

ON AUTOMORPHISMS IN PRIME RINGS WITH APPLICATIONS

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ABSTRACT. The notions of skew-commuting/commuting/semi-commuting/skew-centralizing/semi-centralizing mappings play an important role in ring theory. \mathcal{C}^* -algebras with these properties have been studied considerably less and the existing results are motivating the researchers. This article elaborates the structure of prime rings and \mathcal{C}^* -algebras satisfying certain functional identities involving automorphisms.

1. Motivation

The theme of this manuscript is to investigate the n -skew-commuting automorphisms on prime rings and \mathcal{C}^* -algebras. The fact that the structure of prime rings and \mathcal{C}^* -algebras are describable, has been one of the main motivations for this research. In 1993, Brešar [11] set up the concept of skew-commuting mappings on rings. Specifically, he established that “If \mathcal{R} is a prime ring of characteristic not 2, and $\xi : \mathcal{R} \rightarrow \mathcal{R}$ is an additive mapping which is skew-commuting on an ideal \mathcal{I} of \mathcal{R} , i.e., $\xi(a)a + a\xi(a) = 0$ for all $a \in \mathcal{I}$, then $\xi(\mathcal{I}) = (0)$ ”. Also he proved that “Zero is the only additive mapping which is skew-commuting on a 2-torsion free semiprime ring”. On the other hand, in this direction, Brešar [12] characterized commuting additive mappings in this heading by demonstrating that “Every commuting additive mapping $\xi : \mathcal{R} \rightarrow \mathcal{R}$ (where \mathcal{R} is a prime ring, even it is also true for semiprime ring) is of the form $\xi(a) = \lambda a + \gamma(a)$, where $\lambda \in \mathcal{Z}(\mathcal{R})$ and $\gamma : \mathcal{R} \rightarrow \mathcal{Z}(\mathcal{R})$ is an additive mapping”. Additional documentation on skew-commuting/commuting mappings and its implementations can be found in [6, 9, 18, 30, 35–38].

Bell and Lucier [8] amplified the idea of skew-commuting/skew-centralizing mappings and they expressed as, “For $n \in \mathbb{Z}^+$, a mapping ξ of a ring \mathcal{R} into itself is called n -skew-commuting (resp. n -skew-centralizing) on S if $\xi(a)a^n + a^n\xi(a) = 0$ (resp. $\xi(a)a^n + a^n\xi(a) \in \mathcal{Z}(\mathcal{R})$) for every $a \in S$ ”. In particular, “for $n = 1, 2$ described as 1-skew-commuting and 2-skew-commuting, respectively”.

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In the case of 2-skew-commuting mappings, Fošner [20] analyzed it on 2-torsion free semiprime rings. Specifically, she proved that “Let \mathcal{R} be a 2-torsion free semiprime ring. Suppose that an additive mapping $\xi : \mathcal{R} \rightarrow \mathcal{R}$ satisfies the relation $\xi(a)a^2 + a^2\xi(a) = 0$ for all $a \in \mathcal{R}$. In this case $\xi = 0$ ”. In [33] Nadeem *et al.* demonstrated that “If \mathcal{R} is a prime ring with $\text{char}(\mathcal{R}) \neq 2, 3$, \mathcal{I} is an ideal of \mathcal{R} and $\xi : \mathcal{R} \rightarrow \mathcal{R}$ an additive mapping such that $\xi(a)a^2 + a^2\xi(a) = 0$ for every $a \in \mathcal{I}$, then $\xi = 0$ on \mathcal{I} ”. Readers have referred to much more detail about skew-commuting mappings and their extension such as skew-centralizing/semi-centralizing/semi-commuting mappings to [4,17,24,25,29,34]. In [20] one could evaluate the conjecture herein.

Conjecture 1 ([20], Conjecture 6). *Let $n \in \mathbb{Z}^+$ and let \mathcal{R} be a semiprime ring with suitable torsion restrictions. Suppose that an additive mapping $\xi : \mathcal{R} \rightarrow \mathcal{R}$ satisfies the relation*

$$\xi(x)x^n + x^n\xi(x) = 0$$

for all $x \in \mathcal{R}$. What can be say about ξ or \mathcal{R} ?

In [2], Ali set out the following theorem concerning Conjecture 1.

