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FRACTIONAL DIFFERENTIATIONS AND INTEGRATIONS OF QUADRUPLE HYPERGEOMETRIC SERIES

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Abstract. The hypergeometric series of four variables are introduced and studied by Bin-Saad and Younis recently. In this line, we derive several fractional derivative formulas, integral representations and operational formulas for new quadruple hypergeometric series.

1. Introduction

Hypergeometric functions have been attracted the attention of many researchers due to their importance and applications in diverse areas of mathematical, physical, engineering and statistical sciences [2, 4, 5]. Multiple hypergeometric functions occur in various fields of pure and applied mathematics such as approximation theory, partition theory, representation theory, group theory, mirror symmetry, difference equations and mathematical physics etc. They possess important properties such as recurrence and explicit relations, summation formulae, symmetric and convolution identities, algebraic properties etc. In recent years, several multivariable hypergeometric functions and their properties have been considered by many authors [1, 3, 7–11, 15, 19, 23].

For our purpose, we begin by recalling some known functions and earlier works.

The Gaussian hypergeometric function defined by [22],

$$ {}_2F_1\left(a,b;c;x\right)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!},\ (|x|<1), \tag{1.1}$$

where, $(a)_m$ is the Pochhammer symbol defined by

$$(a)_m=\frac{\Gamma(a+m)}{\Gamma(a)}=a(a+1)\cdots(a+m-1)$$

for $m\geq 1$, $(a)_0=1$, Γ being the well-known Gamma function.

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The Appell hypergeometric functions F_1, F_3 and F_4 in double variables [22], defined by

$$F_1\left(a,b,c;d;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m(c)_n}{(d)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}, \tag{1.2}$$

$$F_3\left(a,b,c,d;e;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_m(b)_n(c)_m(d)_n}{(e)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}, \tag{1.3}$$

$$F_4\left(a,b;c,d;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_{m+n}}{(c)_m(d)_n}\frac{x^m}{m!}\frac{y^n}{n!}. \tag{1.4}$$

The Lauricella functions of three variables $F_D^{(3)}, F_N, F_R, F_S$ and F_T [16] are defined as:

$$\begin{aligned}&F_D^{(3)}\left(a,b_1,b_2,b_3;c;x,y,z\right)\\&=\sum_{m,n,p=0}^{\infty}\frac{(a)_{m+n+p}(b_1)_m(b_2)_n(b_3)_p}{(c)_{m+n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p},\end{aligned} \tag{1.5}$$

$$\begin{aligned}&F_N\left(a_1,a_2,a_3,b_1,b_2,b_1;c_1,c_2,c_2;x,y,z\right)\\&=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_m(a_2)_n(a_3)_p(b_1)_{m+p}(b_2)_n}{(c_1)_m(c_2)_{n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p},\end{aligned} \tag{1.6}$$

$$\begin{aligned}&F_R\left(a_1,a_2,a_1,b_1,b_2,b_1;c_1,c_2,c_2;x,y,z\right)\\&=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{m+p}(a_2)_n(b_1)_{m+p}(b_2)_n}{(c_1)_m(c_2)_{n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p},\end{aligned} \tag{1.7}$$

$$\begin{aligned}&F_S\left(a_1,a_2,a_2,b_1,b_2,b_3;c,c,c;x,y,z\right)\\&=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_m(a_2)_{n+p}(b_1)_m(b_2)_n(b_3)_p}{(c)_{m+n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p},\end{aligned} \tag{1.8}$$

$$\begin{aligned}&F_T\left(a_1,a_2,a_2,b_1,b_2,b_1;c,c,c;x,y,z\right)\\&=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_m(a_2)_{n+p}(b_1)_{m+p}(b_2)_n}{(c)_{m+n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p}.\end{aligned} \tag{1.9}$$

Exton's functions $X_6, X_{13}, X_{18}, X_{19}$ and X_{20} are defined by [13]

$$\begin{aligned}&X_6\left(a,b,c;d,e;x,y,z\right)\\&=\sum_{m,n,p=0}^{\infty}\frac{(a)_{2m+n+p}(b)_n(c)_p}{(d)_{m+n}(e)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},\end{aligned} \tag{1.10}$$

$$X_{13}\left(a,b,c;d;x,y,z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+n}(b)_{n+p}(c)_p}{(d)_{m+n+p}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}, \tag{1.11}$$

$$X_{18}\left(a_1,a_2,a_3,a_4;c;x,y,z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_n(a_3)_p(a_4)_p}{(c)_{m+n+p}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}, \tag{1.12}$$

$$X_{19}\left(a_1,a_2,a_3,a_4;c_1,c_2;x,y,z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_n(a_3)_p(a_4)_p}{(c_1)_m(c_2)_{n+p}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}, \tag{1.13}$$

$$X_{20}\left(a_1,a_2,a_3,a_4;c_1,c_2;x,y,z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_n(a_3)_p(a_4)_p}{(c_1)_{m+p}(c_2)_n} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}. \tag{1.14}$$

