Lemma 1.1. *Let $f(z) \in \mathcal{N}(\alpha)$ and $z_1, z_2, \ldots, z_n$ be the zeros of the function $f(z) - z$ in D that are different from zero. Then we have the inequality*

$$|H_2(1)| \leq \left(1 - |1-\alpha|^2\right) \prod_{i=1}^{n} |z_i| . \tag{1.2}$$

This result is sharp.

Several studies on Schwarz lemma exist in literature as it has a wide applicability area. Some examples are about being estimated from below the modulus of the derivative of the function at some boundary point of the unit disc which is also called as boundary version of Schwarz lemma where it is given as follows [17]:

Lemma 1.2. *Let $g : D \to D$ be an analytic function with $g(z) = c_p z^p + c_{p+1} z^{p+1} + \cdots$, $p \geq 1$. Assume that there is a $z_0 \in \partial D$ so that g extends continuously to z_0, $|g(z_0)| = 1$ and $g'(z_0)$ exists. Then*

$$|g'(z_0)| \geq p + \frac{1 - |c_p|}{1 + |c_p|} \tag{1.3}$$

and

$$|g'(z_0)| \geq p. \tag{1.4}$$

Inequalities (1.3) *and* (1.4) *are sharp.*

Inequalities (1.3), (1.4) and their generalizations have important applications in geometric theory of functions and they are still hot topics in the mathematics literature [1–4, 7, 8, 14–17]. Mercer considers some Schwarz and Carathéodory inequalities at the boundary, as consequences of a lemma due to Rogosinski [9]. In addition, he obtains a new boundary Schwarz lemma, for analytic functions mapping the unit disk to itself [10].

The following lemma, known as the Julia-Wolff lemma, is needed in the sequel (see, [19])

Lemma 1.3 (Julia-Wolff lemma)**.** *Let g be an analytic function in D, $g(0) = 0$ and $g(D) \subset D$. If, in addition, the function g has an angular limit $g(z_0)$ at $z_0 \in \partial D$, $|g(z_0)| = 1$, then the angular derivative $g'(z_0)$ exists and $1 \leq |g'(z_0)| \leq \infty$.*

Corollary 1.4. *The analytic function g has a finite angular derivative $g'(z_0)$ if and only if g' has the finite angular limit $g'(z_0)$ at $z_0 \in \partial D$.*

2. Main results

In this section, we discuss different versions of the boundary Schwarz lemma and Hankel determinant for $\mathcal{N}(\alpha)$ class. Second derivative of the module of $f(z)$ is evaluated below by including the $z_1, z_2, \ldots, z_n$ zeros and $H_2(1)$ Hankel determinant of the function $f(z) - z$ which are different from zero. In the inequalities obtained, the relationship between the Hankel determinant and the second angular derivative of the $f(z)$ function was established.

Theorem 2.1. *Let $f(z) \in \mathcal{N}(\alpha)$. Suppose that, for some $z_0 \in \partial D$, f has an angular limit $f(z_0)$ at z_0, $f(z_0) = \frac{z_0}{1+\alpha}$ and $f'(z_0) = \frac{1}{1+\alpha}$. Let $z_1, z_2, \ldots, z_n$ be the zeros of the function $f(z) - z$ in D that are different from zero. Then we have the inequality*

$$|f''(z_0)| \geq \frac{|\alpha|^2}{\left(1-|1-\alpha|^2\right)|1+\alpha|^2}\left(2+\sum_{i=1}^{n}\frac{1-|z_i|^2}{|z_0-z_i|^2}\right. \tag{2.1}$$

$$\left.+\frac{2\left(\left(1-|1-\alpha|^2\right)\prod_{i=1}^{n}|z_i|-|H_2(1)|\right)^2}{\left(\left(1-|1-\alpha|^2\right)\prod_{i=1}^{n}|z_i|\right)^2-|H_2(1)|^2+\left(1-|1-\alpha|^2\right)\prod_{i=1}^{n}|z_i|\left|2\left(c_4-c_2\left(c_2^2+2H_2(1)\right)\right)-H_2(1)\sum_{i=1}^{n}\frac{1-|z_i|^2}{z_i}\right|}\right).$$

This result is sharp for $\alpha \in \mathbb{R}$ and $z_1, z_2, \ldots, z_n \in \mathbb{R}^+$.