Theorem 1.1 ([2], Theorem 1.1). *Let $n \in \mathbb{Z}^+$ and let \mathcal{R} be a prime ring with $\text{char}(\mathcal{R}) = 0$ or $\text{char}(\mathcal{R}) \geq n$. Suppose that an additive mapping $\xi : \mathcal{R} \rightarrow \mathcal{R}$ satisfies the relation*

$$\xi(x)x^n + x^n\xi(x) = 0$$

for every $x \in \mathcal{R}$. In this case $\xi = 0$.

In a recent paper Fošner *et al.* [21] answered in the favour of Conjecture 1 and proved the following

Theorem 1.2 ([21], Theorem 4). *Let \mathcal{R} be an n -torsion free semiprime ring. Suppose that an additive mapping $\xi : \mathcal{R} \rightarrow \mathcal{R}$ satisfies the relation*

$$\xi(x)x^n + x^n\xi(x) = 0$$

for every $x \in \mathcal{R}$. In this case $\xi = 0$.

In 2018, Ali *et al.* [3] discussed n -skew-commuting automorphisms on Lie ideals of prime rings. More exactly they proved that:

Theorem 1.3 ([3], Theorem 1.3). *Let $n \in \mathbb{Z}^+$ and \mathcal{R} be a 2-torsion free semiprime ring. If \mathcal{R} admits an automorphism ξ such that*

$$\xi(x)x^n + x^n\xi(x) = 0$$

for every $x \in \mathcal{R}$, then \mathcal{R} contains a non-zero central ideal.

Given the above-mentioned Conjecture 1 and Theorems 1.1, 1.2, 1.3, it is reasonable to ask what happens if in these results we take ξ as an automorphism/anti-automorphism?

Formulating many results in this setting is achievable by considering suitable conditions on the subset $\mathcal{B}(f, \mathcal{I}) = \{f(x)x^n + x^n f(x) : x \in \mathcal{I}\}$, where \mathcal{I} is

an appropriate subset of \mathcal{R} , while f is an additive mapping of \mathcal{R} and $n \in \mathbb{Z}^+$. Forasmuch as, numerous researchers have studied $\mathcal{B}(f, \mathcal{S}) = 0$ on well-suited subset of rings involving additive mappings/derivations/automorphisms of \mathcal{R} . It would seem natural to interrogate what come into existence if $\mathcal{B}(\xi, \mathcal{S})$ is m -commuting, i.e., $[\mathcal{B}(\xi, \mathcal{S}), x^m] = 0$, where $\mathcal{S} = \mathcal{G}$, a non-central Lie ideal of \mathcal{R} and ξ is an automorphism/anti-automorphism of \mathcal{R} . The primary aim of this paper is really to address the above-mentioned question. Especially, for $[\mathcal{B}(\xi, \mathcal{S}), x^m] = 0$, when the prime ring having an automorphism on a non-central Lie ideal of \mathcal{R} . Furthermore, the structure of \mathcal{C}^* -algebras as an application is identified. The main results of the empirical article are as follows.

Theorem 1.4. *Let \mathcal{R} be a prime ring of characteristic different from 2, ξ be a non-identity automorphism of \mathcal{R} , \mathcal{G} be a non-central Lie ideal of \mathcal{R} and $n, m \in \mathbb{Z}^+$. If $[\xi(x)x^n + x^n\xi(x), x^m] = 0$ holds for each $x \in \mathcal{G}$, then \mathcal{R} satisfies s_4 , the standard identity in four variables.*

Theorem 1.5. *Let \mathcal{A} be a primitive \mathcal{C}^* -algebra and $m, n \in \mathbb{Z}^+$. If $\xi : \mathcal{A} \rightarrow \mathcal{A}$ is an anti-automorphism of \mathcal{A} such that*

$$[\xi(x)x^{*n} + x^{*n}\xi(x), x^{*m}] = 0$$

for every $x \in \mathcal{A}$, then \mathcal{A} is a \mathcal{C}^ - \mathcal{W}_4 -algebra.*

The following example indicates that the presupposition of primeness is necessary in Theorem 1.4.

