Commun. Korean Math. Soc. **40** (2025), No. 1, pp. 71–84
https://doi.org/10.4134/CKMS.c240136
pISSN: 1225-1763 / eISSN: 2234-3024

DERIVATIONS ON DISTRIBUTIVE BILATTICES

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Abstract. In this paper, a derivation is defined on a reduct or two reducts of a distributive bilattice, with and without a relationship between derivations on reducts. The concept of a differential distributive bilattice is introduced. The algebraic structure and properties are investigated. Characterization and Construction theorems are proved.

1. Introduction

In [23], G. Szász defined a derivation on lattice as an extension of derivation concept on rings. After that X. L. Xin et al. in [26] and [25] modified the divination of derivation to compatible with some applications in information science. Many researches covered derivations, e.g. [13, 16, 21, 24] and its generalizations like an (F, G)-derivation [1], a symmetric bi-derivation [5], a higher derivation [6], an (n, m) derivation [7], and others. M. L. Ginsberg [14] and M. Fitting [11, 12] introduced bilattice as an algebra with two distinct lattice structures. Bilattices used for algebraic representation of inferences in AI and logical programming. The structure and applications of a bilattice are studied by many authors e.g. [2, 17–19, 22]. The bilattice theory is growing very fast. There are many relationships between two lattice structures of reducts of a bilattice are introduced in many articles for example of an interlaced bilattice, a modular bilattice, a distributive bilattice, a bilattice with negation, pseudo-complemented bilattices, bi-double Stone algebra and bi-concept algebra refer to [1,8–10,17–20]. This article is restricted on a distributive bilattice. A derivation is defined and some algebraic properties are investigated. A distributive bilattice with differential reducts and a differential bilattice are defined and important properties are proved. We consider that the reader is obligated to lattice theory and for more details refer to [3, 4, 15].

Section 2 recalls the basics of a bilattice and a derivation used in the next sections. In Section 3, a derivation on one reduct of a distributive bilattice is defined and some related algebraic results are proved. Section 4 introduces the

Received May 27, 2024; Accepted October 2, 2024.
2020 *Mathematics Subject Classification.* 06B35, 06B75, 06B99, 06F99, 03G10.
Key words and phrases. Bilattice, derivation, distributive bilattice, differential lattice, isotone map, isomorphism.

concept of a distributive bilattice with differential reducts. The necessary and sufficient conditions for differential reducts are given and associated properties are shown. Finally, in Section 5 a differential distributive bilattice is defined and its algebraic structure is investigated.

2. Preliminaries

Here we present the most important terms and results related to bilattices and derivations, on which the rest of the article depends.

2.1. Terminology of a bilattice

Definition ([14])**.** An algebra $\mathfrak{B} = (B; \wedge, \vee, 0_1, 1_1, \bullet, +, 0_2, 1_2)$ is a bounded bilattice if $\mathfrak{B}_1 = (B; \wedge, \vee, 0_1, 1_1)$ and $\mathfrak{B}_2 = (B; \bullet, +, 0_2, 1_2)$ are bounded lattices.

$\mathfrak{B}_1$ is the first reduct of a bilattice $\mathfrak{B}$ associate with order relation $\leq_1$ and $\mathfrak{B}_2$ is the second reduct associate with order relation $\leq_2$. Both $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are called the main reducts of a bilattice $\mathfrak{B}$.

Definition ([17])**.** An interlaced bilattice is a bounded bilattice that satisfies that: for all $u, v, u', v' \in \mathfrak{B}$, such that $u \leq_i u'$, and $v \leq_i v'$, and $i = 1, 2$

$$u \wedge v \leq_i u' \wedge v', \ u \vee v \leq_i u' \vee v',$$

$$u \bullet v \leq_i u' \bullet v', \text{ and } u + v \leq_i u' + v'.$$

A bilattice $\mathfrak{B}$ is distributive if and only if any operation of the set $\{\wedge, \vee, \bullet, +\}$ of binary operations distributes over the others. Every distributive bilattice is interlaced.

Let $\mathfrak{L}_1 = (L_1; \wedge_1, \vee_1, 0, 1)$ and be bounded $\mathfrak{L}_2 = (L_2; \wedge_2, \vee_2, 0', 1')$ lattices. The product bilattice associated with $\mathfrak{L}_1$ and $\mathfrak{L}_2$ is defined as: $\mathfrak{B}(\mathfrak{L}_1, \mathfrak{L}_2) = (L_1 \times L_2; \wedge, \vee, 0_1, 1_1, \bullet, +, 0_2, 1_2)$ such that:

$$\begin{aligned}
(u, v) \wedge (u', v') &= (u \wedge_1 u', v \vee_2 v'), \\
(u, v) \vee (u', v') &= (u \vee_1 u', v \wedge_2 v'), \\
(u, v) \bullet (u', v') &= (u \wedge_1 u', v \wedge_2 v'), \\
(u, v) + (u', v') &= (u \vee_1 u', v \vee_2 v'),
\end{aligned}$$

for all $(u, v), (u', v') \in L_1 \times L_2$, see [26].

To deduce the structure and representation theorems of interlaced bilattices positive and negative elements are defined as follows.

Definition ([17])**.** Let $\mathfrak{B} = (B; \wedge, \vee, 0_1, 1_1, \bullet, +, 0_2, 1_2)$ be an interlaced bilattice. Then:

i) $u \in \mathfrak{B}$ is a positive element if $u \leq_1 v \Longrightarrow u \leq_2 v$ for any $v \in \mathfrak{B}$;
ii) $u \in \mathfrak{B}$ is a negative element if $u \leq_1 v \Longrightarrow v \leq_2 u$ for any $v \in \mathfrak{B}$.

$POS(\mathfrak{B})$ indicates to the set of all positive elements and $NEG(\mathfrak{B})$ indicates to the set of all negative elements. In general $POS(\mathfrak{B}) = [0_2, 1_1]_{\leq_1} = [0_2, 1_1]_{\leq_2}$ and $NEG(\mathfrak{B}) = [0_2, 0_1]_{\geq_1} = [0_2, 0_1]_{\leq_2}$, see [17] and [2].

Theorem 2.1 ([2,11])**.** (***Characterization Theorem***) *A bilattice* $\mathfrak{B} = (B; \wedge, \vee, 0_1, 1_1, \bullet, +, 0_2, 1_2)$ *is an interlaced* (*distributive*) *if and only if there exist bounded* (*distri-butive*) *lattices* $\mathfrak{L}_1 = (L_1; \wedge_1, \vee_1 0, 1)$ *and* $\mathfrak{L}_2 = (L_1; \wedge_2, \vee_2 0', 1')$ *such that* $\mathfrak{B} \cong \mathfrak{B}(\mathfrak{L}_1, \mathfrak{L}_2)$.