Proof. Let $z_1, z_2, \ldots, z_n$ be the zeros of the function $f(z) - z$ in D that are different from zero. Consider the following functions

$$\vartheta(z) = \frac{p(z) - p(0)}{1 - \overline{p(0)}p(z)}$$

and

$$B(z) = z^2 \prod_{i=1}^{n} \frac{z - z_i}{1 - \overline{z_i}z}.$$

By the maximum principle, for each $z \in D$, we have $|\vartheta(z)| \leq |B(z)|$. The function

$$r(z) = \frac{\vartheta(z)}{B(z)}$$

is analytic in D, and $|r(z)| < 1$ for $z \in D$. In particular, we have

$$|r(0)| = \frac{\left|c_3 - c_2^2\right|}{\left(1-|1-\alpha|^2\right)\prod_{i=1}^{n}|z_i|} = \frac{|H_2(1)|}{\left(1-|1-\alpha|^2\right)\prod_{i=1}^{n}|z_i|}$$

and

$$|r'(0)| = \frac{\left|2c_4 - 4c_2c_3 + 2c_2^3 - H_2(1)\sum_{i=1}^{n}\frac{1-|z_i|^2}{z_i}\right|}{\left(1-|1-\alpha|^2\right)\prod_{i=1}^{n}|z_i|}$$

$$=\frac{\left|2\left(c_4-c_2\left(c_2^2+2H_2(1)\right)\right)-H_2(1)\sum\limits_{i=1}^{n}\frac{1-|z_i|^2}{z_i}\right|}{\left(1-|1-\alpha|^2\right)\prod\limits_{i=1}^{n}|z_i|}.$$

It is obvious that

$$\frac{z_0\vartheta'(z_0)}{\vartheta(z_0)}=|\vartheta'(z_0)|\geq\frac{z_0B'(z_0)}{B(z_0)}=|B'(z_0)|$$

and

$$\frac{z_0B'(z_0)}{B(z_0)}=|B'(z_0)|=2+\sum_{i=1}^{n}\frac{1-|z_i|^2}{|z_0-z_i|^2}.$$

The auxiliary function

$$\Phi(z)=\frac{r(z)-r(0)}{1-\overline{r(0)}r(z)}$$

is analytic in the unit disc D, $\Phi(0)=0$, $|\Phi(z)|<1$ for $z\in D$ and $|\Phi(z_0)|=1$ for $z_0\in\partial D$. From (1.3) for $p=1$, we obtain

$$\begin{aligned}\frac{2}{1+|\Phi'(0)|}\leq\ &|\Phi'(z_0)|=\frac{1-|r(0)|^2}{\left|1-\overline{r(0)}r(z_0)\right|^2}|r'(z_0)|\\ &\leq\frac{1+|r(0)|}{1-|r(0)|}\left\{|\vartheta'(z_0)|-|B'(z_0)|\right\}\\ &=\frac{\left(1-|1-\alpha|^2\right)\prod\limits_{i=1}^{n}|z_i|+|H_2(1)|}{\left(1-|1-\alpha|^2\right)\prod\limits_{i=1}^{n}|z_i|-|H_2(1)|}\left\{\frac{1-|1-\alpha|^2}{|\alpha|^2}|f''(z_0)|\,|1+\alpha|^2\right.\\ &\qquad\left.-\left(2+\sum_{i=1}^{n}\frac{1-|z_i|^2}{|z_0-z_i|^2}\right)\right\}.\end{aligned}$$