Example 1.1. Let \mathcal{R}_1 be any prime ring not satisfying s_4 , the standard identity in four variables and $\mathcal{R}_2 = \mathcal{M}_2(\mathcal{F})$, the ring of 2×2 matrices over a field \mathcal{F} . Let $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$ and $\mathcal{G} = \mathcal{M}_2(\mathcal{F}) \oplus 0$. Then \mathcal{G} is a nonzero Lie ideal of \mathcal{R} , a semiprime ring. The following is constructed $\zeta : \mathcal{R} \rightarrow \mathcal{R}$ such that $\zeta(x_1, x_2) = (x_2, x_1)$. It can be easily seen that ζ is a non-identity automorphism which satisfies Theorem 1.4 but \mathcal{R} does not satisfies s_4 .

2. Preliminaries

Some conceptual notions are necessarily demonstrated to develop the proof of the key theorems. Roughly, these ideas are well known and written compactly. The standard identity s_4 in four variables is defined as follows:

$$s_4(X_1, X_2, X_3, X_4) = \sum (-1)^\eta X_{\eta(1)}X_{\eta(2)}X_{\eta(3)}X_{\eta(4)},$$

where $(-1)^\eta$ is the sign of a permutation η of the symmetric group of degree 4.

Throughout the manuscript unless otherwise mention, “ \mathcal{R} is a prime ring, \mathcal{Q} is the maximal right ring of quotients of \mathcal{R} with center \mathcal{C} , and \mathcal{RC} is the central closure of \mathcal{R} . It is known that \mathcal{Q} is also prime and \mathcal{C} is a field which is called extended centroid of \mathcal{R} (see [7, Chapter 2] for further key definitions and related properties)”. “An automorphism ξ is called X -inner if there exists invertible element $g \in \mathcal{Q}$ such that $\xi(s) = gsg^{-1}$ for every $s \in \mathcal{R}$ ”. “An automorphism ξ is called X -outer if it is not X -inner”. From [5, 7, 16], let \mathcal{A} be

a ring with a simple left module M and set $\mathcal{D} = \text{End}({}_{\mathcal{A}}M)$. By Schur's Lemma, \mathcal{D} is a division ring. Next, let $\text{End}(M)$ be a ring of all endomorphism of an additive group M acting on M from the left and consider the ring $\text{End}(M_{\mathcal{D}})$ as a subring of $\text{End}(M)$. Given $a \in \mathcal{A}$, define a map $L_a : M \rightarrow M$ by $L_ax = ax$ for all $x \in M$. "An automorphism ξ of \mathcal{A} is called M -inner if there exists an invertible element $T \in \text{End}(M)$ such that $TL_aT^{-1} = L_{\xi(a)}$ for all $a \in \mathcal{A}$; otherwise it is called M -outer" (see [16] for details). A few well known results can be collected which is able be utilized all through the text.

Fact 2.1 ([5, Lemma 7.1]). Let $\mathcal{V}_{\mathcal{D}}$ be a vector space over a division ring \mathcal{D} with $\dim \mathcal{V}_{\mathcal{D}} \geq 2$ and $\mathcal{P} \in \text{End}(\mathcal{V})$. If s and $\mathcal{P}s$ are \mathcal{D} -dependent for every $s \in \mathcal{V}$, then there exists $\chi \in \mathcal{D}$ such that $\mathcal{P}s = \chi s$ for every $s \in \mathcal{V}$.

Fact 2.2 ([14, Theorem 1]). Let \mathcal{R} be a prime ring and \mathcal{I} be a two sided ideal of \mathcal{R} . Then \mathcal{I}, \mathcal{R} and \mathcal{D} satisfy the same generalized polynomial identities (GPIs) with automorphisms.

Fact 2.3 ([28, Proposition]). Let \mathcal{R} be a prime algebra over an infinite field k and let \mathcal{K} be a field extension over k . Then \mathcal{R} and $\mathcal{R} \otimes_k \mathcal{K}$ satisfy the same generalized polynomial identities with coefficients in \mathcal{R} .

Fact 2.4 ([38, Lemma 2.1]). Let \mathcal{R} be a prime ring with extended centroid \mathcal{C} . Then the following conditions are equivalent:

- (i) $\dim_{\mathcal{C}} \mathcal{RC} \leq 4$.
- (ii) \mathcal{R} satisfies s_4 , the standard identity in four variables.
- (iii) \mathcal{R} is commutative or \mathcal{R} embeds in $\mathcal{M}_2(\mathcal{F})$ for \mathcal{F} a field.
- (iv) \mathcal{R} is algebraic of bounded degree 2 over \mathcal{C} .
- (v) \mathcal{R} satisfies $[[x^2, y], [x, y]]$.