The Sharma's and Parihar's hypergeometric functions of four variables $F_{27}^{(4)}$, $F_{30}^{(4)}$, $F_{47}^{(4)}$ and $F_{48}^{(4)}$ defined by [21]

$$F_{27}^{(4)}\left(a_1,a_1,a_2,a_2,a_3,a_3,a_4,a_4;c_1,c_2,c_1,c_3;x,y,z,u\right) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_{p+q}(a_3)_{m+n}(a_4)_{p+q}}{(c_1)_{m+p}(c_2)_n(c_3)_q} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!}, \tag{1.15}$$

$$F_{30}^{(4)}\left(a_1,a_1,a_2,a_2,a_3,a_3,a_4,a_5;c_1,c_2,c_1,c_3;x,y,z,u\right) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_{p+q}(a_3)_{m+n}(a_4)_p(a_5)_q}{(c_1)_{m+p}(c_2)_n(c_3)_q} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!}, \tag{1.16}$$

$$F_{47}^{(4)}\left(a_1,a_1,a_2,a_2,a_3,a_3,a_4,a_4;c_1,c_2,c_1,c_2;x,y,z,u\right) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_{p+q}(a_3)_{m+n}(a_4)_{p+q}}{(c_1)_{m+p}(c_2)_{n+q}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!}, \tag{1.17}$$

$$F_{48}^{(4)}\left(a_1,a_1,a_2,a_2,a_3,a_3,a_4,a_5;c_1,c_2,c_1,c_2;x,y,z,u\right) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_{p+q}(a_3)_{m+n}(a_4)_p(a_5)_q}{(c_1)_{m+p}(c_2)_{n+q}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!}. \tag{1.18}$$

Very recently, Bin-Saad and Younis [6] defined thirty functions of four variables denoted these by $X_1^{(4)}, X_2^{(4)}, \ldots, X_{30}^{(4)}$. Here, we give four of them as follows:

$$X_{19}^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_2;c_1,c_1,c_1,c_2;x,y,z,u\right) \tag{1.19}$$

$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n+p+q}(a_2)_{q+n}(a_3)_p}{(c_1)_{m+n+p}(c_2)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},$$

(1.20)
$$X_{21}^{(4)}(a_1,a_1,a_2,a_1,a_1,a_2,a_3,a_2;c_1,c_1,c_2,c_3;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n+q}(a_2)_{q+n+p}(a_3)_p}{(c_1)_{m+n}(c_2)_p(c_3)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},$$

(1.21)
$$X_{23}^{(4)}(a_1,a_1,a_2,a_1,a_1,a_2,a_3,a_2;c_1,c_1,c_1,c_2;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n+q}(a_2)_{q+n+p}(a_3)_p}{(c_1)_{m+n+p}(c_2)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},$$

(1.22)
$$X_{28}^{(4)}(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_4;c_1,c_1,c_2,c_3;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n+p+q}(a_2)_n(a_3)_p(a_4)_q}{(c_1)_{m+n}(c_2)_p(c_3)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}.$$

Here, we consider a further quadruple hypergeometric functions as follows:

(1.23)
$$X_{80}^{(4)}(a_1,a_1,a_2,a_2,a_1,a_3,a_4,a_5;c_1,c_2,c_1,c_2;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{p+q}(a_3)_n(a_4)_p(a_5)_q}{(c_1)_{m+p}(c_2)_{n+q}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},$$

(1.24)
$$X_{81}^{(4)}(a_1,a_1,a_2,a_2,a_1,a_3,a_4,a_5;c_1,c_2,c_1,c_1;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{p+q}(a_3)_n(a_4)_p(a_5)_q}{(c_1)_{m+p+q}(c_2)_n} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},$$