It can be seen that

$$|\Phi'(z)|=\frac{1-|r(0)|^2}{\left|1-\overline{r(0)}r(z)\right|^2}|r'(z)|$$

and

$$|\Phi'(0)|=\frac{|r'(0)|}{1-|r(0)|^2}=\frac{\dfrac{\left|2\left(c_4-c_2\left(c_2^2+2H_2(1)\right)\right)-H_2(1)\sum\limits_{i=1}^{n}\frac{1-|z_i|^2}{z_i}\right|}{\left(1-|1-\alpha|^2\right)\prod\limits_{i=1}^{n}|z_i|}}{1-\left(\dfrac{|H_2(1)|}{\left(1-|1-\alpha|^2\right)\prod\limits_{i=1}^{n}|z_i|}\right)^2}$$

$$=\left(1-|1-\alpha|^2\right)\prod_{i=1}^{n}|z_i|\frac{\left|2\left(c_4-c_2\left(c_2^2+2H_2(1)\right)\right)-H_2(1)\sum\limits_{i=1}^{n}\frac{1-|z_i|^2}{z_i}\right|}{\left(\left(1-|1-\alpha|^2\right)\prod\limits_{i=1}^{n}|z_i|\right)^2-|H_2(1)|^2}.$$

Let us substitute the values of $|\Phi'(0)|$. Thus, we obtain

$$\frac{2}{1+\left(1-|1-\alpha|^2\right)\prod\limits_{i=1}^{n}|z_i|\frac{\left|2\left(c_4-c_2\left(c_2^2+2H_2(1)\right)\right)-H_2(1)\sum\limits_{i=1}^{n}\frac{1-|z_i|^2}{z_i}\right|}{\left(\left(1-|1-\alpha|^2\right)\prod\limits_{i=1}^{n}|z_i|\right)^2-|H_2(1)|^2}}$$

$$\leq\frac{\left(1-|1-\alpha|^2\right)\prod\limits_{i=1}^{n}|z_i|+|H_2(1)|}{\left(1-|1-\alpha|^2\right)\prod\limits_{i=1}^{n}|z_i|-|H_2(1)|}\left\{\frac{1-|1-\alpha|^2}{|\alpha|^2}|f''(z_0)|\,|1+\alpha|^2-\left(2+\sum_{i=1}^{n}\frac{1-|z_i|^2}{|z_0-z_i|^2}\right)\right\},$$

$$\frac{2\left(\left(\left(1-|1-\alpha|^2\right)\prod\limits_{i=1}^{n}|z_i|\right)^2-|H_2(1)|^2\right)}{\left(\left(1-|1-\alpha|^2\right)\prod\limits_{i=1}^{n}|z_i|\right)^2-|H_2(1)|^2+\left(1-|1-\alpha|^2\right)\prod\limits_{i=1}^{n}|z_i|\left|2\left(c_4-c_2\left(c_2^2+2H_2(1)\right)\right)-H_2(1)\sum\limits_{i=1}^{n}\frac{1-|z_i|^2}{z_i}\right|}$$

$$\leq\frac{\left(1-|1-\alpha|^2\right)\prod\limits_{i=1}^{n}|z_i|+|H_2(1)|}{\left(1-|1-\alpha|^2\right)\prod\limits_{i=1}^{n}|z_i|-|H_2(1)|}\left\{\frac{1-|1-\alpha|^2}{|\alpha|^2}|f''(z_0)|\,|1+\alpha|^2-\left(2+\sum_{i=1}^{n}\frac{1-|z_i|^2}{|z_0-z_i|^2}\right)\right\},$$

$$\frac{2\left(\left(1-|1-\alpha|^2\right)\prod\limits_{i=1}^{n}|z_i|-|H_2(1)|\right)^2}{\left(\left(1-|1-\alpha|^2\right)\prod\limits_{i=1}^{n}|z_i|\right)^2-|H_2(1)|^2+\left(1-|1-\alpha|^2\right)\prod\limits_{i=1}^{n}|z_i|\left|2\left(c_4-c_2\left(c_2^2+2H_2(1)\right)\right)-H_2(1)\sum\limits_{i=1}^{n}\frac{1-|z_i|^2}{z_i}\right|}$$

$$\leq\frac{1-|1-\alpha|^2}{|\alpha|^2}|f''(z_0)|\,|1+\alpha|^2-\left(2+\sum_{i=1}^{n}\frac{1-|z_i|^2}{|z_0-z_i|^2}\right)$$

and

$$|f''(z_0)|\geq\frac{|\alpha|^2}{\left(1-|1-\alpha|^2\right)|1+\alpha|^2}\left(2+\sum_{i=1}^{n}\frac{1-|z_i|^2}{|z_0-z_i|^2}\right.$$