Fact 2.5 ([5, Theorem 3.2]). Let \mathcal{A} be a primitive ring with nonzero socle and let M be a faithful simple left \mathcal{A} -module. Then every automorphism of \mathcal{A} is M -inner.

Fact 2.6. Let \mathcal{R} be a prime ring and \mathcal{L} a be non-central Lie ideal of \mathcal{R} . If $\text{char}(\mathcal{R}) \neq 2$, by [10, Lemma 1] there exists a nonzero ideal \mathcal{I} of \mathcal{R} such that $0 \neq [\mathcal{I}, \mathcal{R}] \subseteq \mathcal{L}$. If $\text{char}(\mathcal{R}) = 2$ and $\dim_{\mathcal{C}} \mathcal{RC} > 4$, i.e., $\text{char}(\mathcal{R}) = 2$ and \mathcal{R} does not satisfy s_4 , then by [27, Theorem 13] there exists a nonzero ideal \mathcal{I} of \mathcal{R} such that $0 \neq [\mathcal{I}, \mathcal{R}] \subseteq \mathcal{L}$. Thus if either $\text{char}(\mathcal{R}) \neq 2$ or \mathcal{R} does not satisfy s_4 , then we may conclude that there exists a nonzero ideal \mathcal{I} of \mathcal{R} such that $[\mathcal{I}, \mathcal{I}] \subseteq \mathcal{L}$.

3. Results on prime rings

Beginning with the following propositions which are needed for the key results to develop the proofs.

Proposition 3.1. Let \mathcal{R} be a dense subring of $\text{End}(\mathcal{V}_{\mathcal{D}})$, ξ be an automorphism of \mathcal{R} and $n, m \in \mathbb{Z}^+$. If \mathcal{R} satisfies $[\xi([x, y])[x, y]^n + [x, y]^n \xi([x, y]), [x, y]^m] = 0$

for every $x, y \in \mathcal{R}$, then either $\dim(\mathcal{V}_{\mathcal{D}}) \leq 2$ or ξ is an identity map on $\text{End}(\mathcal{V}_{\mathcal{D}})$.

Proof. Since \mathcal{R} is a primitive ring with nonzero socle. By Jacobson [22, Isomorphism Theorem, p. 79], there exists a semi-linear automorphism $\mathcal{P} \in \text{End}(\mathcal{V}_{\mathcal{D}})$ such that $\xi(s) = \mathcal{P}s\mathcal{P}^{-1}$ for every $s \in R$. Moreover, for every $v \in \mathcal{V}$, $\lambda \in \mathcal{D}$, $\mathcal{P}(v\lambda) = (\mathcal{P}v)\eta(\lambda)$, where η is an automorphism of \mathcal{D} . Using the hypotheses, following can be obtained

$$\begin{aligned} 0 &= [\xi([x, y])[x, y]^n + [x, y]^n \xi([x, y]), [x, y]^m] \\ &= [\mathcal{P}[x, y]\mathcal{P}^{-1}[x, y]^n + [x, y]^n \mathcal{P}[x, y]\mathcal{P}^{-1}, [x, y]^m] \end{aligned}$$

for all $x, y \in \text{End}(\mathcal{V}_{\mathcal{D}})$. Expect to begin with that v and $\mathcal{P}^{-1}v$ are \mathcal{D} -dependent for each $v \in \mathcal{V}$. Chuang *et al.*, [15, Lemma 1] provided $\mathcal{P}^{-1}v = v\theta$, where $\theta \in \mathcal{D}$ and $v \in \mathcal{V}$. Hence for each $s \in \text{End}(\mathcal{V}_{\mathcal{D}})$, $\mathcal{P}^{-1}(sv) = sv\theta$ and consequently $sv = \mathcal{P}(sv\theta) = \mathcal{P}(s(v\theta)) = \mathcal{P}s\mathcal{P}^{-1}(v) = \xi(s)v$ for all $v \in \mathcal{V}$, $s \in \text{End}(\mathcal{V}_{\mathcal{D}})$. In addition $(\xi(s) - s)\mathcal{V} = (0)$ for each $s \in \text{End}(\mathcal{V}_{\mathcal{D}})$. In this way in all, $\xi(s) = s$ for each $s \in \text{End}(\mathcal{V}_{\mathcal{D}})$. This appears that ξ is an identity map on $\text{End}(\mathcal{V}_{\mathcal{D}})$, as required.