(1.25)
$$X_{82}^{(4)}(a_1,a_1,a_2,a_2,a_1,a_3,a_4,a_5;c_2,c_1,c_1,c_1;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{p+q}(a_3)_n(a_4)_p(a_5)_q}{(c_1)_{n+p+q}(c_2)_m} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},$$

(1.26)
$$X_{83}^{(4)}(a_1,a_1,a_2,a_2,a_1,a_3,a_4,a_5;c,c,c,c;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{p+q}(a_3)_n(a_4)_p(a_5)_q}{(c)_{m+n+p+q}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}.$$

The aim of this paper is to investigate certain properties of four new hypergeometric functions of four variables $X_{80}^{(4)}, X_{81}^{(4)}, X_{82}^{(4)}, X_{83}^{(4)}$. In Section 2, some fractional derivative formulas involving the functions $X_i^{(4)}$ $(i = 80, 81, 82, 83)$ are obtained. In Section 3, we consider several integral representations of Euler-type for our series involving triple and quadruple hypergeometric functions. Section 4 deals with the Laplace integrals for each new quadruple functions.

Section 5 gives some operational formulas with the help of certain inverse pairs of symbolic operators.

2. Fractional derivatives

Here, in this section, we consider some fractional derivative formulas for the quadruple series $X_{80}^{(4)}, X_{81}^{(4)}, X_{82}^{(4)}, X_{83}^{(4)}$. The fractional derivative operator D_w^k is defined by [18]

$$D_w^k w^a = \frac{\Gamma(a+1)}{\Gamma(a-k+1)} w^{a-k},\ (Re(a) > -1). \tag{2.1}$$

Theorem 2.1. *The following fractional derivative formulas holds*:

$$\begin{aligned}
&D_w^{a_2-c}\left[w^{a_2-1}X_{80}^{(4)}\left(a_1,a_1,c,c,a_1,a_3,a_4,a_5;c_1,c_2,c_1,c_2;\right.\right.\\
&\left.\left. x,y,wz,wu\right)\right]\\
&=\frac{\Gamma(a_2)}{\Gamma(c)}w^{c-1}X_{80}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_4,a_5;c_1,c_2,c_1,c_2;\right.\\
&\left. x,y,wz,wu\right),
\end{aligned} \tag{2.2}$$

$$\begin{aligned}
&D_{w_1}^{a_2-c}D_{w_2}^{a_3-c'}\left[w_1^{a_2-1}w_2^{a_3-1}X_{80}^{(4)}\left(a_1,a_1,c,c,a_1,c',a_4,a_5;\right.\right.\\
&\left.\left. c_1,c_2,c_1,c_2;x,w_2y,w_1z,w_1u\right)\right]\\
&=\frac{\Gamma(a_2)\Gamma(a_3)}{\Gamma(c)\Gamma(c')}w_1^{c-1}w_2^{c'-1}X_{80}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_4,a_5;\right.\\
&\left. c_1,c_2,c_1,c_2;x,w_2y,w_1z,w_1u\right),
\end{aligned} \tag{2.3}$$

$$\begin{aligned}
&D_w^{a_1-c}D_y^{a_3-c'}\left[w^{a_1-1}y^{a_3-1}X_{80}^{(4)}\left(c,c,a_2,a_2,c,c',a_4,a_5;\right.\right.\\
&\left.\left. c_1,c_2,c_1,c_2;w^2x,wy,z,u\right)\right]\\
&=\frac{\Gamma(a_1)\Gamma(a_3)}{\Gamma(c)\Gamma(c')}w^{c-1}y^{c'-1}X_{80}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_4,a_5;\right.\\
&\left. c_1,c_2,c_1,c_2;w^2x,wy,z,u\right),
\end{aligned} \tag{2.4}$$

$$\begin{aligned}
&D_y^{a_3-c}D_z^{a_4-c'}D_u^{a_5-c''}\left[y^{a_3-1}z^{a_4-1}u^{a_5-1}X_{80}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,\right.\right.\\
&\left.\left. c,c',c'';c_1,c_2,c_1,c_2;x,y,z,u\right)\right]\\
&=\frac{\Gamma(a_3)\Gamma(a_4)\Gamma(a_5)}{\Gamma(c)\Gamma(c')\Gamma(c'')}y^{c-1}z^{c'-1}u^{c''-1}X_{80}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_4,a_5;\right.\\
&\left. c_1,c_2,c_1,c_2;x,y,z,u\right).
\end{aligned} \tag{2.5}$$