$$\left.+\frac{2\left(\left(1-|1-\alpha|^2\right)\prod\limits_{i=1}^{n}|z_i|-|H_2(1)|\right)^2}{\left(\left(1-|1-\alpha|^2\right)\prod\limits_{i=1}^{n}|z_i|\right)^2-|H_2(1)|^2+\left(1-|1-\alpha|^2\right)\prod\limits_{i=1}^{n}|z_i|\left|2\left(c_4-c_2\left(c_2^2+2H_2(1)\right)\right)-H_2(1)\sum\limits_{i=1}^{n}\frac{1-|z_i|^2}{z_i}\right|}\right).$$

Now, we shall show that the inequality (2.1) is sharp. Let

$$\frac{p(z)-p(0)}{1-\overline{p(0)}p(z)}=z^3\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}$$

and

$$p(z)=\frac{z^3\prod\limits_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}+p(0)}{1+\overline{p(0)}z^3\prod\limits_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}}.$$

If we take the derivative of both sides of the last equation, for $z=1$ and $z_1, z_2, \ldots, z_n \in \mathbb{R}^+$, we obtain

$$p'(1) = \frac{1-|1-\alpha|^2}{(2-\overline{\alpha})^2}\left(3+\sum_{i=1}^{n}\frac{1+z_i}{1-z_i}\right).$$

Since $p'(1) = f''(1)\,(1+\alpha)^2$ and $\alpha \in \mathbb{R}$, we have

$$f''(1) = \frac{\alpha}{(2-\alpha)\,(1+\alpha)^2}\left(3+\sum_{i=1}^{n}\frac{1+z_i}{1-z_i}\right).$$

On the other hand, we obtain

$$p(z) = \frac{z^3\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}+p(0)}{1+\overline{p(0)}z^3\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}},$$

$$1-\alpha+\left(c_3-c_2^2\right)z^2+\left(2c_4-4c_2c_3+2c_2^3\right)z^3+\cdots = \frac{z^3\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}+1-\alpha}{1+(1-\overline{\alpha})\,z^3\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}},$$

$$\left(c_3-c_2^2\right)z^2+\left(2c_4-4c_2c_3+2c_2^3\right)z^3+\cdots = \frac{\left(1-|1-\alpha|^2\right)z^3\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}}{1+(1-\overline{\alpha})\,z^3\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}},$$

and

$$\left(c_3-c_2^2\right)+\left(2c_4-4c_2c_3+2c_2^3\right)z+\cdots = \frac{\left(1-|1-\alpha|^2\right)z\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}}{1+(1-\overline{\alpha})\,z^3\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}}.$$

Passing to limit in the last equality yields $c_3-c_2^2 = H_2(1) = 0$. Similarly, using straightforward calculations, we take

$$\left|2c_4-4c_2c_3+2c_2^3\right| = \left(1-|1-\alpha|^2\right)\prod_{i=1}^{n}|z_i|.$$

Thus, for $\alpha \in \mathbb{R}$, we get

$$\frac{|\alpha|^2}{\left(1-|1-\alpha|^2\right)|1+\alpha|^2}\left(2+\sum_{i=1}^{n}\frac{1-|z_i|^2}{|z_0-z_i|^2}\right.$$
$$\left.+\frac{2\left(\left(1-|1-\alpha|^2\right)\prod_{i=1}^{n}|z_i|-|H_2(1)|\right)^2}{\left(\left(1-|1-\alpha|^2\right)\prod_{i=1}^{n}|z_i|\right)^2-|H_2(1)|^2+\left(1-|1-\alpha|^2\right)\prod_{i=1}^{n}|z_i|\left|2\left(c_4-c_2\left(c_2^2+2H_2(1)\right)\right)-H_2(1)\sum_{i=1}^{n}\frac{1-|z_i|^2}{z_i}\right|}\right)$$

