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ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO 3D CONVECTIVE BRINKMAN-FORCHHEIMER EQUATIONS WITH FINITE DELAYS

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Abstract. In this paper we prove the existence of global weak solutions, the exponential stability of a stationary solution and the existence of a global attractor for the three-dimensional convective Brinkman-Forchheimer equations with finite delay and fast growing nonlinearity in bounded domains with homogeneous Dirichlet boundary conditions.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. In this paper we consider the following convective Brinkman-Forchheimer (BF) equations with finite delays

$$(1)\quad \begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p + f(u) = G(u(t-\rho(t))) + h(x), & x\in\Omega, t>0,\\ \nabla\cdot u = 0, & x\in\Omega, t>0,\\ u(x,t) = 0, & x\in\partial\Omega, t>0,\\ u(x,0) = u_0(x), & x\in\Omega,\\ u(x,t) = \phi(x,t), & x\in\Omega,\\ & t\in(-r,0), \end{cases}$$

where $u = u(x,t) = (u_1, u_2, u_3)$ is the velocity field of the fluid, $\nu > 0$ is the kinematic viscocity, p is the pressure, h is a nondelayed external force field, G is another external force term and contains some memory effects during a fixed interval of time of length $r > 0$, ρ is an adequate given delay function, u_0 is the initial velocity and ϕ is the initial datum on the interval.

In the special case $f(u) \equiv 0$ the equations (1) turn to be the Navier-Stokes equation with delay. Equations of Navier-Stokes type with delay have been extensively studied in [4–7] for the case of finite delay and in [1, 12, 16–18] for the case of infinite delay.

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In order to study problem (1), we make the following assumptions:

- The nonlinearity $f \in C^2(\mathbb{R}^3, \mathbb{R}^3)$ satisfies the following conditions:

$$\begin{cases} 1)\ f'(u)v \cdot v \geq (-K + \kappa |u|^{\beta-1})|v|^2, \ \ \forall u, v \in \mathbb{R}^3, \\ 2)\ |f'(u)| \leq C_f(1 + |u|^{\beta-1}), \ \ \forall u \in \mathbb{R}^3, \end{cases} \tag{2}$$

where K, κ, C_f, are some positive constants, $\beta \geq 1$ ($\beta > 3$ to ensure the uniqueness of solutions) and $u \cdot v$ is the inner product in $\mathbb{R}^3$.
A typical example for such a nonlinear term $f(u)$ is the following

$$f(u) = au + b|u|^{\beta-1}u, \beta \in [1, \infty), \tag{3}$$

where $a \in \mathbb{R}$ and $b > 0$ are the Darcy and Forchheimer coefficients, respectively.

- $G : \mathbb{R}^3 \to \mathbb{R}^3$ is a function satisfying $G(0) = 0$, and assume that there exists $L_G > 0$ such that

$$|G(u) - G(v)|_{\mathbb{R}^3} \leq L_G |u - v|_{\mathbb{R}^3}, \forall u, v \in \mathbb{R}^3. \tag{4}$$

Consider a function $\rho(\cdot) \in C^1(\mathbb{R})$ such that $\rho(t) \geq 0$ for all $t \in \mathbb{R}$, $\sup_{t\in\mathbb{R}} \rho(t) = r \in (0, \infty)$, and $\rho_* = \sup_{t\in\mathbb{R}} \rho'(t) < 1$.

The convective Brinkman-Forchheimer equations describes the motion of fluid flow in a saturated porous medium and have been studied in [14]. The Brinkman-Forchheimer model, that is equation (1) without the convective term $(u \cdot \nabla)u$, have been studied extensively in [8, 11, 13, 19–22]. For this model, the case of the so-called subcritical growth rate of the nonlinearity f (i.e., $\beta \leq 3$ in (3)) has been widely considered. The main contribution of [14] is to remove this growth restriction and verify the global existence, uniqueness and dissipativity of smooth solutions for a large class of nonlinearity f with an arbitrary growth exponent $\beta > 3$.

In this paper, we consider problem (1) when the nonlinear term $f(u)$ satisfied (2) and the forcing term with bounded variable delay $G(\cdot)$ satisfied (4). We will discuss the existence and long-time behavior of solutions in terms of the stability of stationary solutions and the existence of a global attractor. Here the existence and uniqueness of solutions are studied by combining the Galerkin approximation method and the energy method. The existence of a stationary solution is established by a corollary of the Brouwer fixed point theorem, while its exponential stability is proved by using the Gronwall-like lemma. Finally, we use the energy method to show the existence of a global attractor in the phase space $L^2(-r, 0; H) \times H$.

The paper is organized as follows. In Section 2, we recall some function spaces and lemmas which will be used frequently later. Section 3 is devoted to the existence and uniqueness of weak solutions. In Section 4, we study the existence and exponential stability of a stationary solution. The existence of a global attractor for the continuous semigroup generated by problem (1) is shown in the last section.

2. Preliminaries

Let us recall function spaces, operators, inequalities and notations which are frequently used in the paper.

Putting

$$\mathcal{V} = \{u \in (C_0^\infty(\Omega))^3 \ : \ \nabla \cdot u = 0\}.$$

Denote H as the closure of $\mathcal{V}$ in $(L^2(\Omega))^3$ with the norm $|\cdot|$ and the inner product $(\cdot,\cdot)$ defined by

$$(u,v) = \sum_{j=1}^{3} \int_\Omega u_j(x)v_j(x)dx \text{ for } u,v \in (L^2(\Omega))^3.$$

We also denote V as the closure of $\mathcal{V}$ in $(H_0^1(\Omega))^3$ with the norm $\|\cdot\|$ and the associated scalar product $((,))$ defined by

$$((u,v)) = \sum_{i,j=1}^{3} \int_\Omega \frac{\partial u_j}{\partial x_j}\frac{\partial v_j}{\partial x_i}dx \text{ for } u,v \in (H_0^1(\Omega))^3.$$

We use $\|\cdot\|_*$ for the norm in V' and $\langle\cdot,\cdot\rangle_{V,V'}$ for the dual pairing between V and V'. We recall the Stokes operator $A : V \to V'$ by $\langle Au, v\rangle = ((u,v))$. Denote by P the Helmholtz-Leray orthogonal projection in $(H_0^1(\Omega))^3$ onto the space V. Then $Au = -P\Delta u$ for all $u \in D(A) = (H^2(\Omega))^3 \cap V$. The Stokes operator A is a positive self-adjoint operator with compact inverse. Hence there exists a complete orthonormal set of eigenfunctions $\{w_j\}_{j=1}^\infty \subset H$ such that $Aw_j = \lambda_j w_j$ and

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots, \ \lambda_j \to +\infty \text{ as } j \to \infty.$$

We have the following Poincaré inequalities

$$\|u\|^2 \ge \lambda_1 |u|^2, \ \forall u \in V, \tag{5}$$

$$|u|^2 \ge \lambda_1 \|u\|_*^2, \ \forall u \in H.$$

We define the trilinear form b on $V \times V \times V$ by

$$b(u,v,w) = \sum_{i,j=1}^{3} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j dx, \forall u,v,w \in V,$$

and $B : V \times V \to V'$ by $\langle B(u,v), w\rangle = b(u,v,w)$. We can write $B(u,v) = P[(u\cdot\nabla)v]$. It is easy to check that if $u,v,w \in V$, then $b(u,v,w) = -b(u,w,v)$, and in particular,

$$b(u,v,v) = 0, \quad \forall u,v \in V. \tag{6}$$

Using Hölder's and Ladyzhenskaya's inequalities, we can choose the best positive constant c_0 such that

$$|b(u,v,w)| \le c_0 \|u\|\|v\||w|^{1/2}\|w\|^{1/2}, \ \forall u,v,w \in V. \tag{7}$$

From (7) and using Poincaré's inequality (5), we obtain that

$$|b(u,v,w)| \le c_0\lambda_1^{-1/4}\|u\|\|v\|\|w\|, \quad \forall u,v,w \in V. \tag{8}$$

We also use the following inequality in [10]

$$|b(u,v,u)| \le c_1|u|\|u\|\|v\| \text{ for all } u,v \in V. \tag{9}$$

To prove the existence of a stationary solution, we need the following lemma.