Proof. To prove (2.2). Making an appeal to the formula (2.1), we obtain

$$D_w^{a_2-c}\left[w^{a_2-1}X_{80}^{(4)}\left(a_1,a_1,c,c,a_1,a_3,a_4,a_5;c_1,c_2,c_1,c_2;x,y,wz,wu\right)\right]$$

$$= w^{c-1}\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+n}(c)_{p+q}(a_3)_n(a_4)_p(a_5)_q\Gamma(a_2+p+q)}{(c_1)_{m+p}(c_2)_{n+q}\Gamma(c+p+q)}\frac{x^m}{m!}\frac{y^n}{n!}\frac{(wz)^p}{p!}\frac{(wu)^q}{q!}.$$

In view of (1.1) to the above equation, we get the required result. A similar argument will establish (2.3) to (2.5). The proofs of the following theorems would run parallel to the result (2.2). Therefore, we omit the details. □

Theorem 2.2. *The following fractional derivative formulas holds*:

$$\text{(2.6)}\quad D_{w_1}^{a_1-c}D_{w_2}^{a_2-c'}D_{w_3}^{a_3-c''}\Big[w_1^{a_1-1}w_2^{a_2-1}w_3^{a_3-1}X_{81}^{(4)}\Big(c,c,c',c',c,c'',a_4,a_5;$$
$$c_1,c_2,c_1,c_1;w_1^2x,w_1w_3y,w_2z,w_2u\Big)\Big]$$
$$=\frac{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}{\Gamma(c)\Gamma(c')\Gamma(c'')}w_1^{c-1}w_2^{c'-1}w_3^{c''-1}X_{81}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_4,a_5;\right.$$
$$\left.c_1,c_2,c_1,c_1;w_1^2x,w_1w_3y,w_2z,w_2u\right),$$

$$\text{(2.7)}\quad D_y^{a_1-c}D_z^{a_2-c'}\Big[y^{a_1-1}z^{a_2-1}X_{81}^{(4)}\Big(c,c,c',c',c,a_3,a_4,a_5;$$
$$c_1,c_2,c_1,c_1;x,y,z,u\Big)\Big]$$
$$=\frac{\Gamma(a_1)\Gamma(a_2)}{\Gamma(c)\Gamma(c')}y^{c-1}z^{c'-1}X_{81}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_4,a_5;\right.$$
$$\left.c_1,c_2,c_1,c_1;x,y,z,u\right),$$

$$\text{(2.8)}\quad D_x^{a_1-c}\Big[x^{a_1-1}X_{81}^{(4)}\left(c,c,a_2,a_2,c,a_3,a_4,a_5;c_1,c_2,c_1,c_1;x^2,xy,z,u\right)\Big]$$
$$=\frac{\Gamma(a_1)}{\Gamma(c)}x^{c-1}X_{81}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_4,a_5;c_1,c_2,c_1,c_1;x^2,xy,z,u\right),$$

$$\text{(2.9)}\quad D_u^{a_5-c}\Big[u^{a_5-1}X_{81}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_4,c;c_1,c_2,c_1,c_1;x,y,z,u\right)\Big]$$
$$=\frac{\Gamma(a_5)}{\Gamma(c)}u^{c-1}X_{81}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_4,a_5;c_1,c_2,c_1,c_1;x,y,z,u\right).$$

Corollary 2.1. *In formula (2.7), if we set $x=0=u$, we get the following result.*

$$\text{(2.10)}\quad D_y^{a_1-c}y^{a_1-1}{}_2F_1\left(c,a_3;c_2;y\right)D_z^{a_2-c'}z^{a_2-1}{}_2F_1\left(c',a_4;c_1;z\right)$$
$$=\frac{\Gamma(a_1)\Gamma(a_2)}{\Gamma(c)\Gamma(c')}y^{c-1}z^{c'-1}{}_2F_1\left(a_1,a_3;c_2;y\right){}_2F_1\left(a_2,a_4;c_1;z\right).$$