$$
\begin{aligned}
&= \frac{\alpha}{(2-\alpha)(1+\alpha)^2}\left(2+\sum_{i=1}^{n}\frac{1+z_i}{1-z_i}\right.\\
&\quad\left.+\frac{2\left(\left(1-|1-\alpha|^2\right)\prod_{i=1}^{n}z_i\right)^2}{\left(\left(1-|1-\alpha|^2\right)\prod_{i=1}^{n}z_i\right)^2+\left(1-|1-\alpha|^2\right)\prod_{i=1}^{n}z_i\left(1-|1-\alpha|^2\right)\prod_{i=1}^{n}|z_i|}\right)\\
&= \frac{\alpha}{(2-\alpha)(1+\alpha)^2}\left(3+\sum_{i=1}^{n}\frac{1+z_i}{1-z_i}\right).
\end{aligned}
$$

□

If $f(z)-z$ has no zeros different from $z=0$ in Theorem 2.1, this is given by the following theorem.

Theorem 2.2. *Let $f(z)\in\mathcal{N}(\alpha)$ and $c_3>c_2^2$ $(c_2>0,\ c_3>0)$. Also, $f(z)-z$ has no zeros in D except $z=0$. Suppose that, for some $z_0\in\partial D$, f has an angular limit $f(z_0)$ at z_0, $f(z_0)=\frac{z_0}{1+\alpha}$ and $f'(z_0)=\frac{1}{1+\alpha}$. Then we have*

(2.2)

$$
|f''(z_0)| \geq \frac{|\alpha|^2}{\left(1-|1-\alpha|^2\right)|1+\alpha|^2}\left(2-\frac{\ln^2\left(\frac{H_2(1)}{1-|1-\alpha|^2}\right)|H_2(1)|}{\ln\left(\frac{H_2(1)}{1-|1-\alpha|^2}\right)|H_2(1)|-\left|c_4-c_2\left(c_2^2+2H_2(1)\right)\right|}\right)
$$

and

$$
\left|c_4-c_2\left(c_2^2+2H_2(1)\right)\right| \leq \left||H_2(1)|\ln\left(\frac{H_2(1)}{1-|1-\alpha|^2}\right)\right|. \tag{2.3}
$$

The inequality (2.3) is sharp.

Proof. Let $\vartheta(z)$ be as in the proof of Theorem 2.1. Also, let $s(z)=z^2$. By the maximum principle, for each $z\in D$, we have $|\vartheta(z)|\leq|s(z)|$. The function

$$
t(z)=\frac{\vartheta(z)}{s(z)}
$$

is analytic in D and $|t(z)|<1$ for $z\in D$. In particular, we have

$$
\begin{aligned}
t(z) &= \frac{p(z)-p(0)}{\left(1-\overline{p(0)}p(z)\right)z^2}\\
&= \frac{1-\alpha+\left(c_3-c_2^2\right)z^2+\left(2c_4-4c_2c_3+2c_2^3\right)z^3+\cdots-(1-\alpha)}{\left[1-(1-\overline{\alpha})\left(1-\alpha+\left(c_3-c_2^2\right)z^2+\left(2c_4-4c_2c_3+2c_2^3\right)z^3+\cdots\right)\right]z^2}\\
&= \frac{\left(c_3-c_2^2\right)+\left(2c_4-4c_2c_3+2c_2^3\right)z+\cdots}{1-(1-\overline{\alpha})\left(1-\alpha+\left(c_3-c_2^2\right)z^2+\left(2c_4-4c_2c_3+2c_2^3\right)z^3+\cdots\right)},
\end{aligned}
$$

$$
|t(0)|=\frac{|H_2(1)|}{1-|1-\alpha|^2} \tag{2.4}
$$

and

$$|t'(0)| = \frac{\left|2c_4 - 4c_2c_3 + 2c_2^3\right|}{1-|1-\alpha|^2} = \frac{2\left|c_4 - c_2\left(c_2^2 + 2H_2(1)\right)\right|}{1-|1-\alpha|^2}.$$