In this way, there exists $v \in \mathcal{V}$ such that v and $\mathcal{P}^{-1}v$ are straightly \mathcal{D} -independent. Firstly, one can accept that $\dim(\mathcal{V}_{\mathcal{D}}) \geq 4$. At that point take $u, w \in \mathcal{V}$ such that $\{u, v, w, \mathcal{P}^{-1}v\}$ is \mathcal{D} -independent. For this, let $x, y \in \text{End}(\mathcal{V}_{\mathcal{D}})$ such that

$$\begin{aligned} xv = 0, \quad x\mathcal{P}^{-1}v = 0, \quad xu = \mathcal{P}^{-1}w, \quad xw = u; \\ yv = 0, \quad y\mathcal{P}^{-1}v = u, \quad yu = -w, \quad yw = 0. \end{aligned}$$

It is taken note that $[x, y]\mathcal{P}^{-1}v = \mathcal{P}^{-1}w$, $[x, y]w = w$, $[x, y]v = 0$ and subsequently, the presumption yields

$$\begin{aligned} 0 &= ([\mathcal{P}[x, y]\mathcal{P}^{-1}[x, y]^n + [x, y]^n \mathcal{P}[x, y]\mathcal{P}^{-1}, [x, y]^m)v \\ &= -w, \end{aligned}$$

a contradiction, implying that $\dim(\mathcal{V}_{\mathcal{D}}) \leq 3$.

Secondly, one could assume that $\dim(\mathcal{V}_{\mathcal{D}}) = 3$. Take $w \in \mathcal{V}$ such that $\{v, w, \mathcal{P}^{-1}v\}$ is \mathcal{D} -independent and then $\{v, w, \mathcal{P}^{-1}v\}$ forms a \mathcal{D} -basis of \mathcal{V} . If $\mathcal{P}(v + \mathcal{P}^{-1}v + w) \in v\mathcal{D}$ and $\mathcal{P}(\mathcal{P}^{-1}v + w) \in v\mathcal{D}$, then $\mathcal{P}v, \mathcal{P}(\mathcal{P}^{-1}v + w) \in v\mathcal{D}$ and then $v, \mathcal{P}^{-1}v + w \in \mathcal{P}^{-1}(v\mathcal{D}) = \mathcal{P}^{-1}(v)\xi^{-1}(\mathcal{D}) = \mathcal{P}^{-1}v\mathcal{D}$, contradicting the fact that $\{v, \mathcal{P}^{-1}v + w\}$ is \mathcal{D} -independent. Therefore, one can pick $\rho \in \{0, 1\}$ such that $u = \rho v + \mathcal{P}^{-1}v + w$ and $\mathcal{P}u \notin v\mathcal{D}$. Write $\mathcal{P}u = v\alpha + \mathcal{P}^{-1}v\beta + w\gamma$, where $\alpha, \beta, \gamma \in \mathcal{D}$ and β, γ both are not zero. By density of theorem, there exist $x, y \in \text{End}(\mathcal{V}_{\mathcal{D}})$ such that

$$\begin{aligned} xv = 0, \quad x\mathcal{P}^{-1}v = w, \quad xw = 0; \\ yv = 0, \quad y\mathcal{P}^{-1}v = 0, \quad yw = -u. \end{aligned}$$

That is $xu = w$, $yu = -u$, $x\mathcal{P}u = w\beta$ and $y\mathcal{P}u = -u\gamma$. Therefore, it can be seen that $[x, y]v = 0$, $[x, y]\mathcal{P}^{-1}v = u$, $[x, y]w = -w$. Also, $[x, y]u = u - w$,