Theorem 2.3. *The following fractional derivative formulas holds*:

$$\text{(2.11)}\quad D_{w_1}^{a_2-c}D_{w_2}^{a_3-c'}D_{w_3}^{a_4-c''}D_{w_4}^{a_5-c'''}\Big[w_1^{a_2-1}w_2^{a_3-1}w_3^{a_4-1}w_4^{a_5-1}$$
$$\times X_{82}^{(4)}\Big(a_1,a_1,c,c,a_1,c',c'',c''';c_2,c_1,c_1,c_1;x,w_2y,w_1w_3z,w_1w_4u\Big)\Big]$$
$$=\frac{\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)\Gamma(a_5)}{\Gamma(c)\Gamma(c')\Gamma(c'')\Gamma(c''')}w_1^{c-1}w_2^{c'-1}w_3^{c''-1}w_4^{c'''-1}$$

$$\times X_{82}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_4,a_5;c_2,c_1,c_1,c_1;\right.$$
$$\left.w_1^2x,w_2y,w_1w_3z,w_1w_4u\right),$$

$$\begin{aligned}(2.12)\qquad & D_x^{a_1-c}D_{w_1}^{a_2-c'}D_{w_2}^{a_3-c''}\left[x^{a_1-1}w_1^{a_2-1}w_2^{a_3-1}\right.\\ & \left.\times X_{82}^{(4)}\left(c,c,c',c',c,c'',a_4,a_5;c_2,c_1,c_1,c_1;x^2,w_2xy,w_1z,w_1u\right)\right]\\ &=\frac{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}{\Gamma(c)\Gamma(c')\Gamma(c'')}w_1^{c-1}w_2^{c'-1}w_3^{c''-1}X_{82}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_4,\right.\\ &\left.a_5;c_2,c_1,c_1,c_1;x^2,w_2xy,w_1z,w_1u\right),\end{aligned}$$

$$\begin{aligned}(2.13)\qquad & D_w^{a_1-c}D_u^{a_2-c'}D_z^{a_4-c''}\left[w^{a_1-1}u^{a_2-1}z^{a_4-1}\right.\\ & \left.\times X_{82}^{(4)}\left(c,c,c',c',c,a_3,c'',a_5;c_2,c_1,c_1,c_1;w^2x,wy,zu,u\right)\right]\\ &=\frac{\Gamma(a_1)\Gamma(a_2)\Gamma(a_4)}{\Gamma(c)\Gamma(c')\Gamma(c'')}w^{c-1}u^{c'-1}z^{c''-1}X_{82}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_4,a_5;\right.\\ &\left.c_2,c_1,c_1,c_1;w^2x,wy,zu,u\right),\end{aligned}$$

$$\begin{aligned}(2.14)\qquad & D_x^{a_1-c}D_y^{a_3-c'}\left[x^{a_1-1}y^{a_3-1}\right.\\ & \left.\times X_{82}^{(4)}\left(c,c,a_2,a_2,c,c',a_4,a_5;c_2,c_1,c_1,c_1;x^2,xy,z,u\right)\right]\\ &=\frac{\Gamma(a_1)\Gamma(a_3)}{\Gamma(c)\Gamma(c')}x^{c-1}y^{c'-1}X_{82}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_4,a_5;c_2,c_1,c_1,c_1;\right.\\ &\left.x^2,xy,z,u\right).\end{aligned}$$

Theorem 2.4. *The following fractional derivative formulas holds*:

$$\begin{aligned}(2.15)\quad & D_{w_1}^{a_1-c'}D_{w_2}^{a_2-c''}D_{w_3}^{a_3-c'''}D_{w_4}^{a_4-c''''}\left[w_1^{a_1-1}w_2^{a_2-1}w_3^{a_3-1}w_4^{a_4-1}\right.\\ & \left.\times X_{83}^{(4)}\left(c',c',c'',c'',c',c''',c'''',a_5;c,c,c,c;w_1^2x,w_1w_3y,w_2w_4z,w_2u\right)\right]\\ &=\frac{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)}{\Gamma(c')\Gamma(c'')\Gamma(c''')\Gamma(c'''')}w_1^{c'-1}w_2^{c''-1}w_3^{c'''-1}w_4^{c''''-1}\\ &\times X_{83}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_4,a_5;c,c,c,c;w_1^2x,w_1w_3y,w_2w_4z,w_2u\right),\end{aligned}$$

$$\begin{aligned}(2.16)\qquad & D_w^{a_5-c'}\left[w^{a_5-1}X_{83}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_4,c';c,c,c,c;x,y,z,wu\right)\right]\\ &=\frac{\Gamma(a_5)}{\Gamma(c')}w^{c'-1}X_{83}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_4,a_5;c,c,c,c;x,y,z,wu\right),\end{aligned}$$