In particular, $\mathfrak{B}(POS(\mathfrak{B}), NEG(\mathfrak{B})) = (POS(\mathfrak{B}) \times NEG(\mathfrak{B}); \wedge', \vee', 0_1', 1_1', \bullet, +', 0_2', 1_2')$ such that:

$$\begin{aligned}
(u, v) \wedge' (u', v') &= (u \wedge u', v \vee v') = (u \bullet u', v \bullet v'),\\
(u, v) \vee' (u', v') &= (u \vee u', v \wedge v') = (u + u', v + v'),\\
(u, v) \bullet (u', v') &= (u \wedge u', v \wedge v') = (u \bullet u', v + v'),\\
(u, v) +' (u', v') &= (u \vee u', v \vee v') = (u + u', v \bullet v')
\end{aligned}$$

for all $(u, v), (u', v') \in POS(\mathfrak{B}) \times NEG(\mathfrak{B})$,

$$0_1' = (0_1, 1_2),\ 1_1' = (1_1, 0_2),\ 0_2' = (0_1, 0_2),\ \text{and } 1_2' = (1_1, 1_2).$$

Accordingly, $\mathfrak{B} \cong \mathfrak{B}(POS(\mathfrak{B}), NEG(\mathfrak{B}))$.

In the following

$$\begin{aligned}
POS(\mathfrak{B})_{\leq_1} &= (POS(\mathfrak{B}); \wedge, \vee, 0_2, 1_1),\\
POS(\mathfrak{B})_{\leq_2} &= (POS(\mathfrak{B}); \bullet, +, 0_2, 1_1),\\
NEG(\mathfrak{B})_{\leq_1} &= (NEG(\mathfrak{B}); \wedge, \vee, 0_1, 0_2), \text{ and}\\
NEG(\mathfrak{B})_{\leq_2} &= (NEG(\mathfrak{B}); \bullet, +, 0_2, 0_1).
\end{aligned}$$

2.2. Terminology of a derivation

Definition ([26])**.** Let $\mathfrak{L} = (L; \wedge, \vee)$ be a lattice and $\vartheta : L \longrightarrow L$ be a map. Then ϑ is called a derivation on $\mathfrak{L}$ if it satisfies that:

$$\vartheta(u \wedge v) = (\vartheta(u) \wedge v) \vee (u \wedge \vartheta(v)),$$

for any $u, v \in \mathfrak{L}$.

A differential lattice $\mathfrak{L} = (L; \wedge, \vee, \vartheta, 0, 1)$ is a bounded lattice $(L; \wedge, \vee, 0, 1)$ with derivation ϑ. An isotone derivation is satisfying that:

$$\text{If } u \leq v, \text{ then } \vartheta(u) \leq \vartheta(v).$$

Proposition 2.2 ([26])**.** *Let* $\mathfrak{L} = (L; \wedge, \vee, \vartheta, 0, 1)$ *be a differential lattice. Then*:

i) $\vartheta(u) \leq u$ *and* $\vartheta(0) = 0$;
ii) $\vartheta(u) \wedge \vartheta(v) \leq \vartheta(u \wedge v) \leq \vartheta(u) \vee \vartheta(v)$;
iii) I *is an ideal of* $\mathfrak{L} \implies \vartheta(I) \subseteq I$;
iv) $\vartheta(u) = (u \wedge \vartheta(1)) \vee \vartheta(u)$;
v) $v \leq u = \vartheta(u) \implies \vartheta(v) = v$;
vi) $\vartheta(u) = \vartheta(u) \vee (u \wedge \vartheta(u \vee v))$;

vii) ϑ *is idempotent i.e.,* $\vartheta(\vartheta(u)) = \vartheta(u)$.

Corollary 2.3 ([26])**.** *Let* $\mathfrak{L} = (L; \wedge, \vee, \vartheta, 0, 1)$ *be a differential lattice. Then*:

i) $u \geq \vartheta(1) \implies \vartheta(u) \geq \vartheta(1)$;
ii) $u \leq \vartheta(1) \implies \vartheta(u) = u$;
iii) $\vartheta(1) = 1$ *if and only if* ϑ *is an identity derivation.*

3. Distributive bilattices with one differential reduct

In this section, the derivation on a reduct of a distributive bilattice is defined. Some algebraic properties are proved.

Definition. A distributive bilattice with first differential reduct $\mathfrak{B} = (B; \wedge, \vee, \vartheta, 0_1, 1_1, \bullet, +, 0_2, 1_2)$ is a distributive bilattice with a derivation ϑ on the first reduct $\mathfrak{B}_1 = (B; \wedge, \vee, 0_1, 1_1)$.

Example 3.1. Consider the distributive bilattice in Figure 1. A map $\vartheta : \mathfrak{B}_1 \longrightarrow \mathfrak{B}_1$ defined as:

$$\vartheta(u) = \begin{cases} 0_1 & \text{for} \quad u = 0_1, 0_2, a, b \ \text{ and } \ c \\ 1_2 & \text{for} \quad u = 1_1, 1_2, d, e \ \text{ and } \ f \end{cases}$$

is a derivation on the first reduct $\mathfrak{B}_1$. So, it is a distributive bilattice with the first differential reduct.

Consider two differential lattices $\mathfrak{L} = (L; \wedge, \vee, \vartheta, 0, 1)$ and $\mathfrak{K} = (K; \wedge, \vee, \vartheta', 0', 1')$, then a homomorphism $\phi : \mathfrak{L} \longrightarrow \mathfrak{K}$ is preserving derivation if $\phi(\vartheta(u)) = \vartheta'(\phi(u))$ for all $u \in L$. Assume derivations $\vartheta_P : POS(\mathfrak{B})_{\leq_1} \longrightarrow POS(\mathfrak{B})_{\leq_1}$ and $\vartheta_N : NEG(\mathfrak{B})_{\leq_1} \longrightarrow NEG(\mathfrak{B})_{\leq_1}$ are defined as $\vartheta_P(u \bullet 1_1) = \vartheta|_{POS(\mathfrak{B})}(u \bullet 1_1)$ and $\vartheta_N(u \bullet 0_1) = \vartheta|_{NEG(\mathfrak{B})}(u \bullet 0_1)$. Then we get the following results.