Lemma 2.1 ([3]). *Let X be a finite dimensional Hilbert space with scalar product $[\cdot,\cdot]$ and norm $[\cdot]$ and let P be a continuous mapping from X into itself such that*

$$[P(\xi),\xi] > 0 \text{ for } [\xi] = k > 0.$$

Then there exists $\xi \in X$, $[\xi] < k$, such that

$$P(\xi) = 0.$$

The following lemma is the Gronwall-like lemma (see [9]).

Lemma 2.2. *Let $y(\cdot) : [-r,+\infty) \to [0,+\infty)$ be a function. Assume that there exist positive numbers γ, α_1 and α_2 such that the following inequality holds*

$$y(t) \le \begin{cases} \alpha_1 e^{-\gamma t} + \alpha_2 \int_0^t e^{-\gamma(t-s)} \sup_{\theta\in[-r,0]} y(s+\theta)ds, & t \ge 0, \\ \alpha_2 e^{-\gamma t}, & t \in [-r,0]. \end{cases}$$

Then

$$y(t) \le \alpha_1 e^{-\nu t} \text{ for } t \ge -r,$$

where $\nu \in (0,\gamma)$ is the unique root of the equation $\frac{\alpha_2}{\gamma-\nu}e^{\nu r} = 1$ in this interval.

We can rewrite the 3D convective Brinkman-Forchheimer equations (1) in the following functional form

$$\begin{cases} \partial_t u + \nu Au + B(u,u) + Pf(u) & = PG(u(t-\rho(t))) + Ph, \\ u(0) & = u_0, \\ u(\theta) & = \phi(\theta), \theta \in (-r,0). \end{cases} \tag{10}$$

3. Existence and uniqueness of weak solutions

We first give the definition of weak solutions.

Definition. A function u is said to be a weak solution of problem (1) if $u(0) = u_0, u(t) = \phi(t)$ for a.e. $t \in (-r,0)$,

$$u \in L^2(-r,T;V) \cap L^\infty(0,T;H) \cap L^{\beta+1}(0,T;L^{\beta+1}(\Omega)) \text{ for all } T > 0,$$

and

$$\frac{d}{dt}(u(t),v) + \nu((u(t),v)) + b(u(t),u(t),v) + \langle f(u),v\rangle = (G(u(t-\rho(t))),v) + (h,v)$$

for all test functions $v \in V$.

We now prove the following theorem.

Theorem 3.1. *Suppose that* (2) *and* (4) *hold, and* $u_0, h \in H, \phi \in L^2(-r,0;H)$ *are given. Then if* $\nu^2 > \frac{2L_G^2}{\lambda_1^2(1-\rho_*)}$, *then there exists a unique weak solution to problem* (1).

Proof. **Existence.** Let $\{w_j\}$ be a basis in $V \cap (H^2(\Omega))^3$, which is orthonormal in H, consisting of all eigenfunctions of the Stokes operator A. Denote $V_m = \text{span}\{w_1, \dots, w_m\}$ and consider the projector $P_m u = \sum_{j=1}^m (u, v_j) w_j$. Define also

$$u_m(t) = \sum_{j=1}^m \gamma_{m,j}(t) w_j,$$

where the coefficients $\gamma_{m,j}$ are required to satisfy the following system

$$\begin{cases} \dfrac{d}{dt}(u_m(t), w_j) + \nu((u_m(t), w_j)) + b(u_m(t), u_m(t), w_j) + \langle f(u_m(t)), w_j \rangle \\ = (G(u_m(t-\rho(t))), w_j) + (h, w_j) \text{ in } D'(0,T), 1 \le j \le m, \\ u_m(0) = P_m u_0, \quad u_m(t) = P_m \phi(t), \quad t \in (-r, 0). \end{cases} \tag{11}$$

Observe that (11) is a system of ordinary functional differential equations in the unknown $\gamma^m(t) = (\gamma_{m1}(t), \dots, \gamma_{mm}(t))$. By a classical result in the theory of ordinary functional differential equations, problem (11) has a solution defined in an interval $[0, t^*]$ with $0 < t^* \le T$. However, by the *a priori* estimates below, we can set $t^* = T$.

Multiplying (11) by $\gamma_{mj}(t)$ then summing in j from 1 to m, and using (6), we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|u_m(t)|^2 + \nu\|u_m(t)\|^2 + \int_\Omega f(u_m(t)) u_m(t) dx \\ &= \int_\Omega G(u_m(t-\rho(t))) u_m(t) dx + \int_\Omega h(x) u_m(t) dx. \end{aligned}$$

Using the inequality $f(u) \cdot u \ge -C + \kappa |u|^{\beta+1}$, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|u_m(t)|^2 + \nu\|u_m(t)\|^2 + \kappa \int_\Omega |u_m|^{\beta+1} dx \\ &\le C + |G(u_m(t-\rho(t)))| \cdot |u_m(t)| + |h| \cdot |u_m(t)|. \end{aligned}$$

Assumption (4) implies that

$$|G(\xi)| \le L_G |\xi|. \tag{12}$$

Then, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|u_m(t)|^2 + \nu\|u_m(t)\|^2 + \kappa \int_\Omega |u_m|^{\beta+1} dx \\ &\le C + L_G |u_m(t-\rho(t))| \cdot |u_m(t)| + |h| \cdot |u_m(t)|. \end{aligned}$$

By the Cauchy inequality,

$$\frac{1}{2}\frac{d}{dt}|u_m(t)|^2 + \nu\|u_m(t)\|^2 + \kappa \int_\Omega |u_m|^{\beta+1} dx$$

$$\le C + \frac{L_G^2}{\lambda_1\nu}|u_m(t-\rho(t))|^2 + \frac{\lambda_1\nu}{4}|u_m(t)|^2 + \frac{1}{\lambda_1\nu}|h|^2 + \frac{\lambda_1\nu}{4}|u_m(t)|^2.$$

Integrating from 0 to t and using (12), we have

$$\begin{aligned}&|u_m(t)|^2 + 2\nu\int_0^t \|u_m(s)\|^2ds + 2\kappa\int_0^t \|u_m(s)\|^{\beta+1}_{L^{\beta+1}(\Omega)}ds\\ &\le 2CT + |u_0|^2 + \frac{2L_G^2}{\lambda_1\nu}\int_0^t |u_m(s-\rho(s))|^2ds + \frac{2}{\lambda_1\nu}\int_0^t |h|^2ds\\ &+ \lambda_1\nu\int_0^t |u_m(s)|^2ds.\end{aligned}$$

From (5) we deduce that

$$\begin{aligned}&|u_m(t)|^2 + \nu\int_0^t \|u_m(s)\|^2ds + 2\kappa\int_0^t \|u_m(s)\|^{\beta+1}_{L^{\beta+1}(\Omega)}ds\\ &\le 2CT + |u_0|^2 + \frac{2L_G^2}{\lambda_1\nu}\int_0^t |u_m(s-\rho(s))|^2ds + \frac{2}{\lambda_1\nu}\int_0^t |h|^2ds.\end{aligned} \tag{13}$$

Let $\tau = s-\rho(s)$, and since $\rho(s)\in[0,r]$ and $\frac{1}{1-\rho'} \le \frac{1}{1-\rho_*}$. Then

$$\begin{aligned}\int_0^t |u_m(s-\rho(s))|^2ds &= \frac{1}{1-\rho'}\int_{-r}^t |u_m(\tau)|^2d\tau\\ &\le \frac{1}{1-\rho_*}\int_{-r}^t |u_m(\tau)|^2d\tau\\ &= \frac{1}{1-\rho_*}\int_{-r}^0 |u_m(\tau)|^2d\tau + \frac{1}{1-\rho_*}\int_0^t |u_m(\tau)|^2d\tau.\end{aligned} \tag{14}$$