$$\begin{aligned}(2.17)\qquad & D_{w_1}^{a_1-c'}D_z^{a_2-c''}D_{w_2}^{a_3-c'''}D_u^{a_5-c''''}\left[w_1^{a_1-1}z^{a_2-1}w_2^{a_3-1}u^{a_5-1}\right.\\ & \left.\times X_{83}^{(4)}\left(c',c',c'',c'',c',c''',a_4,c'''';c,c,c,c;w_1^2x,w_1w_2y,z,uz\right)\right]\end{aligned}$$

$$= \frac{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_5)}{\Gamma(c')\Gamma(c'')\Gamma(c''')\Gamma(c'''')} w_1^{c'-1} z^{c''-1} w_2^{c'''-1} u^{c''''-1}$$
$$\times X_{83}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_3, a_4, a_5; c, c, c, c; w_1^2 x, w_1 w_2 y, z, uz\right),$$

$$D_x^{a_1-c'} D_y^{a_3-c''} D_z^{a_4-c'''} D_u^{a_5-c''''}\left[x^{a_1-1} y^{a_3-1} z^{a_4-1} u^{a_5-1}\right.$$
$$\left.\times X_{83}^{(4)}\left(c', c', a_2, a_2, c', c'', c''', c''''; c, c, c, c; x^2, xy, z, u\right)\right]$$
$$= \frac{\Gamma(a_1)\Gamma(a_3)\Gamma(a_4)\Gamma(a_5)}{\Gamma(c')\Gamma(c'')\Gamma(c''')\Gamma(c'''')} x^{c'-1} y^{c''-1} z^{c'''-1} u^{c''''-1}$$
$$\times X_{83}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_3, a_4, a_5; c, c, c, c; x^2, xy, z, u\right). \tag{2.18}$$

3. Integrals of Euler-type

In this section, we establish several Euler-type integrals involving the quadruple functions $X_{80}^{(4)}, X_{81}^{(4)}, X_{82}^{(4)}, X_{83}^{(4)}$ asserted in the following theorems.

Theorem 3.1. *Each of the following integral representations holds*:

$$X_{80}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_3, a_4, a_5; c_1, c_2, c_1, c_2; x, y, z, u\right) \tag{3.1}$$
$$= \frac{8M_1^{a_1} M_2^{a_4} M_3^{a}\Gamma(a_1+a_3)\Gamma(a_4+a_5)\Gamma(c_2)}{\Gamma(a_1)\Gamma(a_3)\Gamma(a_4)\Gamma(a_5)\Gamma(a)\Gamma(c_2-a)}$$
$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{\cosh\alpha\left(\sinh^2\alpha\right)^{a_1-\frac{1}{2}}}{\left(1+M_1\sinh^2\alpha\right)^{a_1+a_3}} \frac{\cosh\beta\left(\sinh^2\beta\right)^{a_4-\frac{1}{2}}}{\left(1+M_2\sinh^2\beta\right)^{a_4+a_5}}$$
$$\times \frac{\cosh\gamma\left(\sinh^2\gamma\right)^{a-\frac{1}{2}}}{\left(1+M_3\sinh^2\gamma\right)^{c_2}} F_{27}^{(4)}\left(\tfrac{a_1+a_3}{2}, \tfrac{a_1+a_3}{2}, a_2, a_2,\right.$$
$$\tfrac{a_1+a_3+1}{2}, \tfrac{a_1+a_3+1}{2}, a_4+a_5, a_4+a_5; c_1, a, c_1, c_2-a;$$
$$\frac{4M_1^2 x\sinh^4\alpha}{\left(1+M_1\sinh^2\alpha\right)^2}\ \frac{4M_1M_3 y\sinh^2\alpha\sinh^2\gamma}{\left(1+M_1\sinh^2\alpha\right)^2\left(1+M_3\sinh^2\gamma\right)},$$
$$\left.\frac{M_2 z\sinh^2\beta}{\left(1+M_2\sinh^2\beta\right)}, \frac{u}{\left(1+M_2\sinh^2\beta\right)\left(1+M_3\sinh^2\gamma\right)}\right) d\alpha d\beta d\gamma,$$