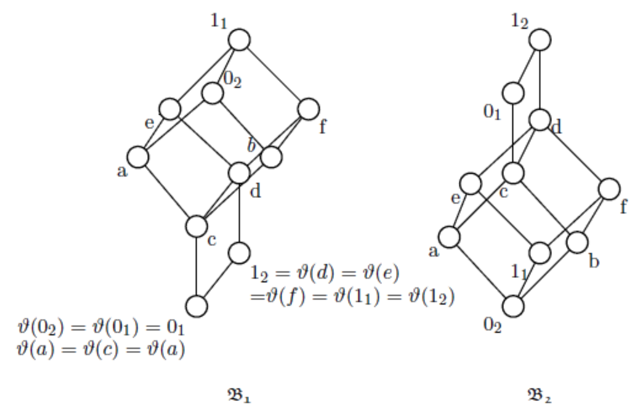

FIGURE 1. Distributive bilattice $\mathfrak{B}$ with first differential reduct

Lemma 3.2. *Let $\mathfrak{B} = (B; \wedge, \vee, \vartheta, 0_1, 1_1, \bullet, +, 0_2, 1_2)$ be a distributive bilattice with first differential reduct $\mathfrak{B}_1 = (B; \wedge, \vee, \vartheta, 0_1, 1_1)$. Then for any $u \in \mathfrak{B}$,*

i) *The onto homomorphism map $\alpha_P : \mathfrak{B}_1 \longrightarrow POS_{\leq_1}(\mathfrak{B})$ defined as $\alpha_P(u) = u \bullet 1_1$ is preserving the derivation if and only if $\vartheta(u \bullet 1_1) = \vartheta(u) \bullet 1_1$;*

ii) *The onto homomorphism map $\alpha_N : \mathfrak{B}_1 \longrightarrow NEG_{\leq_1}(\mathfrak{B})$ defined as $\alpha_N(u) = u \bullet 0_1$ is preserving the derivation if and only if $\vartheta(u \bullet 0_1) = \vartheta(u) \bullet 0_1$.*

Proposition 3.3. *Let $\mathfrak{B} = (B; \wedge, \vee, \vartheta, 0_1, 1_1, \bullet, +, 0_2, 1_2)$ be a distributive bilattice with first differential reduct, α_P and α_N be onto homomorphism maps which preserve derivations. Then for any $u, v \in \mathfrak{B}$,*

$$\vartheta_P(\alpha_P(u) \wedge \alpha_P(v)) = \alpha_P(\vartheta(u \wedge v)) \text{ and}$$
$$\vartheta_N(\alpha_N(u) \wedge \alpha_N(v)) = \alpha_N(\vartheta(u \wedge v)).$$

Proof.

$$\begin{aligned}\vartheta_P(\alpha_P(u) \wedge \alpha_P(v)) &= (\vartheta_P(\alpha_P(u)) \wedge \alpha_P(v)) \vee (\alpha_P(u) \wedge \vartheta_P(\alpha_P(v))) \\ &= (\alpha_P(\vartheta(u)) \wedge \alpha_P(v)) \vee (\alpha_P(u) \wedge \alpha_P(\vartheta(v)) \\ &= \alpha_P(\vartheta(u) \wedge v) \vee \alpha_P(u \wedge \vartheta(v)) \\ &= \alpha_P((\vartheta(u) \wedge v) \vee (u \wedge \vartheta(v))) = \alpha_P(\vartheta(u \wedge v)).\end{aligned}$$

Similarly, the second part can be proven. □

Theorem 3.4. *Let $\mathfrak{B} = (B; \wedge, \vee, \vartheta, 0_1, 1_1, \bullet, +, 0_2, 1_2)$ be a distributive bilattice with first differential reduct, α_P and α_N be onto homomorphisms which preserve derivations. Then for any $u, v \in \mathfrak{B}$;*

i) *$\alpha_P(\vartheta(u)) \leq_1 \alpha_P(u)$ and $\alpha_N(\vartheta(u)) \leq_1 \alpha_N(u)$;*

ii) *$\alpha_P(\vartheta(u) \wedge \vartheta(v)) \leq_1 \alpha_P(u) \wedge \alpha_P(v)$ and $\alpha_N(\vartheta(u) \wedge \vartheta(v)) \leq_1 \alpha_N(u) \wedge \alpha_N(v)$;*

iii) *$\alpha_P(\vartheta(u)) \wedge \alpha_P(\vartheta(v)) \leq_1 \vartheta_P(\alpha_P(u \wedge v)) \leq_1 \alpha_P(u) \vee \alpha_P(v)$ and $\alpha_N(\vartheta(u)) \wedge \alpha_N(\vartheta(v)) \leq_1 \vartheta_N(\alpha_N(u \wedge v)) \leq_1 \alpha_N(u) \vee \alpha_N(v)$;*

iv) *If I is an ideal of $\mathfrak{B}_1$, then $\alpha_P(\vartheta(I)) \subseteq \alpha_P(I)$ and $\alpha_N(\vartheta(I)) \subseteq \alpha_N(I)$;*

v) *If $u \leq_1 v$ and $\vartheta(v) = v$, then $\alpha_P(\vartheta(u)) = \alpha_P(u)$ and $\alpha_N(\vartheta(u)) = \alpha_N(u)$.*

Proof. i) $\alpha_P(\vartheta(u)) = \vartheta_P(\alpha_P(u)) \leq_1 \alpha_P(u)$.

ii) $\alpha_P(\vartheta(u) \wedge \vartheta(v)) = \alpha_P(\vartheta(u)) \wedge \alpha_P(\vartheta(v)) = \vartheta_P(\alpha_P(u)) \wedge \vartheta_P(\alpha_P(v)) \leq_1 \alpha_P(u) \wedge \alpha_P(v)$.

iii) $\alpha_P(\vartheta(u)) \wedge \alpha_P(\vartheta(v)) = \alpha_P(\vartheta(u) \wedge \vartheta(v)) \leq_1 \alpha_P(\vartheta(u \wedge v)) = \vartheta_P(\alpha_P(u \wedge v)) \leq_1 \alpha_P(\vartheta(u) \vee \vartheta(v)) = \alpha_P(\vartheta(u)) \vee \alpha_P(\vartheta(v)) = \vartheta_P(\alpha_P(u)) \vee \vartheta_P(\alpha_P(v)) \leq_1 \alpha_P(u) \vee \alpha_P(v)$.

iv) If I is an ideal of $\mathfrak{B}_1$, then $\alpha_P(\vartheta(I)) = \vartheta_P(\alpha_P(I)) \subseteq \alpha_P(I)$, (from (iii) in Proposition 2.2.

v) Immediately from v) in Proposition 2.2. □

Proposition 3.5. *Let $\mathfrak{B} = (B; \wedge, \vee, \vartheta, 0_1, 1_1, \bullet, +, 0_2, 1_2)$ be a distributive bilattice with first differential reduct, α_P and α_N be onto homomorphisms which preserve derivations. Then for any $u, v \in \mathfrak{B}$;*

i) $\alpha_P(\vartheta(0_2)) = 0_2$ *and* $\alpha_N(\vartheta(0_1)) = 0_1$;
ii) $\alpha_P(\vartheta(1_1)) \leq_1 1_1$ *and* $\alpha_N(\vartheta(0_2)) \leq_1 0_2$;
iii) *If* $\vartheta(1_1) \leq_1 u$, *then* $\vartheta_P(1_1) \leq_1 \alpha_P(u)$ *and* $\vartheta_N(0_2) \leq_1 \alpha_N(u)$;
iv) *If* $u \leq_1 \vartheta(1_1)$, *then* $\alpha_P(\vartheta(u)) = \alpha_P(u)$ *and* $\alpha_N(\vartheta(u)) = \alpha_N(u)$;
v) $\alpha_P(\vartheta(1_1)) = 1_1$ *and* $\alpha_N(\vartheta(1_1)) = 0_2$ *if and only if* ϑ *is the identity derivation.*