Using (13), (14), and the fact that $u(t)=\phi(t)$, $t\in(-r,0)$, we have

$$\begin{aligned}&|u_m(t)|^2 + \nu\int_0^t \|u_m\|^2ds + 2\kappa\int_0^t \|u_m(s)\|^{\beta+1}_{L^{\beta+1}(\Omega)}ds\\ &\le 2CT + |u_0|^2 + \frac{2L_G^2}{\lambda_1\nu(1-\rho_*)}\int_{-r}^0 |\phi(\tau)|^2d\tau\\ &+ \frac{2L_G^2}{\lambda_1\nu(1-\rho_*)}\int_0^t |u_m(\tau)|^2d\tau + \frac{2}{\lambda_1\nu}\int_0^t |h|^2ds.\end{aligned}$$

Using inequality (5) once again, we obtain

$$\begin{aligned}&|u_m(t)|^2 + \Big(\nu - \frac{2L_G^2}{\lambda_1^2\nu(1-\rho_*)}\Big)\int_0^t \|u_m(s)\|^2ds + 2\kappa\int_0^t \|u_m(s)\|^{\beta+1}_{L^{\beta+1}(\Omega)}ds\\ &\le 2CT + |u_0|^2 + \frac{2L_G^2}{\lambda_1\nu(1-\rho_*)}\int_{-r}^0 |\phi(\tau)|^2d\tau + \frac{2}{\lambda_1\nu}\int_0^t |h|^2ds.\end{aligned} \tag{15}$$

Since $\nu^2 > \frac{2L_G^2}{\lambda_1^2(1-\rho_*)}$ and $\phi\in L^2(-r,0;H)$, it follows that $\{u_m\}$ is bounded in $L^2(0,T;V)\cap L^\infty(0,T;H)\cap L^{\beta+1}(0,T;L^{\beta+1}(\Omega))$. Moreover, observe that

$u_m = P_m \phi$ in $(-r, 0)$ and, by the choice of the basis $\{w_j\}$, the sequence $\{u_m\}$ weakly converges to ϕ in $L^2(-r, 0; H)$.

Moreover, $\{G(u_m)\}$ is bounded in $L^2(0, T; H)$ and it is straight forward to bound the nonlinear term $\{b(u_m, u_m, \cdot)\}$. Using (2), we obtain that $|f(u)| \leq C(1 + |u|^\beta)$ with C depending on C_f. Hence,

$$\begin{aligned} \int_0^t \int_\Omega |f(u)|^{\frac{\beta+1}{\beta}} dxdt &\leq C \int_0^t \int_\Omega (1 + |u|^\beta)^{\frac{\beta+1}{\beta}} dxdt \\ &\leq C \int_0^t \int_\Omega (1 + |u|^{\beta+1}) dxdt. \end{aligned}$$

Hence,

$$\{f(u_m)\} \text{ is bounded in } L^{(\beta+1)/\beta}(0, T; L^{(\beta+1)/\beta}(\Omega)).$$

Now, we prove the boundedness of $\{\frac{du_m}{dt}\}$. We have

$$\begin{aligned} \frac{d}{dt} u_m(t) = &- \nu A u_m(t) - P_m B(u_m, u_m) - P f(u_m) \\ &+ P_m h + P_m G(t, u_m(t - \rho(t))). \end{aligned} \tag{16}$$

From (8), (15) and (16), it follows that

$$\begin{aligned} \left\| \frac{d}{dt} u_m \right\|_* &\leq \nu \|A u_m\|_* + \|B(u_m, u_m)\|_* + \|f(u_m)\|_{L^{(\beta+1)/\beta}(\Omega)} + |h| \\ &\quad + \|G(t, u_m(t - \rho(t)))\|_* \\ &\leq \nu \|u_m\| + c_0 \lambda_1^{-1/4} \|u_m\| + \|f(u_m)\|_{L^{(\beta+1)/\beta}(\Omega)} + |h| \\ &\quad + \lambda_1^{-1/2} |G(t, u_m(t - \rho(t)))| \\ &\leq \nu \|u_m\| + c_0 \lambda_1^{-1/4} \|u_m\| + \|f(u_m)\|_{L^{(\beta+1)/\beta}(\Omega)} + |h| \\ &\quad + L_G \lambda_1^{-1/2} |u_m(t - \rho(t))| \\ &\leq C, \ \forall m \geq 1. \end{aligned}$$

This implies that $\{\frac{du_m}{dt}\}$ is bounded in the space

$$L^2(0, T; V') + L^{(\beta+1)/\beta}(0, T; L^{(\beta+1)/\beta}(\Omega)).$$

Using the compactness of the injection of the space $W = \{u \in L^2(0, T; V); \frac{du}{dt} \in L^2(0, T; V') + L^{(\beta+1)/\beta}(0, T; L^{(\beta+1)/\beta}(\Omega))\}$ into $L^2(0, T; H)$, and from the preceding analysis and the assumptions on G, we can deduce that there exist a subsequence (denoted again by $\{u_m\}$) and a function $u \in L^2(0, T; V)$ such that

$$\begin{aligned} &u_m \to u \text{ weakly in } L^2(0, T; V), \\ &u_m \to u \text{ weakly star in } L^\infty(0, T; H), \\ &u_m \to \phi \text{ weakly in } L^2(-r, 0; H), \\ &f(u_m) \to \chi \text{ weakly in } L^{(\beta+1)/\beta}(0, T; L^{(\beta+1)/\beta}(\Omega)), \\ &G(u_m) \to G(u) \text{ weakly in } L^2(0, T; H), \end{aligned}$$

$$\frac{du_m}{dt} \to \frac{du}{dt} \text{ weakly in } L^2(0,T;H).$$

Since $\{u_m\}$ is bounded in $L^2(0,T;V)$, $\{\frac{du_m}{dt}\}$ is bounded in $L^2(0,T;H)$, using the Aubin-Lions compactness lemma we deduce that $u_m \to u$ strongly in $L^2(0,T;(L^2(\Omega))^3)$, up to a subsequence. Thus, we have (up to a subsequence)

$$u_m \to u \text{ a.e. in } \Omega_T.$$

From the continuity of $f(\cdot)$ we obtain that

$$f(u_m) \to f(u) \text{ a.e. in } \Omega_T.$$

Since the uniqueness of the weak limit, we have $f(u) \equiv \chi$.

Arguing now as in the non-delay case, we can take the limits in (11) to show that u is a weak solution to problem (1).

Uniqueness. Let u and v be two weak solutions of problem (1) and let $w = u - v$. Then, we have

$$\left\{ \begin{array}{l} \dfrac{dw}{dt} - \nu\Delta w + (w\cdot\nabla)u + (v\cdot\nabla)w + f(u) - f(v) \\ = G(u(t-\rho(t))) - G(v(t-\rho(t))), \\ \nabla\cdot w = 0, \\ w(\theta) = 0, \quad \theta \in (-r, 0]. \end{array} \right. \tag{17}$$

It is well known (see, e.g. [2]) that there exist two nonnegative constants $\alpha = \alpha(\beta)$ and C_f such that

$$2\int_\Omega (f(u)-f(v))(u-v)dx \ge -C_f|u-v|^2 + \alpha\int_\Omega (|u|^{\beta-1}+|v|^{\beta-1})|u-v|^2 dx. \tag{18}$$

Multiplying the first equation in (17) by w and integrating by parts, and then noticing that f satisfies (18) and using the definition of $b(w,v,w)$, we have

$$\begin{aligned} &\frac{d}{dt}|w|^2 + 2\nu\|w\|^2 + \alpha\int_\Omega (|u|^{\beta-1}+|v|^{\beta-1})|u-v|^2 dx \\ &\le C_f|w|^2 + 2\int_\Omega |((w\cdot\nabla)u)\cdot w|dx \\ &\quad + 2\int_\Omega |G(u(t-\rho(t))) - G(v(t-\rho(t)))|\cdot|w(t)|dx. \end{aligned} \tag{19}$$