Having in mind inequality (2.4), we denote by $\ln t(z)$ the analytic branch of the logarithm normed by the condition

$$\ln t(0) = \ln\left(\frac{H_2(1)}{1-|1-\alpha|^2}\right) < 0.$$

Consider the function

$$w(z) = \frac{\ln t(z) - \ln t(0)}{\ln t(z) + \ln t(0)}.$$

$w(z)$ is analytic in the unit disc D, $|w(z)| < 1$ for $|z| < 1$, $w(0) = 0$ and $|w(z_0)| = 1$ for $z_0 \in \partial D$. From (1.3) for $p = 1$, we obtain

$$\begin{aligned}\frac{2}{1+|w'(0)|} \leq |w'(z_0)| &= \frac{|2\ln t(0)|}{|\ln t(z_0) + \ln t(0)|^2}\left|\frac{t'(z_0)}{t(z_0)}\right| \\ &= \frac{-2\ln t(0)}{\arg^2 t(z_0) + \ln^2 t(0)}\left\{|\vartheta'(z_0)| - |s'(z_0)|\right\}.\end{aligned}$$

Since

$$|\vartheta'(z_0)| = \frac{1-|p(0)|^2}{\left|1-\overline{p(0)}p(z_0)\right|^2}\left|p'\left(z_0\right)\right| = \frac{1-|1-\alpha|^2}{|\alpha|^2}\left|f''(z_0)\right||1+\alpha|^2$$

and

$$\begin{aligned}|w'(0)| &= \frac{|2\ln t(0)|}{|\ln t(0) + \ln t(0)|^2}\left|\frac{t'(0)}{t(0)}\right| \\ &= \frac{-1}{2\ln\left(\frac{H_2(1)}{1-|1-\alpha|^2}\right)}\frac{2\left|c_4 - c_2\left(c_2^2 + 2H_2(1)\right)\right|}{|H_2(1)|},\end{aligned}$$

we take

$$\frac{2}{1-\frac{\left|c_4-c_2\left(c_2^2+2H_2(1)\right)\right|}{\ln\left(\frac{H_2(1)}{1-|1-\alpha|^2}\right)|H_2(1)|}} \leq \frac{-2\ln t(0)}{\arg^2 t(z_0) + \ln^2 t(0)}\left\{\frac{1-|1-\alpha|^2}{|\alpha|^2}\left|f''(z_0)\right||1+\alpha|^2 - 2\right\}.$$

Replacing $\arg^2 t(z_0)$ by zero, we obtain

$$\frac{1}{1-\frac{\left|c_4-c_2\left(c_2^2+2H_2(1)\right)\right|}{\ln\left(\frac{H_2(1)}{1-|1-\alpha|^2}\right)|H_2(1)|}} \leq \frac{-1}{\ln\left(\frac{H_2(1)}{1-|1-\alpha|^2}\right)}\left\{\frac{1-|1-\alpha|^2}{|\alpha|^2}\left|f''(z_0)\right||1+\alpha|^2 - 2\right\},$$

$$2 - \frac{\ln^2\left(\frac{H_2(1)}{1-|1-\alpha|^2}\right)|H_2(1)|}{\ln\left(\frac{H_2(1)}{1-|1-\alpha|^2}\right)|H_2(1)| - \left|c_4 - c_2\left(c_2^2 + 2H_2(1)\right)\right|} \leq \frac{1-|1-\alpha|^2}{|\alpha|^2}\left|f''(z_0)\right||1+\alpha|^2$$

and

$$|f''(z_0)| \geq \frac{|\alpha|^2}{\left(1-|1-\alpha|^2\right)|1+\alpha|^2}\left(2-\frac{\ln^2\left(\frac{H_2(1)}{1-|1-\alpha|^2}\right)|H_2(1)|}{\ln\left(\frac{H_2(1)}{1-|1-\alpha|^2}\right)|H_2(1)|-\left|c_4-c_2\left(c_2^2+2H_2(1)\right)\right|}\right).$$

Similarly, the function $w(z)$ satisfies the assumptions of the Schwarz lemma [6], we obtain

$$\begin{aligned}
1 \geq |w'(0)| &= \frac{|2\ln t(0)|}{|\ln t(0)+\ln t(0)|^2}\left|\frac{t'(0)}{t(0)}\right| \\
&= \frac{-1}{2\ln\left(\frac{H_2(1)}{1-|1-\alpha|^2}\right)}\frac{2\left|c_4-c_2\left(c_2^2+2H_2(1)\right)\right|}{|H_2(1)|} \\
&= -\frac{\left|c_4-c_2\left(c_2^2+2H_2(1)\right)\right|}{|H_2(1)|\ln\left(\frac{H_2(1)}{1-|1-\alpha|^2}\right)}
\end{aligned}$$

and

$$\left|c_4-c_2\left(c_2^2+2H_2(1)\right)\right| \leq \left||H_2(1)|\ln\left(\frac{H_2(1)}{1-|1-\alpha|^2}\right)\right|.$$