$[x, y]\mathcal{P}u = u\beta - w\gamma$ and $[x, y]^2\mathcal{P}u = (u - w)\beta + w\gamma$. If $m + n$ is even, then $[x, y]^{m+n}\mathcal{P}u = (u - w)\beta + w\gamma$ and $[x, y]^{m+n}\mathcal{P}u = (u\beta - w\gamma)$, when $m + n$ is odd. As β, γ are not both zero and u, w are \mathcal{D} -dependent, so it is easy to see that $[x, y]^{m+n}\mathcal{P}u \neq 0$. Thus in all, it is that

$$\begin{aligned} 0 &= ((\mathcal{P}[x, y]\mathcal{P}^{-1}[x, y]^n + [x, y]^n\mathcal{P}[x, y]\mathcal{P}^{-1}, [x, y]^m)v \\ &= -[x, y]^2\mathcal{P}u, \end{aligned}$$

a contradiction, implying that $\dim(\mathcal{V}_{\mathcal{D}}) \leq 2$. \square

Theorem 3.1. *Let \mathcal{R} be a non-commutative prime ring, ξ be a non-identity automorphism of \mathcal{R} and $n, m \in \mathbb{Z}^+$. If $[\xi([x, y])[x, y]^n + [x, y]^n\xi([x, y]), [x, y]^m] = 0$ holds for each $x, y \in \mathcal{R}$, then \mathcal{R} satisfies s_4 , the standard identity in four variables.*

Proof. Firstly, the assumption can be made that ξ is X-inner, i.e., $s^\xi = psp^{-1}$ for every $s \in \mathcal{R}$. As ξ is the non-identity map, so $p \notin \mathcal{C}$. Then $[p[x, y]p^{-1}[x, y]^n + [x, y]^n p[x, y]p^{-1}, [x, y]^m] = 0$ is a non-trivial generalized polynomial identity (GPI) of \mathcal{R} . By Fact 2.2, $[p[x, y]p^{-1}[x, y]^n + [x, y]^n p[x, y]p^{-1}, [x, y]^m] = 0$ is also a generalized polynomial identity (GPI) of \mathcal{Q} . Let \mathcal{F} be the algebraic closure of \mathcal{C} if \mathcal{C} is infinite and set $\mathcal{F} = \mathcal{C}$ for \mathcal{C} finite. By Fact 2.3, $[p[x, y]p^{-1}[x, y]^n + [x, y]^n p[x, y]p^{-1}, [x, y]^m] = 0$ is also a generalized polynomial identity (GPI) of $\mathcal{Q} \otimes_{\mathcal{C}} \mathcal{F}$. In addition, $\mathcal{Q} \otimes_{\mathcal{C}} \mathcal{F}$ is a prime ring with \mathcal{F} as its extended centroid, in view of [19, theorem 3.5]. Then $\mathcal{Q} \otimes_{\mathcal{C}} \mathcal{F}$ is a prime ring that satisfies a non-trivial generalized polynomial identity (GPI), and its extended centroid \mathcal{F} is either a closed algebraic field or a finite field. By Martindale's theorem [5, Corollary 6.1.7], $\mathcal{Q} \otimes_{\mathcal{C}} \mathcal{F}$ may be a primitive ring having nonzero socle with the field \mathcal{F} as its related division ring. By Jacobson theorem [22, p. 75], $\mathcal{Q} \otimes_{\mathcal{C}} \mathcal{F}$ is isomorphic to a dense subring of the ring of linear transformations on a vector space \mathcal{V} over \mathcal{F} , containing nonzero linear transformations of finite rank. By suspicion, \mathcal{R} is non-commutative so $\mathcal{Q} \otimes_{\mathcal{C}} \mathcal{F}$ is non-commutative.. By Proposition 3.1, $\dim \mathcal{V}_{\mathcal{F}} \leq 2$. This implies that $\mathcal{R} \subseteq \mathcal{Q} \cong \mathcal{Q} \otimes_{\mathcal{C}} \mathcal{F} \subseteq \mathcal{Q} \otimes_{\mathcal{C}} \mathcal{F} \cong \mathcal{M}_2(\mathcal{F})$. We get the appropriate conclusion in view of the Fact 2.4.