Proof. i) $\alpha_P(\vartheta(0_2)) = \vartheta_P(\alpha_P(0_2)) = \vartheta_P(0_2) = 0_2$.

ii) $\alpha_P(\vartheta(1_1)) = \vartheta_P(\alpha_P(1_1)) \leq_1 \alpha_P(1_1) = 1_1$.

iii) If $\vartheta(1_1) \leq_1 u$, then $\alpha_P(\vartheta(1_1)) \leq_1 \alpha_P(u)$. From i) in Corollary 2.3 we have $\vartheta(1_1) \leq_1 \vartheta(u)$. Hence $\alpha_P(\vartheta(1_1)) \leq_1 \alpha_P(u)$. But $\alpha_P(\vartheta(1_1)) = \vartheta_P(\alpha_P(1_1)) = \vartheta_P(1_1)$. So $\vartheta_P(1_2) \leq_1 \alpha_P(u)$.

iv) Immediately from ii) in Corollary 2.3.

v) Using iii) in Corollary 2.3, we get ϑ is identity derivation if and only if $\vartheta(1_1) = 1_1$ if and only if $\alpha_P(\vartheta(1_1)) = \vartheta(1_1) \bullet 1_1 = 1_1 \bullet 1_1 = 1_1$ and $\alpha_N(\vartheta(1_1)) = \vartheta(1_1) \bullet 0_1 = 0_2$. □

The following example explains the above results.

Example 3.6. Consider the distributive bilattice with first differential reduct in Example 3.1. If we consider derivations ϑ_P and ϑ_N which are defined as:

$$\vartheta_P(u) = \begin{cases} 0_2 & \text{for} \quad u = 0_2 \\ 1_1 & \text{for} \quad u = 1_1 \end{cases}$$

and

$$\vartheta_N(u) = 0_1.$$

Then the following onto homomorphisms:

$$\alpha_P(u) = u \bullet 1_1 = \begin{cases} 0_2 & \text{for} \quad u = 0_1, 0_2, a, b \;\text{ and }\; c \\ 1_1 & \text{for} \quad u = 1_1, 1_2, d, e \;\text{ and }\; f \end{cases}$$

and

$$\alpha_N(u) = u \bullet 0_1 = \begin{cases} 0_1 & \text{for} \quad u = 0_1 \quad \text{and} \;\; 1_2 \\ a & \text{for} \quad u = a \quad \text{and} \;\; e \\ b & \text{for} \quad u = b \quad \text{and} \;\; f \\ c & \text{for} \quad u = c \quad \text{and} \;\; d \\ 0_2 & \text{for} \quad u = 0_2 \quad \text{and} \;\; 1_1 \end{cases}$$

are preserving derivations ϑ_P and ϑ_N on $POS(\mathfrak{B})_{\leq_1}$ and $NEG(\mathfrak{B})_{\leq_1}$ respectively to ϑ on first reduct $\mathfrak{B}_1$, for example:

$$\alpha_P(\vartheta(c)) = 0_2 = \vartheta_P(\alpha_P(c)) = \vartheta_P(0_2);$$
$$\alpha_P(\vartheta(e)) = 1_1 = \vartheta_P(\alpha_P(e)) = \vartheta_P(1_1);$$

and so on.

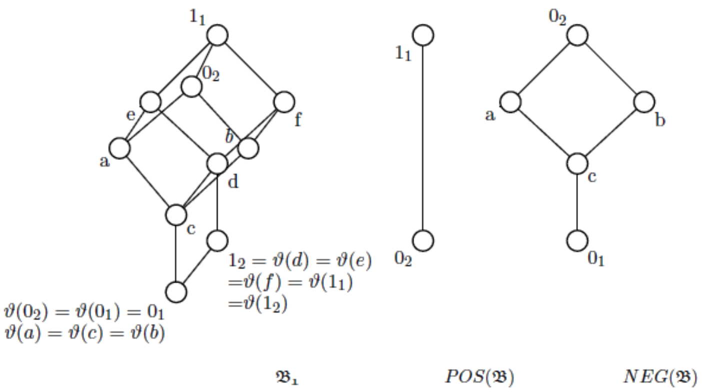

FIGURE 2. Differential reduct $\mathfrak{B}_1$, $POS(\mathfrak{B})$, $NEG(\mathfrak{B})$

If we consider other derivations on $POS(\mathfrak{B})_{\leq_1}$ and $NEG(\mathfrak{B})_{\leq_1}$, the given homomorphisms α_P and α_N do not preserve these derivations. For example, for derivations $\vartheta_P(u) = 0_2$ and $\vartheta_N(u) = u$ on $POS(\mathfrak{B})_{\leq_1}$ and $NEG(\mathfrak{B})_{\leq_1}$, respectively, we get $\alpha_P(\vartheta(e)) = 1_1 \neq \vartheta_P(\alpha_P(e)) = \vartheta_P(1_1) = 0_2$.

Definition. A distributive bilattice with second differential reduct $\mathfrak{B} = (B; \wedge, \vee, 0_1, 1_1, \bullet, +, \vartheta', 0_2, 1_2)$ is a distributive bilattice with derivation ϑ' on the second reduct $\mathfrak{B}_2 = (B; \bullet, +, 0_2, 1_2)$.

Consider derivations $\vartheta'_P : POS(\mathfrak{B})_{\leq_2} \longrightarrow POS(\mathfrak{B})_{\leq_2}$ and $\vartheta'_N : NEG(\mathfrak{B})_{\leq_2} \longrightarrow NEG(\mathfrak{B})_{\leq_2}$ defined as: $\vartheta'_P(u \bullet 1_1) = \vartheta'|_{POS(\mathfrak{B})}(u \bullet 1_1)$ and $\vartheta'_N(u \bullet 0_1) = \vartheta'|_{NEG(\mathfrak{B})}(u \bullet 0_1)$. Thus, by using onto homomorphisms $\beta_P : \mathfrak{B}_2 \longrightarrow POS_{\leq_2}(\mathfrak{B})$ and $\beta_N : \mathfrak{B}_2 \longrightarrow NEG_{\leq_2}(\mathfrak{B})$ defined as: $\beta_P(u) = u \bullet 1_1$ and $\beta_N(u) = u \bullet 0_1$, corresponding results about a distributive bilattice with second differential reduct can be proved.