Now, we shall show that the inequality (2.3) is sharp. Let

$$\frac{\ln t(z)-\ln t(0)}{\ln t(z)+\ln t(0)} = z,$$

$$p(z) = \frac{1-\alpha+z^2e^{\frac{1+z}{1-z}\ln t(0)}}{1+(1-\overline{\alpha})z^2e^{\frac{1+z}{1-z}\ln t(0)}} = 1-\alpha+\frac{\left(1-|1-\alpha|^2\right)z^2e^{\frac{1+z}{1-z}\ln t(0)}}{1+(1-\overline{\alpha})z^2e^{\frac{1+z}{1-z}\ln t(0)}},$$

$$\frac{1-\alpha+\left(c_3-c_2^2\right)z^2+\left(2c_4-4c_2c_3+2c_2^3\right)z^3+\cdots-(1-\alpha)}{z^2} = \frac{\left(1-|1-\alpha|^2\right)e^{\frac{1+z}{1-z}\ln t(0)}}{1+(1-\overline{\alpha})z^2e^{\frac{1+z}{1-z}\ln t(0)}}$$

and

$$\left(c_3-c_2^2\right)+\left(2c_4-4c_2c_3+2c_2^3\right)z+\cdots = \frac{\left(1-|1-\alpha|^2\right)e^{\frac{1+z}{1-z}\ln t(0)}}{1+(1-\overline{\alpha})z^2e^{\frac{1+z}{1-z}\ln t(0)}}.$$

If we take the derivative of both sides of the last equation, we take

$$\begin{aligned}
&\left(2c_4-4c_2c_3+2c_2^3\right)+\cdots \\
&= \left(1-|1-\alpha|^2\right)\left(\frac{\frac{2}{(1-z)^2}\ln t(0)e^{\frac{1+z}{1-z}\ln t(0)}\left(1+(1-\overline{\alpha})z^2e^{\frac{1+z}{1-z}\ln t(0)}\right)}{\left(1+(1-\overline{\alpha})z^2e^{\frac{1+z}{1-z}\ln t(0)}\right)^2}\right. \\
&\quad \left.+\frac{(1-\overline{\alpha})\left(2ze^{\frac{1+z}{1-z}\ln t(0)}+\frac{2}{(1-z)^2}\ln t(0)e^{\frac{1+z}{1-z}\ln t(0)}z^2\right)e^{\frac{1+z}{1-z}\ln t(0)}}{\left(1+(1-\overline{\alpha})z^2e^{\frac{1+z}{1-z}\ln t(0)}\right)^2}\right).
\end{aligned}$$

Passing to limit $(z \to 0)$ in the last equality yields

$$\begin{aligned} 2c_4 - 4c_2c_3 + 2c_2^3 &= \left(1 - |1-\alpha|^2\right) 2t(0) \ln t(0) \\ &= \left(1 - |1-\alpha|^2\right) 2 \frac{H_2(1)}{1-|1-\alpha|^2} \ln\left(\frac{H_2(1)}{1-|1-\alpha|^2}\right) \\ &= 2H_2(1) \ln\left(\frac{H_2(1)}{1-|1-\alpha|^2}\right), \end{aligned}$$

$$2\left|c_4 - c_2\left(c_2^2 + 2H_2(1)\right)\right| = 2\left|H_2(1)\ln\left(\frac{H_2(1)}{1-|1-\alpha|^2}\right)\right|$$

and

$$\left|c_4 - c_2\left(c_2^2 + 2H_2(1)\right)\right| = \left|H_2(1)\ln\left(\frac{H_2(1)}{1-|1-\alpha|^2}\right)\right|. \qquad \square$$

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