By the Holder inequality and the Young inequality, we have

$$2\int_\Omega |((w\cdot\nabla)u)\cdot w|dx \le 2|u||w||\nabla w| \le \frac{\nu}{2}|\nabla w|^2 + C|u|^2|w|^2,$$

where $C = C(\nu)$. Assuming that $\beta - 1 > 2$ and using the Young inequality again, we obtain

$$2\int_\Omega |((w\cdot\nabla)u)\cdot w|dx \le \frac{\nu}{2}|\nabla w|^2 + \alpha\int_\Omega (|u|^{\beta-1}+|v|^{\beta-1})|w|^2dx + C|w|^2,$$

where $C = C(\nu, \alpha)$. Thus, (19) implies that

$$\frac{d}{dt}|w|^2 + 2\nu\|w\|^2 \le C_f|w|^2 + \frac{\nu}{2}|\nabla w|^2 + C|w|^2 \\ + 2\int_\Omega |G(u(t-\rho(t))) - G(v(t-\rho(t)))| \cdot |w(t)|dx.$$

Combining with (4), we get

$$\frac{d}{dt}|w|^2 + 2\nu\|w\|^2 \le (C_f + C)|w|^2 + \frac{\nu}{2}|\nabla w|^2 \\ + 2\int_\Omega L_G|u(t-\rho(t)) - v(t-\rho(t))| \cdot |w(t)|dx.$$

By the Cauchy inequality, we obtain

$$\frac{d}{dt}|w|^2 + 2\nu\|w\|^2 \le (C_f + C)|w|^2 + \frac{\nu}{2}|\nabla w|^2 + \frac{2L_G^2}{\lambda_1\nu}|w(t-\rho(t))|^2 + \frac{\lambda_1\nu}{2}|w|^2.$$

Using inequality (5), we have

$$\frac{d}{dt}|w|^2 + \nu\|w\|^2 \le (C_f + C)|w|^2 + \frac{2L_G^2}{\lambda_1\nu}|w(t-\rho(t))|^2.$$

Intergrating from 0 to t, we get

$$|w|^2 + \nu\int_0^t \|w\|^2 ds \\ \le |w(0)|^2 + (C_f + C)\int_0^t |w|^2 ds + \frac{2L_G^2}{\lambda_1\nu}\int_0^t |w(s-\rho(s))|^2 ds.$$

Using (14) again, we have

$$|w|^2 + \nu\int_0^t \|w\|^2 ds \\ \le |w(0)|^2 + (C_f + C)\int_0^t |w|^2 ds + \frac{2L_G^2}{\lambda_1\nu(1-\rho_*)}\int_{-r}^t |w(\tau)|^2 d\tau.$$

Note that $w(s) = 0$ for $s \in (-r, 0)$ and (5), we obtain

$$|w(t)|^2 + \nu\int_0^t \|w(s)\|^2 ds \\ \le |w(0)|^2 + (C_f + C)\int_0^t |w(s)|^2 ds + \frac{2L_G^2}{\lambda_1^2\nu(1-\rho_*)}\int_0^t \|w(s))\|^2 ds.$$

Thus,

$$|w(t)|^2 + \Big(\nu - \frac{2L_G^2}{\lambda_1^2\nu(1-\rho_*)}\Big)\int_0^t \|w(s)\|^2 ds \le |w(0)|^2 + (C_f + C)\int_0^t |w(s)|^2 ds.$$

Note that $\nu^2 > \frac{2L_G^2}{\lambda_1^2(1-\rho_*)}$ and $w(0) = 0$, we get the uniqueness of solutions by using the Gronwall lemma. □

4. Existence and exponential stability of a stationary solution

Let us recall the definition of stationary solutions to problem (1).

Definition. A weak stationary solution to problem (1) is an element $u^* \in V$ such that

$$\nu((u^*, v)) + b(u^*, u^*, v) + \langle f(u^*), v\rangle = (G(u^*), v) + (h, v)$$

for all test functions $v \in V$.

Theorem 4.1. *Suppose that G satisfies* (4) *and $2L_G < \nu\lambda_1$. Then, there exists a weak stationary solution of problem* (1)*. Moreover, if*

$$\nu > \frac{C_f}{\lambda_1} + \frac{L_G}{\lambda_1} + \frac{c_1}{\sqrt{\lambda_1}}\sqrt{\frac{2C_f\lambda_1\nu + |h|^2}{\nu(\nu\lambda_1 - 2L_G)}}, \tag{20}$$

where c_1 is the positive constant in inequality (9)*, then this stationary solution is unique.*

Proof. Let $\{w_j\}$ be a Hilbert basis of $(L^2(\Omega))^3$ such that $V_m = \mathrm{span}\{w_j\}_{j\geq 1}$ is dense in $(H_0^1(\Omega))^2 \cap (L^{\beta+1}(\Omega))^3$. For each integer $m \geq 1$, we find the approximate stationary solution in the form

$$u_m(t) = \sum_{j=1}^{m} \gamma_{mj}(t) w_j,$$

where

$$\begin{aligned} &\nu((u_m(t), w_j)) + b(u_m(t), u_m(t), w_j) + \langle f(u_m(t)), w_j\rangle \\ &= (G(u_m), w_j) + (h, w_j) \end{aligned} \tag{21}$$

for all $j = 1, \dots, m$. We apply Lemma 2.1 to prove the existence of u_m as follows.

Let $X = (H_0^1(\Omega))^3 \cap (L^{\beta+1}(\Omega))^3$ and $R_m : V_m \to V_m$ be defined by

$$((R_m u, v)) = \nu((u, v)) + b(u, u, v) + \langle f(u), v\rangle - (G(u), v) - (h, v), \quad \forall u, v, \in V_m.$$

For all $u \in V_m$, we have

$$\begin{aligned} ((R_m u, u)) &\geq \nu\|u\|^2 + b(u, u, u) + \kappa\|u\|_{L^{\beta+1}}^{\beta+1} - C_f - L_G|u|\cdot|u| - |h|\cdot|u| \\ &\geq \nu\|u\|^2 + \kappa\|u\|_{L^{\beta+1}}^{\beta+1} - C_f - \frac{L_G}{\lambda_1}\|u\|^2 - \frac{1}{2\lambda_1\nu}|h|^2 - \frac{\nu}{2}\|u\|^2 \\ &\geq \Big(\frac{\nu}{2} - \frac{L_G}{\lambda_1}\Big)\|u\|^2 + \kappa\|u\|_{L^{\beta+1}}^{\beta+1} - C_f - \frac{1}{2\lambda_1\nu}|h|^2. \end{aligned}$$

It follows that $((R_m u, u)) \geq 0$ for $\|u\|_X = \|u\| + \|u\|_{L^{p+1}} = k$ sufficiently large, and thus we obtain

$$k = \Big(\frac{2C_f\lambda_1\nu + |h|^2}{\nu(\lambda_1\nu - 2L_G)}\Big)^{1/2} + \Big(\frac{2C_f\lambda_1\nu + |h|^2}{2\lambda_1\nu\kappa}\Big)^{1/(\beta+1)},$$

where $L_G < \frac{\nu\lambda_1}{2}$. Thus, there exists a solution $u_m \in V_m$ satisfying $R_m(u_m) = 0$.

Now multiplying (21) by γ_{mj} and adding resulting equalities for $j = 1, \dots, m$, we obtain

$$\nu\|u_m\|^2 + \langle f(u_m), u_m\rangle = (G(u_m), u_m) + (h, u_m).$$

Hence we have the estimate

$$\Big(\frac{\nu}{2} - \frac{L_G}{\lambda_1}\Big)\|u_m\|^2 + \kappa\|u_m\|^{\beta+1}_{L^{\beta+1}} \le C_f + \frac{1}{2\lambda_1\nu}|h|^2. \tag{22}$$

Then $\{u_m\}$ is bounded in $(H_0^1(\Omega))^3 \cap (L^{\beta+1}(\Omega))^3$, and therefore there exists some u^* in $(H_0^1(\Omega))^3 \cap (L^{\beta+1}(\Omega))^3$ and a subsequence $n \to \infty$ such that

$$u_n \rightharpoonup u^* \text{ weakly in } (H_0^1(\Omega))^3 \cap (L^{\beta+1}(\Omega))^3.$$

On the other hand, using (2) and applying the Aubin-Lions lemma (see [15]), we can conclude that

$$f(u_m) \rightharpoonup f(u^*) \text{ weakly in } (L^{(\beta+1)/\beta}(\Omega))^3.$$

Finally, using (4), we have

$$G(u_m) \rightharpoonup G(u^*) \text{ weakly in } (L^2(\Omega))^3.$$

Combining the above, we conclude that u^* is a weak stationary solution to problem (1).