4. Distributive bilattices with differential reducts

In the previous section, we explore that the derivation can be defined on one reduct of a distributive bilattice. In this section, we will discuss the derivations on both reducts.

Definition. A distributive bilattice with differential reducts $\mathfrak{B} = (B; \wedge, \vee, \vartheta, 0_1, 1_1, \bullet, +, \vartheta', 0_2, 1_2)$ is a distributive bilattice with two derivations ϑ and ϑ' on the first and the second reducts, respectively.

Theorem 4.1. (***Characterization Theorem***) *Let* $\mathfrak{B} = (B; \wedge, \vee, \vartheta, 0_1, 1_1, \bullet, +, \vartheta', 0_2, 1_2)$ *be a distributive bilattice with differential reducts. Then:* ϑ (ϑ') *is a derivation on the first reduct* $\mathfrak{B}_1$ (*the second reduct* $\mathfrak{B}_2$) *if and only if* *there exist two derivations on $POS(\mathfrak{B})_{\leq_1}$ and $NEG(\mathfrak{B})_{\leq_1}$ ($POS(\mathfrak{B})_{\leq_2}$ and $NEG(\mathfrak{B})_{\leq_2}$) which are preserved under an onto homomorphism.*

Proof. Let ϑ be a derivation on the first reduct $\mathfrak{B}_1$, α_P and α_N be the onto homomorphisms defined in Lemma 3.2 and $u \in \mathfrak{B}_1$. Then $\alpha_P(\vartheta(u)) = \vartheta(u) \bullet 1_1$ and $\vartheta|_{POS(\mathfrak{B})}(u \bullet 1_1) = \vartheta(u) \bullet 1_1$. So, $\vartheta|_{POS(\mathfrak{B})}(\alpha_P(u)) = \alpha_P(\vartheta(u))$ and the restriction of derivation ϑ on $POS(\mathfrak{B})$ is appropriate considered derivation on $POS(\mathfrak{B})$. Also $\alpha_N(\vartheta(u)) = \vartheta(u) \bullet 0_1$ and $\vartheta|_{NEG(\mathfrak{B})}(\alpha_N(u)) = \vartheta(u) \bullet 0_1$. Thus $\vartheta|_{NEG(\mathfrak{B})}(\alpha_N(u)) = \alpha_N(\vartheta(u))$ and the restriction of derivation ϑ on $NEG(\mathfrak{B})$ is appropriate considered a derivation on $NEG(\mathfrak{B})$.

For the other direction, assume ϑ_P and ϑ_N are two derivations on $POS(\mathfrak{B})$ and $NEG(\mathfrak{B})$, respectively. Define a map ϑ on first reduct $\mathfrak{B}_1$ as $\vartheta(u) = \vartheta_P(\alpha_P(u)) + \vartheta_N(\alpha_N(u))$. Now we prove that ϑ is a derivation on $\mathfrak{B}_1$.

$$
\begin{aligned}
&\vartheta(u \wedge w)\\
&= \vartheta_P(\alpha_P(u \wedge w)) + \vartheta_N(\alpha_N(u \wedge w))\\
&= \vartheta_P((u \wedge w) \bullet 1_1) + \vartheta_N((u \wedge w) \bullet 0_1)\\
&= \vartheta_P((u \bullet 1_1) \wedge (w \bullet 1_1)) + \vartheta_N((u \bullet 0_1) \wedge (w \bullet 0_1))\\
&= [(\vartheta_P(u \bullet 1_1) \wedge (w \bullet 1_1)) \vee ((u \bullet 1_1) \wedge \vartheta_P(w \bullet 1_1))]\\
&\quad + [(\vartheta_N(u \bullet 0_1) \wedge (w \bullet 0_1)) \vee ((u \bullet 0_1) \wedge \vartheta_N(w \bullet 0_1))]\\
&= [(\vartheta_P(u \bullet 1_1) \wedge w) \vee (u \wedge \vartheta_P(w \bullet 1_1))]\\
&\quad + [(\vartheta_N(u \bullet 0_1) \wedge w)) \vee ((u \wedge \vartheta_N(w \bullet 0_1))]\\
&= [(\vartheta_P(u \bullet 1_1) \wedge w) + (\vartheta_N(u \bullet 0_1) \wedge w)]\\
&\quad \vee [(\vartheta_P(u \bullet 1_1) \wedge w) + ((u \wedge \vartheta_N(w \bullet 0_1))]\\
&\quad \vee [(u \wedge \vartheta_P(w \bullet 1_1)) + (\vartheta_N(u \bullet 0_1) \wedge w]\\
&\quad \vee [(u \wedge \vartheta_P(w \bullet 1_1)) + (u \wedge \vartheta_N(w \bullet 0_1)]\\
&= [w \wedge (\vartheta_P(u \bullet 1_1) + \vartheta_N(u \bullet 0_1))] \vee [(\vartheta_P(u \bullet 1_1) \wedge w) + ((u \wedge \vartheta_N(w \bullet 0_1))]\\
&\quad \vee [(u \wedge \vartheta_P(w \bullet 1_1)) + (\vartheta_N(u \bullet 0_1) \wedge w] \vee [u \wedge (\vartheta_P(w \bullet 1_1) + \vartheta_N(w \bullet 0_1))]\\
&= [w \wedge (\vartheta(u)] \vee [u \wedge (\vartheta(w)] \vee [(\vartheta_P(u \bullet 1_1) \wedge w) + (u \wedge \vartheta_N(w \bullet 0_1))]\\
&\quad \vee [(u \wedge \vartheta_P(w \bullet 1_1)) + (\vartheta_N(u \bullet 0_1) \wedge w]\\
&= (w \wedge (\vartheta(u)) \vee (u \wedge (\vartheta(w)).
\end{aligned}
$$

To show the derivation ϑ is preserved under the onto homomorphism, $\vartheta_P(\alpha_P(u)) = \vartheta_P(u \wedge 1_2)$ and $\alpha_P(\vartheta(u)) = \vartheta(u) \bullet 1_1 = (\vartheta_P(\alpha_P(u)) + \vartheta_N(\alpha_N(u))) \bullet 1_1 = (\vartheta_P(u \bullet 1_1) + \vartheta_N(u \bullet 0_1)) \bullet 1_1 = (\vartheta_P(u \bullet 1_1) \bullet 1_1) + (\vartheta_N(u \bullet 0_1) \bullet 1_1)$. Since $\vartheta_N(u \bullet 0_1) \leq_1 0_2$ meeting inequalities by 1_1 we get $\vartheta_N(u \bullet 0_1) \bullet 1_1 \leq_1 0_2 \bullet 1_1 = 0_2$. Therefore, $(\vartheta_P(u \bullet 1_1) \bullet 1_1) + (\vartheta_N(u \bullet 0_1) \bullet 1_1) = \vartheta_P(u \bullet 1_1) + 0_2 = \vartheta_P(u \bullet 1_1)$ and so $\alpha_P(\vartheta(u)) = \vartheta_P(\alpha_P(u))$. Similarly, we can obtain $\alpha_N(\vartheta(u)) = \vartheta_N(\alpha_N(u))$. □