$(Re(a_i) > 0, (i = 1, 3, 4, 5), Re(a) > 0, Re(c_2 - a) > 0, M_j > 0, j = (1, 2, 3)),$

$$X_{80}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_3, a_4, a_5; c_1, c_2, c_1, c_2; x, y, z, u\right) \tag{3.2}$$
$$= \frac{\Gamma(a_1+a_3)\Gamma(c_2)}{\Gamma(a_1)\Gamma(a_3)\Gamma(a)\Gamma(c_2-a)}$$
$$\int_0^\infty\int_0^\infty \left(e^{-\alpha}\right)^a\left(1-e^{-\alpha}\right)^{c_2-a-1}\left(e^{-\beta}\right)^{a_1}\left(1-e^{-\beta}\right)^{a_3-1}$$
$$\times F_{30}^{(4)}\left(\tfrac{a_1+a_3}{2}, \tfrac{a_1+a_3}{2}, a_2, a_2, \tfrac{a_1+a_3+1}{2}, \tfrac{a_1+a_3+1}{2}, a_4, a_5; c_1, a, c_1, c_2-a;\right.$$

$$4xe^{-2\beta}, 4ye^{-(\alpha+\beta)}\left(1-e^{-\beta}\right), z, u\left(1-e^{-\alpha}\right)\Big) d\alpha d\beta,$$
$$\left(Re(a_1)>0, Re(a_3)>0, Re(a)>0, Re(c_2-a)>0\right),$$

$$\begin{aligned}(3.3)\quad & X_{80}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_4,a_5;c_1,c_2,c_1,c_2;x,y,z,u\right)\\ &= \frac{\Gamma(a_1+a_3)\Gamma(a_4+a_5)\Gamma(c_2)}{\Gamma(a_1)\Gamma(a_3)\Gamma(a_4)\Gamma(a_5)}\int_0^\infty\int_0^\infty \frac{\alpha^{a_3}}{(1+\alpha)^{a_1+a_3}}\frac{\beta^{a_5}}{(1+\beta)^{a_4+a_5}}\\ &\times F_{47}^{(4)}\Big(\tfrac{a_1+a_3}{2},\tfrac{a_1+a_3}{2},a_2,a_2,\tfrac{a_1+a_3+1}{2},\tfrac{a_1+a_3+1}{2},\\ &a_4+a_5,a_4+a_5;c_1,c_2,c_1,c_2;\tfrac{4x}{(1+\alpha)^2},\tfrac{4\alpha y}{(1+\alpha)^2},\tfrac{z}{(1+\beta)},\tfrac{\beta u}{(1+\beta)}\Big)d\alpha d\beta,\\ &\left(Re(a_i)>0,(i=1,3,4,5)\right),\end{aligned}$$

$$\begin{aligned}(3.4)\quad & X_{80}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_4,a_5;c_1,c_2,c_1,c_2;x,y,z,u\right)\\ &= \frac{\Gamma(a_1+a_3)\left(1+M\right)^{a_3}}{\Gamma(a_1)\Gamma(a_3)}\int_0^1 \frac{\alpha^{a_3-1}\left(1-\alpha\right)^{a_1-1}}{(1+M\alpha)^{a_1+a_3}}\\ &\times F_{48}^{(4)}\Big(\tfrac{a_1+a_3}{2},\tfrac{a_1+a_3}{2},a_2,a_2,\tfrac{a_1+a_3+1}{2},\tfrac{a_1+a_3+1}{2},a_4,a_5;c_1,c_2,c_1,c_2;\ ,\\ &\tfrac{4(1-\alpha)^2x}{(1+M\alpha)^2},\tfrac{4(1+M)\alpha(1-\alpha)y}{(1+M\alpha)^2},z,u\Big)d\alpha,\\ &\left(Re(a_1)>0,Re(a_3)>0,M>-1\right).\end{aligned}$$

Corollary 3.1. *By considering equation* (3.1) *we obtain, when* $x=0$.