Now let u and v be two stationary solutions to problem (1). Denote $w = u - v$, we have

$$\nu\|u-v\|^2 + (f(u) - f(v), u-v) = (G(u) - G(v), u-v) + 2\int_\Omega |((w\cdot\nabla)v)\cdot w|dx.$$

By inequality (9), we have

$$2\int_\Omega |((w\cdot\nabla)v)\cdot w|dx \le c_1|w|\cdot\|w\|\cdot\|v\| \le \frac{c_1}{\sqrt{\lambda_1}}\|v\|\cdot\|w\|^2.$$

From inequality $\int_\Omega (f(u) - f(v))(u-v)dx \ge -C_f|u-v|^2$ and (4), we obtain

$$\nu\|w\|^2 \le C_f|w|^2 + L_G|w|^2 + \frac{c_1}{\sqrt{\lambda_1}}\|v\|\cdot\|w\|^2.$$

Hence,

$$\begin{aligned}\nu\|w\|^2 &\le (C_f + L_G)|w|^2 + \frac{c_1}{\sqrt{\lambda_1}}\|v\|\cdot\|w\|^2\\ &\le (\frac{C_f}{\lambda_1} + \frac{L_G}{\lambda_1} + \frac{c_1}{\sqrt{\lambda_1}}\|v\|)\|w\|^2.\end{aligned}$$

Finally, we get

$$(\nu - \frac{C_f}{\lambda_1} - \frac{L_G}{\lambda_1} - \frac{c_1}{\sqrt{\lambda_1}}\|v\|)\|u-v\|^2 \le 0,$$

where $\|v\|$ satisfies the *a priori* estimate like (22). Hence we get the uniqueness of stationary solutions. □

Theorem 4.2. *Assume that the assumptions of Theorem 4.1 and* (20) *hold. Then the unique stationary solution u^* of problem* (1) *is exponentially stable.*

Proof. Notice that we can write the solution $u(t)$ to problem (1) in the form $u(t) = u^* + v(t)$, for $v(t)$ satisfies

$$\frac{dv}{dt} - \nu\Delta v + (u\cdot\nabla)u - (u^*\cdot\nabla)u^* + f(u(t)) - f(u^*) = G(u(t-\rho(t)) - G(u^*).$$

Multiplying this equation by v and an exponential term $e^{\lambda t}$ with a positive λ to be fixed later on, we obtain

$$\begin{aligned}&\frac{d}{dt}(e^{\lambda t}|v(t)|^2) - \lambda e^{\lambda t}|v(t)|^2 + 2\nu e^{\lambda t}\|v(t)\|^2 + 2e^{\lambda t}(f(u(t) - f(u^*), u(t) - u^*)\\ &\le 2e^{\lambda t}(G(u(t-\rho(t))) - G(u^*), u(t) - u^*) - 2e^{\lambda t}b(u-u^*, u^*, u-u^*).\end{aligned}$$

Using the facts that $\int_\Omega (f(u) - f(v))(u-v)dx \ge -C_f|u-v|^2$ and that

$$\begin{aligned}b(u-u^*, u^*, u-u^*) &\le c_1|u-u^*|\cdot\|u-u^*\|\cdot\|u^*\|\\ &\le \frac{c_1}{\sqrt{\lambda_1}}\|u^*\|\cdot\|u-u^*\|^2,\end{aligned}$$

we then have

$$\begin{aligned}&\frac{d}{dt}(e^{\lambda t}|v(t)|^2) + 2\nu e^{\lambda t}\|v(t)\|^2\\ &\le \lambda e^{\lambda t}|v(t)|^2 + 2C_f e^{\lambda t}|v(t)|^2 + 2e^{\lambda t}(G(u(t-\rho(t))) - G(u^*), u(t) - u^*)\\ &\quad + 2e^{\lambda t}\frac{c_1}{\sqrt{\lambda_1}}\|u^*\|\cdot\|v(t)\|^2.\end{aligned}$$

From (4), we get

$$\begin{aligned}&\frac{d}{dt}(e^{\lambda t}|v(t)|^2) + 2\nu e^{\lambda t}\|v(t)\|^2\\ &\le \lambda e^{\lambda t}|v(t)|^2 + 2C_f e^{\lambda t}|v(t)|^2 + 2L_G e^{\lambda t}|v(t-\rho(t))|\cdot|v(t)|\\ &\quad + 2e^{\lambda t}\frac{c_1}{\sqrt{\lambda_1}}\|u^*\|\cdot\|v(t)\|^2.\end{aligned}$$

Using the Cauchy inequality, we have

$$\begin{aligned}&\frac{d}{dt}(e^{\lambda t}|v(t)|^2)\\ &\le (\lambda + 2C_f)e^{\lambda t}|v(t)|^2 - 2\Big(\nu - \frac{c_1}{\sqrt{\lambda_1}}\|u^*\|\Big)e^{\lambda t}\|v(t)\|^2\\ &\quad + L_G e^{\lambda t}|v(t-\rho(t))|^2 + L_G e^{\lambda t}|v(t)|^2\\ &\le (\lambda + 2C_f + L_G)e^{\lambda t}|v(t)|^2 - 2\Big(\nu - \frac{c_1}{\sqrt{\lambda_1}}\|u^*\|\Big)e^{\lambda t}\|v(t)\|^2\\ &\quad + L_G e^{\lambda t}|v(t-\rho(t))|^2\\ &\le (\lambda + 2C_f + L_G)e^{\lambda t}|v(t)|^2 - 2\lambda_1\Big(\nu - \frac{c_1}{\sqrt{\lambda_1}}\|u^*\|\Big)e^{\lambda t}|v(t)|^2\\ &\quad + L_G e^{\lambda t}|v(t-\rho(t))|^2\\ &\le (\lambda + 2C_f + L_G + 2\sqrt{c_1\lambda_1}\|u^*\| - 2\nu\lambda_1)e^{\lambda t}|v(t)|^2 + L_G e^{\lambda t}|v(t-\rho(t))|^2.\end{aligned}$$

Integrating from 0 to t, we obtain

$$
\begin{aligned}
e^{\lambda t}|v(t)|^2 \leq |v(0)|^2 &+ (\lambda + 2C_f + L_G + 2\sqrt{c_1\lambda_1}\|u^*\| - 2\nu\lambda_1)\int_0^t e^{\lambda s}\|v(s)\|^2 \\
&+ L_G \int_0^t e^{\lambda s}|v(s-\rho(s))|^2 ds.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&|v(t)|^2 \\
&\leq e^{-\lambda t}|v(0)|^2 \\
&\quad + (\lambda + 2C_f + 2L_G + 2\sqrt{c_1\lambda_1}\|u^*\| - 2\nu\lambda_1)\int_0^t e^{-\lambda(t-s)} \sup_{\theta\in[-r;0]}|v(s+\theta)|^2 ds.
\end{aligned}
$$

If $2\nu\lambda_1 > 2C_f + 2L_G + 2\sqrt{c_1\lambda_1}\|u^*\|$, then there exists $\lambda > 0$ such that

$$
\lambda + 2C_f + 2L_G + 2\sqrt{c_1\lambda_1}\|u^*\| - 2\nu\lambda_1 > 0.
$$

By Lemma 2.2, it follows that

$$
|u(t) - u^*|^2 \leq Me^{-\gamma t}, \quad t \geq 0,
$$

where $\gamma \in (0, \lambda)$. The proof is complete. □

5. Existence of a global attractor

By Theorem 3.1, we can define a semigroup $S(t) : L^2(-r, 0; H) \times H \to L^2(-r, 0; H) \times H$ by

$$
S(t)(\phi, u_0) = (u_t, u(t)),
$$

where $u(\cdot)$ is the unique weak solution of problem (1) with the initial datum (ϕ, u_0).