Theorem 4.2. *Let* $\mathfrak{B} = (B; \wedge, \vee, \vartheta, 0_1, 1_1, \bullet, +, \vartheta', 0_2, 1_2)$ *be a distributive bilattice with differential reducts. Then*:

i) $\vartheta(\vartheta')$ *is isotone if and only if corresponding derivations on* $POS(\mathfrak{B})$ *and* $NEG(\mathfrak{B})$ *are isotones*;

ii) $\vartheta(\vartheta')$ *is one-to-one if and only if corresponding derivations on* $POS(\mathfrak{B})$ *and* NEG $(\mathfrak{B})$ *are one-to-one*;

iii) $\vartheta(\vartheta')$ *is onto if and only if corresponding derivations on* $POS(\mathfrak{B})$ *and* $NEG(\mathfrak{B})$ *are onto.*

Proof. i) Let ϑ be isotone and $u, v \in \mathfrak{B}_1$ such that $u \leq_1 v$. Then $u \bullet 1_1 \leq v \bullet 1_1$, $u \bullet 0_1 \leq v \bullet 0_1$, and $\vartheta(u) \leq_1 \vartheta(v)$. By using onto homomorphisms α_N and α_N in Lemma 3.2, we obtain $\alpha_P(\vartheta(u)) \leq_1 \alpha_P(\vartheta(v))$ and $\alpha_N(\vartheta(u)) \leq_1 \alpha_N(\vartheta(v))$. Suppose ϑ_P and ϑ_N are corresponding derivations on $POS(\mathfrak{B}_1)$ and $NEG(\mathfrak{B}_1)$ respectively. But, $\alpha_P(\vartheta(u)) = \vartheta_P(\alpha_P(u)) = \vartheta_P(u \bullet 1_1)$, $\alpha_P(\vartheta(v)) = \vartheta_P(\alpha_P(v)) = \vartheta_P(v \bullet 1_1)$, $\alpha_N(\vartheta(u)) = \vartheta_N(\alpha_P(u)) = \vartheta_N(u \bullet 0_1)$ and $\alpha_N(\vartheta(v)) = \vartheta_N(\alpha_P(v)) = \vartheta_N(v \bullet 0_1)$. Accordingly, $\vartheta_P(u \bullet 1_1) \leq_1 \vartheta_P(v \bullet 1_1)$ and $\vartheta_N(u \bullet 0_1) \leq_1 \vartheta_N(v \bullet 0_1)$ and hence ϑ_P and ϑ_N are isotone. Conversely, let ϑ_P and ϑ_N be isotone and $u, v \in \mathfrak{B}$ such that $u \leq_1 v$. Then $u \bullet 1_1 \leq_1 v \bullet 1_1$ and $u \bullet 0_1 \leq_1 v \bullet 0_1$. Thus $\vartheta_P(u \bullet 1_1) \leq_1 \vartheta_P(v \bullet 1_1)$ and $\vartheta_N(u \bullet 0_1) \leq_1 \vartheta_N(v \bullet 0_1)$. Since $\vartheta(u) = \vartheta_P(\alpha_P(u)) + \vartheta_N(\alpha_N(u)) = \vartheta_P(u \bullet 1_1) + \vartheta_N(u \bullet 0_1)$ and $\vartheta(v) = \vartheta_P(\alpha_P(v)) + \vartheta_N(\alpha_N(v)) = \vartheta_P(v \bullet 1_1) + \vartheta_N(v \bullet 0_1)$. Therefore $\vartheta(u) \leq_1 \vartheta(v)$ and ϑ is isotone.

ii) If ϑ is one-to-one, then $\alpha_P(\vartheta(u)) = \alpha_P(\vartheta(v))$ and $\alpha_N(\vartheta(u)) = \alpha_N(\vartheta(v))$. Thus $\vartheta_P(u \bullet 1_1) = \vartheta_P(v \bullet 1_1)$ and $\vartheta_N(u \bullet 0_1) = \vartheta_N(v \bullet 0_1)$. But, $u \bullet 1_1 = v \bullet 1_1$ and $u \bullet 0_1 = v \bullet 0_1$. Hence ϑ_P and ϑ_N are one-to-one. In the opposite direction, if ϑ_P and ϑ_N are one-to-one, then $\vartheta_P(u \bullet 1_1) = \vartheta_P(v \bullet 1_1)$ and $\vartheta_N(u \bullet 0_1) = \vartheta_N(v \bullet 0_1)$, imply $u \bullet 1_1 = v \bullet 1_1$ and $u \bullet 0_1 = v \bullet 0_1$. So, $\vartheta(u) = \vartheta_P(u \bullet 1_1) + \vartheta_N(u \bullet 0_1) = \vartheta_P(v \bullet 1_1) + \vartheta_N(v \bullet 0_1) = \vartheta(v)$. But $u = (u \bullet 1_1) + (u \bullet 0_1) = (v \bullet 1_1) + (v \bullet 0_1) = v$. Therefore ϑ is one-to-one.

iii) Let for every $v \in \mathfrak{B}_1$ there be $u \in \mathfrak{B}_1$ such that $\vartheta(u) = v$. Then $\alpha_P(\vartheta(u)) = \vartheta_P(u \bullet 1_1) = \alpha_P(v) = v \bullet 1_1$ and $\alpha_N(\vartheta(u)) = \vartheta_N(u \bullet 0_1) = \alpha_N(v) = v \bullet 0_1$. Conversely, let ϑ_P and ϑ_N be onto. Then for every $v \bullet 1_1 POS(\mathfrak{B})$ and $v \wedge 0_2 \in NEG(\mathfrak{B})$ there exist $u \wedge 1_2 \in POS(\mathfrak{B})$ and $u \wedge 0_2 \in NEG(\mathfrak{B})$ such that $\vartheta_P(u \bullet 1_1) = v \bullet 1_1$ and $\vartheta_N(u \bullet 0_1) = v \bullet 0_1$. Thus for arbitrary $u \in \mathfrak{B}_1$ there exists $v \in \mathfrak{B}_1$ such that $\vartheta(u) = \vartheta_P(u \bullet 1_1) + \vartheta_N(u \bullet 0_1) = (v \bullet 1_1) + (v \bullet 0_1) = v$. Hence ϑ is onto. □

The next example illustrates that, if there is a lattice isomorphism between two reducts of distributive bilattice with differential reducts it does not necessarily preserve derivations on the two reducts.