Secondly, suppose that ξ is X-outer. Therefore

$$[\xi([x, y])[x, y]^n + [x, y]^n\xi([x, y]), [x, y]^m] = 0$$

is a nontrivial differential identity with automorphism ξ . Otherwise, if it is trivial we get a contradiction, as ξ is the non-identity map. It follows \mathcal{R} is a GPI ring from Main Theorem in [13]. By exploiting the techniques presented above, one can easily seen that \mathcal{R} could be a primitive ring with a faithful simple (left) \mathcal{R} -module \mathcal{V} and related division ring \mathcal{C} such that $H = \text{soc}(\mathcal{R}) \neq 0$. At that point, by the Jacobson density theorem [23], \mathcal{R} is isomorphic to a dense ring of linear transformations of vector space \mathcal{V} over \mathcal{C} . Take note that $\dim \mathcal{V}_{\mathcal{C}} \geq 2$ (something else, \mathcal{R} is commutative). On the off chance that $\dim \mathcal{V}_{\mathcal{C}} = 2$, at that point $\mathcal{R} \cong \mathcal{M}_2(\mathcal{C})$, i.e., \mathcal{R} fulfills s_4 , as required. In addition, ξ is \mathcal{V} -inner in see of Fact 2.5. Subsequently, there exists an invertible element $T \in \text{End}(\mathcal{V})$

such that $TL_rT^{-1} = L_{\xi(r)}$ for all $r \in \mathcal{R}$. In the event that the ring \mathcal{R} is primitive and \mathcal{V} could be a faithful \mathcal{R} -module, able to consider \mathcal{R} as a subring of $\text{End}(\mathcal{V}_{\mathcal{E}})$ by means of the embedding $r \rightarrow L_r$ (see [16] for points of interest). Hence in see of Proposition 3.1, we get the specified conclusion. \square

Hence, having all the necessary frames to execute the proof of the first fundamental result of this paper.

Proof of Theorem 1.4. The presumption gives $[\xi(x)x^n + x^n\xi(x), x^m] = 0$ for all $x \in \mathcal{G}$. In see of Fact 2.6, there exists an ideal \mathcal{I} of \mathcal{R} such that $(0) \neq [\mathcal{I}, \mathcal{R}] \subseteq \mathcal{G}$ and $[\mathcal{G}, \mathcal{G}] \neq (0)$. Moreover, \mathcal{R} is non-commutative as \mathcal{G} is non-central Lie ideal of \mathcal{R} . In this way by the given speculations, \mathcal{I} as well as \mathcal{R} (Fact 2.2) satisfies $[\xi([x, y])[x, y]^n + [x, y]^n\xi([x, y]), [x, y]^m] = 0$. Hence Theorem 3.1 delivers the specified conclusion. \square

The taking after corollary could be a special case of Theorems 3.1 and 1.4, which is of autonomous intrigued.

Corollary 3.1. *Let \mathcal{R} be a prime ring of characteristic different from 2, ξ be a non-identity automorphism of \mathcal{R} and $n, m \in \mathbb{Z}^+$. If $[\xi(x)x^n + x^n\xi(x), x^m] = 0$ holds for each $x \in \mathcal{R}$, then \mathcal{R} satisfies s_4 , the standard identity in four variables.*

4. Results on \mathcal{C}^* -algebras

This section addresses the applications of strict ring theoretical framework on \mathcal{C}^* -algebras. Let's introduce some well known facts and elementary definitions for the purpose of fullness. "A Banach algebra is a linear associate algebra which, as a vector space, is a Banach space with norm $\|\cdot\|$ satisfying the multiplicative inequality; $\|xy\| \leq \|x\|\|y\|$ for all x and y in \mathcal{A} ".

A Banach algebra \mathcal{A} is a PI-algebra if and only if there exist $n \in \mathbb{N}$ and a polynomial $q \in \mathcal{W}_n$, $q \neq 0$, such that $q(x_1, x_2, \dots, x_n) = 0$ for all $x_1, x_2, \dots, x_n \in \mathcal{A}$, where \mathcal{W}_n is the set of all complex polynomials in n non-commuting variables. "An involution on an algebra \mathcal{A} is a map $x \mapsto x^*$ of \mathcal{A} onto such that the following conditions are hold: (i) $(xy)^* = y^*x^*$, (ii) $(x^*)^* = x$, and (iii) $(x + \lambda y)^* = x^* + \bar{\lambda}y^*$ for all $x, y \in \mathcal{A}$ and $\lambda \in \mathbb{C}$ the field of complex number, where $\bar{\lambda}$ is the conjugate of λ ". Of course the prototypical case of an involution on a Banach algebra is the adjoint operation on $\mathcal{B}(\mathcal{H})$, the set of bounded linear operators on Hilbert space \mathcal{H} . Another important example is a complex conjugation on $\mathbb{C}(\mathbb{X})$, the set of all continuous complex valued functions on \mathbb{X} , a compact Hausdorff space defined as $f^*(x) := \overline{f(x)}$. "An algebra equipped with an involution is called a $*$ -algebra or algebra with involution". A Banach $*$ -algebra is a Banach algebra \mathcal{A} together with an isometric involution $\|x^*\| = \|x\|$ for all $x \in \mathcal{A}$. A Banach $*$ -algebra is called a \mathcal{C}^* -algebra \mathcal{A} if $\|x^*x\| = \|x\|^2$ for all $x \in \mathcal{A}$. "A \mathcal{C}^* -algebra \mathcal{A} is primitive if its zero ideal is primitive, i.e., if \mathcal{A} has a faithful non-zero irreducible representation".