We first prove the following continuity result.

Lemma 5.1. *Under the assumptions of Theorem 3.1, the mapping $S(t)$: $L^2(-r, 0; H) \times H \to L^2(-r, 0; H) \times H$ is continuous for any $t > 0$.*

Proof. Let $(\phi, u_0), (\psi, v_0) \in L^2(-r, 0; H) \times H$ be two pairs of initial data, and u, v are the corresponding solutions to problem (1). We have

$$
\begin{aligned}
&\frac{d}{dt}(u - v) - \nu\Delta(u - v) + (u\cdot\nabla)u - (v\cdot\nabla)v + \nabla(p_u - p_v) + (f(u) - f(v)) \\
&= G(u(t-\rho(t))) - G(v(t-\rho(t))).
\end{aligned}
$$

Setting $w = u - v$ and multiplying the above equality by w, we deduce that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|w|^2 + \nu\|w\|^2 + \langle(u\cdot\nabla)u - (v\cdot\nabla)v, w\rangle + \langle f(u) - f(v), w\rangle \\
&= (G(u(t-\rho(t))) - G(v(t-\rho(t))), w).
\end{aligned}
$$

Therefore, from the assumption of f and (9) we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|w|^2+\nu\|w\|^2\\&\leq C_f|w|^2+c_1|w|\|u\|\|w\|+L_G|u(t-\rho(t))-v(t-\rho(t))|\cdot|w|\\&\leq C_f|w|^2+c_2|w|^2\|u\|^2+\nu\|w\|^2+\frac{L_G}{2}|(u(t-\rho(t))-v(t-\rho(t))|^2+\frac{1}{2}|w|^2.\end{aligned}$$

From (4) we obtain

$$\frac{d}{dt}|w|^2\leq(C_f+2c_2\|u\|^2+1)|w|^2+L_G|w(t-\rho(t))|^2.$$

Using (14) we have

$$\begin{aligned}|w(t)|^2&\leq|w(0)|^2+\int_0^t(C_f+2c_2\|u(s)\|^2+1)|w(s)|^2ds+L_G\int_0^t|w(s-\rho(s))|^2ds\\&\leq|u_0-v_0|^2+\int_0^t(C_f+2c_2\|u(s)\|^2+1)|w(s)|^2ds+\frac{L_G}{1-\rho_*}\int_{-r}^0|w(\tau)|^2d\tau\\&\quad+\frac{L_G}{1-\rho_*}\int_0^t|w(\tau)|^2d\tau.\end{aligned}$$

Consequently,

$$\begin{aligned}|w(t)|^2&\leq|u_0-v_0|^2+\frac{L_G}{1-\rho_*}\int_{-r}^0|\phi-\psi|^2ds\\&\quad+\int_0^t\Big(C_f+2c_2\|u(s)\|^2+1+\frac{L_G}{1-\rho_*}\Big)|w(s)|^2ds\\&\leq|u_0-v_0|^2+\frac{L_G}{1-\rho_*}|\phi-\psi|^2_{L^2(-r,0;H)}\\&\quad+\int_0^t\Big(C_f+2c_2\|u(s)\|^2+1+\frac{L_G}{1-\rho_*}\Big)|w(s)|^2ds.\end{aligned}$$

Using the Gronwall lemma, we have

$$\begin{aligned}|w(t)|^2&=|u(t)-v(t)|^2\\&\leq\Big(|u_0-v_0|^2+\frac{L_G}{1-\rho_*}|\phi-\psi|^2_{L^2(-r,0;H)}\Big)\\&\quad\times\exp\Big(\int_0^t\Big(C_f+2c_2\|u(s)\|^2+1+\frac{L_G}{1-\rho_*}\Big)ds\Big).\end{aligned}$$

For $\theta\in[-r,0]$, assume now that

$$\begin{aligned}|u_t-v_t|^2&\leq\sup_{\theta\in[-r,0]}|u(t+\theta)-v(t+\theta)|^2\\&\leq\Big(|u_0-v_0|^2+\frac{L_G}{1-\rho_*}|\phi-\psi|^2_{L^2(-r,0;H)}\Big)\end{aligned}$$

$$
\begin{aligned}
&\times \exp\Big(\int_0^{t+\theta}\Big(C_f+2c_2\|u(s)\|^2+1+\frac{L_G}{1-\rho_*}\Big)ds\Big)\\
&\le \Big(|u_0-v_0|^2+\frac{L_G}{1-\rho_*}|\phi-\psi|^2_{L^2(-r,0;H)}\Big)\\
&\quad\times \exp\Big(\int_0^{t}\Big(C_f+2c_2\|u(s)\|^2+1+\frac{L_G}{1-\rho_*}\Big)ds\Big).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
|u_t-v_t|^2_{L^2(-r,0;H)} &= \int_{-r}^0 |u_t(\theta)-v_t(\theta)|^2 d\theta\\
&\le \int_{-r}^0 \sup_{\theta\in[-r,0]}|u(t+\theta)-v(t+\theta)|^2 d\theta\\
&\le \int_{-r}^0 \Big(|u_0-v_0|^2+\frac{L_G}{1-\rho_*}|\phi-\psi|^2_{L^2(-r,0;H)}\Big)\\
&\quad\times \exp\Big(\int_0^{t+\theta}\Big(C_f+2c_2\|u(s)\|^2+1+\frac{L_G}{1-\rho_*}\Big)ds\Big)d\theta\\
&\le \int_{-r}^0 \Big(|u_0-v_0|^2+\frac{L_G}{1-\rho_*}|\phi-\psi|^2_{L^2(-r,0;H)}\Big)\\
&\quad\times \exp\Big(\int_0^{t}\Big(C_f+2c_2\|u(s)\|^2+1+\frac{L_G}{1-\rho_*}\Big)ds\Big)d\theta\\
&\le r\Big(|u_0-v_0|^2+\frac{L_G}{1-\rho_*}|\phi-\psi|^2_{L^2(-r,0;H)}\Big)\\
&\quad\times \exp\Big(\int_0^{t}\Big(2c_2\|u(s)\|^2+1+\frac{L_G}{1-\rho_*}\Big)ds\Big).
\end{aligned}
$$

The proof is now complete. □

We now prove the existence of an absorbing set in $L^2(-r,0;H)\times H$.

Lemma 5.2. *Suppose that the assumptions of Theorem 3.1 hold and* $2L_G<\nu\lambda_1$*. Then the semigroup* $S(t)$ *has an absorbing set* $\mathcal{B}_H$ *in* $L^2(-r,0;H)\times H$*.*

Proof. Multiplying the first equation in (1) by u, we obtain

$$
\frac{1}{2}\frac{d}{dt}|u|^2+\nu\|u\|^2+\langle f(u),u\rangle=(h,u)+(G(u(t-\rho(t))),u).
$$

Using the inequality $f(u)\cdot u\ge -C+\kappa|u|^{\beta+1}$ and (4), we have

$$
\frac{1}{2}\frac{d}{dt}|u|^2+\nu\|u\|^2+\kappa\int_\Omega |u|^{\beta+1}dx\le C+L_G|u(t-\rho(t))|\cdot|u|+|h|\cdot|u|.
$$

Choose $\sigma > 0$ small enough such that $\lambda_1 \nu > 2L_G + \sigma$. By the Cauchy inequality, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|u|^2 + \nu\|u\|^2 + \kappa \int_\Omega |u|^{\beta+1} dx \\ &\leq C + \frac{L_G}{8}|u(t-\rho(t))|^2 + 2L_G|u|^2 + \frac{1}{4\sigma}|h|^2 + \sigma|u|^2. \end{aligned}$$