Example 4.3. Consider a distributive bilattice $\mathfrak{B}$ with differential reducts in Figure 3, ϑ is the identity map and ϑ' is defined as

$$\vartheta'(u) = \begin{cases} 0_2 & \text{for} \quad u = 0_2, b, \quad \text{and} \;\; g \\ c & \text{for} \quad u = a, c, \quad \text{and} \;\; 0_1 \\ f & \text{for} \quad u = e, f, \quad \text{and} \;\; 1_2 \\ 1_1 & \text{for} \quad u = d, h, \quad \text{and} \;\; 1_1. \end{cases}$$

Consider an isomorphism map $\phi : \mathfrak{B}_1 \longrightarrow \mathfrak{B}_2$ defined as

$$\phi(u) = \begin{cases} 0_1 & \text{for} \quad u = 0_2, & e & \text{for} \quad u = d, \\ 1_2 & \text{for} \quad u = 1_1, & f & \text{for} \quad u = h, \\ a & \text{for} \quad u = g, & g & \text{for} \quad u = a, \\ b & \text{for} \quad u = c, & h & \text{for} \quad u = f, \\ c & \text{for} \quad u = b, & 0_2 & \text{for} \quad u = 0_1, \\ d & \text{for} \quad u = e, & 1_1 & \text{for} \quad u = 1_2. \end{cases}$$

Note that ϕ does not preserve a derivation, e.g.,

$$\phi(\vartheta(a)) = \phi(a) = g \neq \vartheta'(\phi(a)) = \vartheta'(g) = 0_2.$$

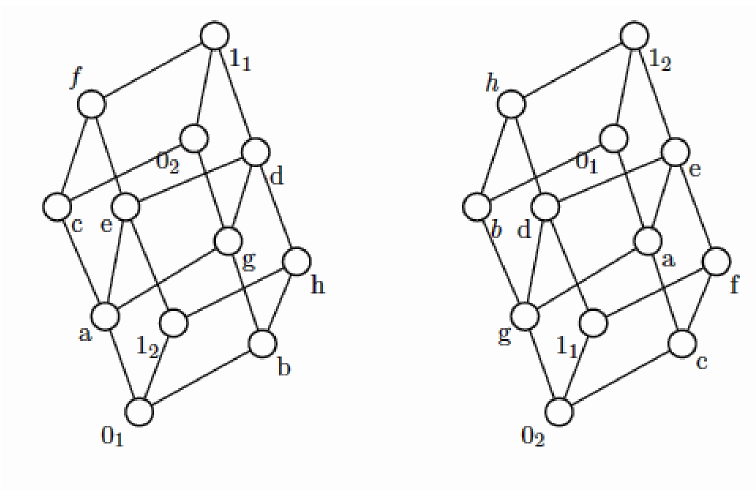

FIGURE 3. Distributive bilattice with differential reducts $\mathfrak{B}$

5. Differential distributive bilattices

In this section we solve the problem: under what conditions the two reducts of a distributive bilattice are isomorphic differential lattices. The concept of a differential distributive bilattice is defined, algebraic properties and construction theorem are proved.

Two differential lattices $\mathfrak{L} = (L; \wedge, \vee, \vartheta, 0, 1)$ and $\mathfrak{K} = (K; \wedge, \vee, \vartheta', 0', 1')$ are isomorphic if there exists a lattice isomorphism $\phi : \mathfrak{L} \longrightarrow \mathfrak{K}$ which is preserving derivations, see [13].

Definition. A differential distributive bilattice $\mathfrak{B} = (B; \wedge, \vee, \vartheta, 0_1, 1_1, \bullet, +, \vartheta', 0_2, 1_2)$ is a distributive bilattice with isomorphic differential reducts $\mathfrak{B}_1 = (B; \wedge, \vee, \vartheta, 0_1, 1_1,)$ and $\mathfrak{B}_2 = (B; \bullet, +, \vartheta', 0_2, 1_2)$.

Let $\mathfrak{B} = (B; \wedge, \vee, \vartheta, 0_1, 1_1, \bullet, +, \vartheta', 0_2, 1_2)$ be a differential distributive bilattice and $\phi : \mathfrak{B}_1 \longrightarrow \mathfrak{B}_2$ is an isomorphism. Consider the isomorphisms $\psi : POS(\mathfrak{B})_{\leq_1} \longrightarrow POS_{\leq_2}(\mathfrak{B})$ and $\lambda : NEG_{\leq_1}(\mathfrak{B}) \longrightarrow NEG_{\leq_2}(\mathfrak{B})$ defined as $\psi(u \bullet 1_1) = \phi(u) \bullet 1_1$ and $\lambda(u \bullet 0_1) = \phi(u) \bullet 0_1$. Then we have the following properties:

Proposition 5.1. i) *ψ, ψ^{-1}, λ and λ^{-1} are preserving derivations;*

ii) *$\psi \circ \alpha_P = \beta_P \circ \phi$, $\psi^{-1} \circ \beta_P = \alpha_P \circ \phi^{-1}$, $\lambda \circ \alpha_N = \beta_N \circ \phi$ and $\lambda^{-1} \circ \beta_N = \alpha_N \circ \phi^{-1}$;*

iii) *$\phi(1_2) = 1_1$ and $\phi(0_2) = 0_1$.*

Proof. i) $\psi(\vartheta_P(u \bullet 1_1)) = \psi(\vartheta(u) \bullet 1_1)) = \phi(\vartheta(u)) \bullet 1_1 = \vartheta'(\phi(u)) \bullet 1_1 = \vartheta'_P(\phi(u) \bullet 1_1) = \vartheta'(\psi(u \bullet 1_1)) = \vartheta'_P(\psi(u \bullet 1_1))$.

ii) For an arbitrary element $u \in \mathfrak{B}_1$, we have that

$$(\psi \circ \alpha_P)(u) = \psi(\alpha_P(u)) = \psi(u \bullet 1_1) = \phi(u) \bullet 1_1 = \beta_P(\phi(u)) = (\beta_P \circ \phi)(u).$$

iii) Assume that $\phi(1_2) \neq 1_1$. Then there exists $v \in \mathfrak{B}_2$ such that $\phi(1_2) = v$, $v \neq 1_1$, and $\phi(u) = 1_1$ for some $u \in \mathfrak{B}_1$. Since ϕ is an isomorphism, $\phi(u) \leq_2 \phi(1_2)$. Consequently, $1_1 = \phi(u) = \phi(1_2) \bullet \phi(u) = v \bullet 1_1$. So $v \leq_2 1_1$ implies $\phi(1_2) \leq_2 \phi(u) = 1_1$, which is a contradiction. Therefore $\phi(1_2) = 1_1$.

Similarly, other parts can be proven. □

The following diagrams in Figure 4 clarify Proposition 5.1.