Let \mathcal{W}_n denote the standard polynomial of degree n in n non-commuting variables, $\mathcal{W}_n(a_1, a_2, \dots, a_n) = \sum_{\sigma \in S_n} \text{sign}(\sigma) a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)}$, where S_n is the set of all permutations of $\{1, 2, 3, \dots, n\}$ and $\text{sign}(\sigma) = \pm 1$ for σ even (odd) (see [26], [31] and references therein). An algebra \mathcal{A} is said to be a \mathcal{C}^* - \mathcal{W}_n -algebra if $\mathcal{W}_n(a_1, a_2, \dots, a_n) = 0$ for each choice of elements $a_1, a_2, \dots, a_n \in \mathcal{A}$. In particular, an algebra is a \mathcal{C}^* - \mathcal{W}_4 -algebra if it satisfies the standard identity $\mathcal{W}_4(a_1, a_2, a_3, a_4) = 0$ for all $a_1, a_2, a_3, a_4 \in \mathcal{A}$. Moreover, an algebra is a \mathcal{C}^* - \mathcal{W}_2 -algebra if and only if it is commutative, i.e., a \mathcal{C}^* - \mathcal{W}_2 -algebra is commutative if it satisfies the standard identity $\mathcal{W}_2(a_1, a_2) = 0$ for all $a_1, a_2 \in \mathcal{A}$. Many researchers discussed Gelfand's theory for Banach algebra and \mathcal{C}^* -algebra namely, Banach- \mathcal{W}_{2n} -algebra and \mathcal{C}^* - \mathcal{W}_{2n} -algebra.

Theorem 4.1. *Let \mathcal{A} be a primitive \mathcal{C}^* -algebra and $m, n \in \mathbb{Z}^+$. If $\xi : \mathcal{A} \rightarrow \mathcal{A}$ is an automorphism of \mathcal{A} such that*

$$[\xi(x)x^n + x^n\xi(x), x^m] = 0$$

for all $x \in \mathcal{A}$, then \mathcal{A} is a \mathcal{C}^ - \mathcal{W}_4 -algebra.*

Proof. In the light of hypothesis given, $\xi : \mathcal{A} \rightarrow \mathcal{A}$ is satisfying $[\xi(x)x^n + x^n\xi(x), x^m] = 0$ for all $x \in \mathcal{A}$. Within the light of [32, Theorem 5.4.5], \mathcal{A} is a prime \mathcal{C}^* -algebra, as it is primitive and hence \mathcal{A} is a prime ring [1]. Therefore, implementation of Corollary 3.1 heeds the needed result, thereby proving the theorem. \square

Proof of Theorem 1.5. The assumption gives $[\xi(x)x^{*m} + x^{*m}\xi(x), x^{*n}] = 0$ for all $x \in \mathcal{A}$. In the last expression, exchanging x^* for x , gives $[\xi(x^*)x^n + x^n\xi(x^*), x^m] = 0$ for every $x \in \mathcal{A}$. Let us set a map $\psi : \mathcal{A} \rightarrow \mathcal{A}$ such that $\psi(x) = \xi(x^*)$ for all $x \in \mathcal{A}$. Then, understandably $\psi(xy) = \psi(x)\psi(y)$ for all $x, y \in \mathcal{A}$, i.e., ψ is an automorphism of \mathcal{A} thereby getting for all $x \in \mathcal{A}$, $[\psi(x)x^n + x^n\psi(x), x^m] = 0$. The conclusion can be drawn from Theorem 4.1. This finishes out theorem's proof. \square

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