By inequality (5), we have

$$\frac{d}{dt}|u|^2 \leq 2C + \frac{1}{2\sigma}|h|^2 + \frac{L_G}{4}|u(t-\rho(t))|^2 - (2\nu\lambda_1 - (2\sigma + 4L_G))|u|^2.$$

We now choose $m \in (0, m_0)$, $m_0 > 0$, such that

$$\nu\lambda_1 > \frac{L_G}{8(1-\rho_*)}e^{mr} + 2L_G + \sigma + \frac{m}{2}.$$

Then

$$\begin{aligned} \frac{d}{dt}(e^{mt}|u|^2) &= me^{mt}|u|^2 + e^{mt}\frac{d}{dt}|u|^2 \\ &\leq me^{mt}|u|^2 + 2Ce^{mt} + \frac{1}{2\sigma}e^{mt}|h|^2 + \frac{L_G}{4}e^{mt}|u(t-\rho(t))|^2 \\ &\quad - (2\nu\lambda_1 - (2\sigma + 4L_G))\, e^{mt}|u|^2. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{d}{dt}(e^{mt}|u|^2) &\leq 2Ce^{mt} + \frac{1}{2\sigma}e^{mt}|h|^2 \\ &\quad + \frac{L_G}{4}e^{mt}|u(t-\rho(t))|^2 + (m - (2\nu\lambda_1 - (\sigma + L_G)))\, e^{mt}|u|^2. \end{aligned}$$

Integrating between 0 and t we obtain

$$\begin{aligned} e^{mt}|u(t)|^2 - |u_0|^2 &\leq \frac{2}{m}Ce^{mt} + \frac{1}{2m\sigma}e^{mt}|h|^2 + \frac{L_G}{4}\int_0^t e^{ms}|u(s-\rho(s))|^2 ds \\ &\quad + (m - (2\nu\lambda_1 - (2\sigma + 4L_G)))\int_0^t e^{ms}|u(s)|^2 ds. \end{aligned} \tag{23}$$

Now, let $\tau = s - \rho(s)$. In view of $\rho(s) \in [0, r]$ and $\frac{1}{1-\rho} < \frac{1}{1-\rho_*}$, then

$$\begin{aligned} \int_0^t e^{ms}|u(s-\rho(s))|^2 ds &\leq \frac{1}{1-\rho_*}\int_{-r}^t e^{m(\tau+r)}u(\tau)d\tau \\ &= \frac{e^{mr}}{1-\rho_*}\int_{-r}^t e^{m\tau}|u(\tau)|d\tau. \end{aligned} \tag{24}$$

Combining (23) and (24), we have

$$\begin{aligned} e^{mt}|u(t)|^2 - |u_0|^2 &\leq \frac{2}{m}Ce^{mt} + \frac{1}{2m\sigma}e^{mt}|h|^2 + \frac{L_G}{4(1-\rho_*)}e^{mr}\int_{-r}^t e^{ms}|u(s)|^2 ds \\ &\quad + (m - (2\nu\lambda_1 - (2\sigma + 4L_G)))\int_0^t e^{ms}|u(s)|^2 ds. \end{aligned}$$

Since $u(t) = \phi(t)$ for $t \in (-r, 0)$, we obtain

$$
\begin{aligned}
&e^{mt}|u(t)|^2 - |u_0|^2 \\
&\le \frac{2}{m}Ce^{mt} + \frac{1}{2m\sigma}e^{mt}|h|^2 + \frac{L_G}{4(1-\rho_*)}e^{mr}\int_{-r}^{0} e^{ms}|\phi(s)|^2 ds \\
&\quad + \frac{L_G}{4(1-\rho_*)}e^{mr}\int_0^t e^{ms}|u(s)|^2 ds \\
&\quad + (m - (2\nu\lambda_1 - (2\sigma + 4L_G)))\int_0^t e^{ms}|u(s)|^2 ds \\
&= \frac{2}{m}Ce^{mt} + \frac{1}{2m\sigma}e^{mt}|h|^2 + \frac{L_G}{4(1-\rho_*)}e^{mr}\int_{-r}^{0} e^{ms}|\phi(s)|^2 ds \\
&\quad + \Big(m + \frac{L_G}{4(1-\rho_*)}e^{mr} - (2\nu\lambda_1 - (2\sigma + 4L_G))\Big)\int_0^t e^{ms}|u(s)|^2 ds \\
&\le \frac{2}{m}Ce^{mt} + \frac{1}{2m\sigma}e^{mt}|h|^2 + \frac{L_G}{4(1-\rho_*)}e^{mr}\int_{-r}^{0} e^{ms}|\phi(s)|^2 ds.
\end{aligned}
$$

Thus,

$$
e^{mt}|u(t)|^2 \le |u_0|^2 + \frac{2}{m}Ce^{mt} + \frac{1}{2m\sigma}|h|^2 + \frac{L_G}{4(1-\rho_*)}e^{mr}\int_{-r}^{0} e^{ms}|\phi(s)|^2 ds
$$

and

$$
|u(t)|^2 \le \frac{2}{m}C + \frac{1}{2m\sigma}|h|^2 + e^{-mt}\Big(|u_0|^2 + \frac{L_G}{4(1-\rho_*)}e^{mr}\int_{-r}^{0} e^{ms}|\phi(s)|^2 ds\Big).
$$

Therefore,

$$
\begin{aligned}
&|u(t-\rho(t))|^2 \\
&\le \frac{2}{m}C + \frac{1}{2m\sigma}|h|^2 + e^{-m(t-\rho(t))}\Big(|u_0|^2 + \frac{L_G}{4(1-\rho_*)}e^{mr}\int_{-r}^{0} e^{ms}|\phi(s)|^2 ds\Big) \\
&\le \frac{2}{m}C + \frac{1}{2m\sigma}|h|^2 + e^{-mt}\cdot e^{m\rho(t)}\Big(|u_0|^2 + \frac{L_G}{4(1-\rho_*)}e^{mr}\int_{-r}^{0} e^{ms}|\phi(s)|^2 ds\Big) \\
&\le \frac{2}{m}C + \frac{1}{2m\sigma}|h|^2 + e^{-mt}\cdot e^{mr}\Big(|u_0|^2 + \frac{L_G}{4(1-\rho_*)}e^{mr}\int_{-r}^{0} e^{ms}|\phi(s)|^2 ds\Big)
\end{aligned}
$$

for $\rho \in [0, r]$. For $\theta \in [-r, 0]$, we have

$$
\begin{aligned}
&|u(t+\theta)|^2 \\
&\le \frac{2}{m}C + \frac{1}{2m\sigma}|h|^2 + e^{-m(t+\theta)}\Big(|u_0|^2 + \frac{L_G}{4(1-\rho_*)}e^{mr}\int_{-r}^{0} e^{ms}|\phi(s)|^2 ds\Big) \\
&\le \frac{2}{m}C + \frac{1}{2m\sigma}|h|^2 + e^{-m(t-r)}\Big(|u_0|^2 + \frac{L_G}{4(1-\rho_*)}e^{mr}\int_{-r}^{0} e^{ms}|\phi(s)|^2 ds\Big)
\end{aligned}
$$

$$= \frac{2}{m}C + \frac{1}{2m\sigma}|h|^2 + e^{-mt}\Big(e^{mr}|u_0|^2 + \frac{L_G}{4(1-\rho_*)}e^{2mr}\int_{-r}^{0} e^{ms}|\phi(s)|^2 ds\Big).$$

Hence,

$$|u_t|^2 \le \frac{2}{m}C + \frac{1}{2m\sigma}|h|^2 + e^{-mt}\Big(e^{mr}|u_0|^2 + \frac{L_G}{4(1-\rho_*)}e^{2mr}\int_{-r}^{0} e^{ms}|\phi(s)|^2 ds\Big).$$

Denoting $\frac{\rho_H}{2} = \frac{2}{m}C + \frac{1}{2m\sigma}|h|^2$, we have

$$|u_t|^2 \le \rho_H. \tag{25}$$

This implies the existence of an absorbing set for the semigroup $S(t)$. □

Lemma 5.3. *Under the assumptions of Lemma 5.2, the semigroup $S(t)$ is asymptotically compact in $L^2(-r,0;H)\times H$.*

Proof. Let B be a bounded set in $L^2(-r,0;H)\times H$ and $u^n(\cdot)$ be a sequence of solutions in $[0,+\infty)$ with initial data $(\phi^n, u_0^n) \in B$. Consider the sequence $\xi^n = S(t_n)(\phi^n, u_0^n)$, where $t_n \to +\infty$ as $n \to +\infty$. We will show that this sequence is relatively compact in $L^2(-r,0;H)\times H$.