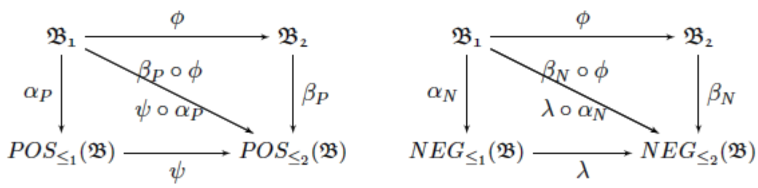

FIGURE 4.

Proposition 5.2. *Let $\mathfrak{B} = (B; \wedge, \vee, \vartheta, 0_1, 1_1, \bullet, +, \vartheta', 0_2, 1_2)$ be a differential distributive bilattice. Then:*

i) *$\phi \circ \vartheta|_{POS(\mathfrak{B})}$ is a derivation on $POS(\mathfrak{B})$ and $\phi \circ \vartheta|_{NEG(\mathfrak{B})}$ is a derivation on $NEG(\mathfrak{B})$;*

ii) *$\phi^{-1} \circ \vartheta'|_{POS(\mathfrak{B})}$ is a derivation on $POS(\mathfrak{B})$ and $\phi^{-1} \circ \vartheta'|_{NEG(\mathfrak{B})}$ is a derivation on $NEG(\mathfrak{B})$;*

iii) *$\vartheta(u) = \vartheta_P(\psi^{-1}(u \bullet 1_1)) + \vartheta_N(\lambda^{-1}(u \bullet 0_1))$;*

iv) *$\vartheta'(u) = \vartheta'_P(\psi(u \bullet 1_1)) + \vartheta'_N(\lambda(u \bullet 1_1))$.*

Proposition 5.3. *Let* $\mathfrak{B} = (B; \wedge, \vee, \vartheta, 0_1, 1_1, \bullet, +, \vartheta', 0_2, 1_2)$ *be a differential distributive bilattice. Then*:

i) ϑ *is isotone if and only if* ϑ' *is isotone*;
ii) ϑ *is one-to-one if and only if* ϑ' *is one-to-one*;
iii) ϑ *is onto if and only if* ϑ' *is onto.*

Theorem 5.4. (***Construction Theorem***) *Let* $(\mathfrak{L}_1; \wedge_1, \vee_1, \vartheta_1, 0, 1)$ *and* $(\mathfrak{L}_2; \wedge_2, \vee_2, \vartheta_2, 0', 1')$ *be two differential distributive lattice. If there exist an isomorphism* $\rho : \mathfrak{L}_2^{\partial} \longrightarrow \mathfrak{L}_2$ *and a derivation* ϑ'_2 *on* $\mathfrak{L}_2^{\partial}$ *such that*:

$$(\rho \circ \vartheta'_2)(a) = (\vartheta_2 \circ \rho)(a), \text{ for all } a \in \mathfrak{L}_2.$$

Then the product bilattice $\mathfrak{B}(\mathfrak{L}_1, \mathfrak{L}_2)$ *is a differential distributive.*

Proof. Assume $(u_1, u_2), (v_1, v_2) \in \mathfrak{B}(\mathfrak{L}_1, \mathfrak{L}_2)$, ϑ is a derivation on the first reduct $\mathfrak{B}_1(\mathfrak{L}_1, \mathfrak{L}_2)$ defined as: $\vartheta((u_1, u_2)) = (\vartheta_1(u_1), \vartheta'_2(u_2))$. Thus:

$$\begin{aligned}
&\vartheta((u_1, u_2) \wedge (v_1, v_2)) \\
&= \vartheta((u_1 \wedge_1 v_1, u_2 \vee_2 v_2)) \\
&= (\vartheta_1(u_1 \wedge_1 v_1), \vartheta'_2(u_2 \vee_2 v_2)) \\
&= ((\vartheta_1(u_1) \wedge_1 v_1) \vee_1 (u_1 \wedge_1 \vartheta_1(v_1)), (\vartheta'_2(u_2) \vee_2 v_2) \wedge_2 (u_2 \vee_2 \vartheta'_2(v_2))) \\
&= (\vartheta_1(u_1) \wedge_1 v_1, \vartheta'_2(u_2) \vee_2 v_2) \vee ((u_1 \wedge_1 \vartheta_1(v_1), (u_2 \vee_2 \vartheta'_2(v_2)) \\
&= (\vartheta(u_1, u_2) \wedge (v_1, v_2)) \vee ((u_1, u_2) \wedge \vartheta(v_1, v_2)).
\end{aligned}$$

Similarly, we can prove that a map $\vartheta'((u_1, u_2)) = (\vartheta_1(u_1), \vartheta_2(u_2))$ is a derivation on the second reduct $\mathfrak{B}_2(\mathfrak{L}_1, \mathfrak{L}_2)$. Define a map $\phi : \mathfrak{B}_1(\mathfrak{L}_1, \mathfrak{L}_2) \longrightarrow \mathfrak{B}_2(\mathfrak{L}_1, \mathfrak{L}_2)$ as:

$$\phi(u_1, u_2) = (u_1, \rho(u_2)), \text{ for all } (u_1, u_2) \in \mathfrak{B}(\mathfrak{L}_1, \mathfrak{L}_2).$$

To prove that ϕ is an isomorphism, it is enough to show that ϕ is a homomorphism preserving the derivation.

$$\begin{aligned}
\phi((u_1, u_2) \wedge (v_1, v_2)) &= \phi((u_1 \wedge_1 v_1, u_2 \vee_2 v_2)) \\
&= (u_1 \wedge_1 v_1, \rho(u_2 \vee_2 v_2)) \\
&= (u_1 \wedge_1 v_1, \rho(u_2) \vee_2 \rho(v_2)) \\
&= (u_1 \wedge_1 v_1, u_2 \wedge_2 v_2),
\end{aligned}$$

$$\begin{aligned}
\phi((u_1, u_2) \vee (v_1, v_2)) &= \phi((u_1 \vee_1 v_1, u_2 \wedge_2 v_2)) \\
&= (u_1 \vee_1 v_1, \rho(u_2 \wedge_2 v_2)) \\
&= (u_1 \vee_1 v_1, \rho(u_2) \wedge_2 \rho(v_2)) \\
&= (u_1 \vee_1 v_1, u_2 \vee_2 v_2),
\end{aligned}$$

$$\begin{aligned}
(\phi \circ \vartheta)((u_1, u_2)) &= \phi(\vartheta((u_1, u_2))) \\
&= \phi((\vartheta_1(u_1), \vartheta'_2(v_2))
\end{aligned}$$

$$
\begin{aligned}
&= (\vartheta_1(u_1), \rho(\vartheta_2'(v_2)) \\
&= (\vartheta_1(u_1), \vartheta_2(\rho(v_2))) \\
&= \vartheta'((u_1, \rho(u_2))) \\
&= \vartheta'(\phi((u_1, u_2))) \\
&= (\vartheta' \circ \phi)((u_1, u_2)).
\end{aligned}
$$
□

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