First, let $T > 0$. We will prove that ξ^n is relatively compact in $L^2(-r,0;H)\times H$. It follows from (25) that there exists n_0 such that $t_n \ge T$ for all $n > n_0$ and

$$|\xi^n|^2_{L^2(-r,0;H)} \le \rho_H. \tag{26}$$

Let $y^n(\cdot) = u^n_{t_n - T}(\cdot) = u^n(\cdot + t_n - T)$. Then for each $n \ge 1$ such that $t_n \ge T$, the function y^n is a solution on $[0,T]$ of a similar problem to (1), namely,

$$\frac{d}{dt}y^n(t) - \nu\Delta y^n + (y^n\cdot\nabla)y^n + f(y^n(t)) = G(y^n(t-\rho(t))) + h,$$

with $y_0^n = u^n_{t_n-T}$, $y_T^n = \xi^n$. Then y_0^n satisfies the estimates in (26) for all $n > n_0$. Using arguments as in the proof of Theorem 3.1, we have

$$y^n(t_n) \rightharpoonup y(t_0) \text{ weakly in } V \text{ if } t_n \to t_0 \in [0,T].$$

Also, by (4), we obtain

$$\int_0^t |G(y^n(t-\rho(t)))|^2 ds \le Ct, \forall 0 \le t \le T,$$

where $C > 0$ does not depend either on n or t. Since $G(y^n(t-\rho(t))) \rightharpoonup \xi - \rho(t)$ in $L^2(0,T;H)$, we get

$$\int_s^t |\xi|^2 d\tau \le \liminf_{n\to+\infty}\int_s^t |G(y^n_\tau - \rho(\tau))|d\tau \le C(t-s), \forall 0 \le s \le t \le T.$$

Thus, we can pass to the limits and prove that y is a solution of a similar problem to (1), that is

$$\frac{d}{dt}(y(t),v) + \nu((y(t),v)) + B(y(t),v) + \int_\Omega \langle f(y(t)),v\rangle dx = (\xi,v) + \langle h,v\rangle$$

for all $v \in L^\infty(0,T;V) \cap L^{\beta+1}(0,T;L^{\beta+1}(\Omega))$. Since

$$\int_s^t \int_\Omega G(z_r - \rho(t)) z(r) dx dr \le \frac{1}{2\lambda_1 \nu} \int_s^t |G(z_r - \rho(t))|^2 dr + \frac{\lambda_1 \nu}{2} \int_s^t |z(r)|^2 dr,$$

we obtain the energy inequality

$$\begin{aligned} |z(t)|^2 + \nu \int_s^t \|z(t)\|^2 dr + 2\int_0^t \langle f(z(r)), z(r)\rangle \\ = |z(s)|^2 + 2\int_s^t \langle h, z(r)\rangle dr + 2C(t-s), \forall 0 \le s \le t \le T, \end{aligned}$$

where $z = y^n$ or $z = y$.

Now, consider two functions $J_n, J : [0,T] \to \mathbb{R}$ defined by

$$\begin{aligned} J_n(t) &= \frac{1}{2}|y_n(t)|^2 + \int_0^t \langle f(y^n(r)), y^n(r)\rangle dr - \int_0^t \langle h, y^n(r)\rangle dr - Ct, \\ J(t) &= \frac{1}{2}|y(t)|^2 + \int_0^t \langle f(y(r)), y(r)\rangle dr - \int_0^t \langle h, y(r)\rangle dr - Ct. \end{aligned}$$

It is clear that J_n and J are non-increasing and continuous functions. Since $y^n(t)$ converges to $y(t)$ for a.e. $t \in (0,T)$, we obtain

$$J_n(t) \to J(t) \text{ for a.e. } t \in [0,T].$$

Analogously as we did in the proof of Theorem 3.1, for a fixed $t_0 > 0$, using a sequence $\{t_k\}$ with $t_k \to t_0$, we are able to establish the convergence of the norms

$$\lim_{n\to\infty} |y^n(t_n)| = |y(t_0)|.$$

And therefore, jointly with the weak convergence already proved, we deduce that $y^n \to y$ in $C([0,T];H)$.

Now, since $T > 0$ and $y^n \to y$ in $C([0,T];H)$, we obtain that $\xi^n \to \varphi$ in $C([0,T];H)$, where $\varphi(s) = y(s+T)$ for $s \in [-r,0]$. Repeating the same procedure for $2T, 3T$, etc., for a diagonal subsequence (relabeled the same) we can obtain a continuous function $\varphi : (-r,0] \to H$ and a subsequence such that $\xi^n \to \varphi$ in $C([-r,0];H)$ on every interval $[-r,0]$. Moreover, for a fixed $T > 0$, we also have

$$|\varphi(s)| \le \rho_H, \forall s \in [-r,0], \forall T > 0.$$

Second, we claim that ξ^n converges to φ in $L^2(-r,0;H)$. Indeed, we have to prove that for every $\varepsilon > 0$, there exists n_ε such that

$$|\xi^n(s) - \varphi(s)|^2 \le \varepsilon, \forall n \ge n_\varepsilon. \tag{27}$$

Fix $T_\varepsilon > 0$ such that $\rho_H^2 \le \frac{\varepsilon}{4}$.

From the first step, we have $\xi^n \to \varphi$ in $L^2(-r,0;H)$, so there exists $n_\varepsilon = n_\varepsilon(T_\varepsilon)$ such that for all $n \ge n_\varepsilon$, we obtain

$$|\xi^n(s) - \varphi(s)|^2 \le \varepsilon, \forall t_n \ge T_\varepsilon.$$

In order to prove (27), we only have to check that

$$|\xi^n(s) - \varphi(s)|^2 \leq \varepsilon, \forall n \geq n_\varepsilon.$$

Because of (26) and the choice of T_ε, we can check that for all $k \in \mathbb{N} \cup \{0\}$ and $s \in [-(T_\varepsilon + k + 1), -(T_\varepsilon + k)]$, the following holds

$$\int_{-r}^{0} |\varphi(s)|^2 ds \leq \int_{-r}^{0} |\varphi(s - T_\varepsilon - k)|^2 ds \leq \frac{\varepsilon}{4}.$$

Thus, it suffices to prove that

$$|\xi^n(s)|^2 \leq \frac{\varepsilon}{4}, \forall n \geq n_\varepsilon.$$

We have

$$\xi^n(s) = \begin{cases} \phi^n(s + t_n), & \text{if } s \in [-r, -t_n], \\ u^n(s + t_n), & \text{if } s \in [-t_n, 0]. \end{cases}$$

Hence the proof is finished if we prove that

$$\max \left\{ \int_{-r}^{0} |\phi^n(s + t_n)|^2 ds, \int_{-r}^{0} |u^n(s + t_n)|^2 ds \right\} \leq \frac{\varepsilon}{4}.$$

The first term above can be estimated as follows

$$\int_{-r}^{0} |\phi^n(s + t_n)|^2 ds \leq \frac{\varepsilon}{4}.$$

And, finally, for the second term, we obtain

$$\int_{-r}^{0} |u^n(s + t_n)|^2 ds \leq \frac{\varepsilon}{4}.$$

This completes the proof. □

From Lemmas 5.2 and 5.3, we obtain the following theorem.

Theorem 5.4. *Under the assumptions of Lemma 5.2, the semigroup $S(t)$ has a compact global attractor in the phase space $L^2(-r, 0; H) \times H$.*

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