$$\begin{aligned}(3.5)\quad & F_N\left(a_4,a_1,a_5,a_2,a_3,a_2;c_1,c_2,c_2;z,y,u\right)\\ &= \frac{8M_1^{a_1}M_2^{a_4}M_3^{a}\Gamma(a_1+a_3)\Gamma(a_4+a_5)\Gamma(c_2)}{\Gamma(a_1)\Gamma(a_3)\Gamma(a_4)\Gamma(a_5)\Gamma(a)\Gamma(c_2-a)}\\ &\int_0^\infty\int_0^\infty\int_0^\infty \frac{\cosh\alpha\left(\sinh^2\alpha\right)^{a_1-\frac12}}{\left(1+M_1\sinh^2\alpha\right)^{a_1+a_3}}\\ &\times\frac{\cosh\beta\left(\sinh^2\beta\right)^{a_4-\frac12}}{\left(1+M_2\sinh^2\beta\right)^{a_4+a_5}}\frac{\cosh\gamma\left(\sinh^2\gamma\right)^{a-\frac12}}{\left(1+M_3\sinh^2\gamma\right)^{c_2}}\\ &\times {}_2F_1\left(\tfrac{a_1+a_3}{2},\tfrac{a_1+a_3+1}{2};a;\tfrac{4M_1M_3y\sinh^2\alpha\sinh^2\gamma}{(1+M_1\sinh^2\alpha)^2(1+M_3\sinh^2\gamma)}\right)\\ &\times F_4\left(a_2,a_4+a_5;c_1,c_2-a;\tfrac{M_2z\sinh^2\beta}{(1+M_2\sinh^2\beta)},\tfrac{u}{(1+M_2\sinh^2\beta)(1+M_3\sinh^2\gamma)}\right)\\ &\times d\alpha d\beta d\gamma,\\ &\left(Re(a_i)>0,(i=1,3,4,5),Re(a)>0,Re(c_2-a)>0,M_j>0,j=(1,2,3)\right).\end{aligned}$$

Theorem 3.2. *Each of the following integral representations holds true*:

$$\begin{aligned}(3.6)\quad & X_{81}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_4,a_5;c_1,c_2,c_1,c_1;x,y,z,u\right)\\ &= \frac{\Gamma(a_1+a_2)\Gamma(a_3+a_5)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_5)}\int_0^1\int_0^1 \alpha^{a_1-1}\beta^{a_3-1}\left(1-\alpha\right)^{a_2-1}\left(1-\beta\right)^{a_5-1}\end{aligned}$$

$$\times X_{19}^{(4)}\left(a_1+a_2, a_1+a_2, a_1+a_2, a_1+a_2, a_1+a_2, a_3+a_5, a_4, a_3+a_5;\right.$$
$$\left.c_1, c_1, c_1, c_2; \alpha^2 x, (1-\alpha)(1-\beta)u, (1-\alpha)z, \alpha\beta y\right) d\alpha d\beta,$$
$$\left(Re(a_i)>0, (i=1,2,3,5)\right),$$

(3.7)
$$X_{81}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_3, a_4, a_5; c_1, c_2, c_1, c_1; x, y, z, u\right)$$
$$= \frac{\Gamma(a_1+a_5)\Gamma(a_2+a_3)\Gamma(c_1)}{2^{a_1+a_2+a_3+a_5+c_1-6}\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_5)\Gamma(a)\Gamma(c_1-a)}$$
$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \frac{\left[(1+\alpha)^2\right]^{a_1-\frac{1}{2}}\left[(1-\alpha)^2\right]^{a_5-\frac{1}{2}}}{(1+\alpha^2)^{a_1+a_5}} \frac{\left[(1+\beta)^2\right]^{a_2-\frac{1}{2}}\left[(1-\beta)^2\right]^{a_3-\frac{1}{2}}}{(1+\beta^2)^{a_2+a_3}}$$
$$\times \frac{\left[(1+\gamma)^2\right]^{a-\frac{1}{2}}\left[(1-\gamma)^2\right]^{c_1-a-\frac{1}{2}}}{(1+\gamma^2)^{c_1}} X_{21}^{(4)}\left(a_1+a_5, a_1+a_5, a_2+a_3,\right.$$
$$a_1+a_5, a_1+a_5, a_2+a_3, a_4, a_2+a_3; a, a, c_1-a, c_2; \tfrac{(1+\alpha)^4(1+\gamma)^2 x}{8(1+\alpha^2)(1+\gamma^2)},$$
$$\left.\tfrac{(1-\alpha)^2(1+\beta)^2(1+\gamma)^2 u}{8(1+\alpha^2)(1+\beta^2)(1+\gamma^2)}, \tfrac{(1+\beta)^2(1-\gamma)^2 z}{4(1+\beta^2)(1+\gamma^2)}, \tfrac{(1+\alpha)^2(1-\beta)^2 y}{4(1+\alpha^2)(1+\beta^2)}\right) d\alpha d\beta d\gamma,$$
$$\left(Re(a_i)>0, (i=1,2,3,5), Re(a)>0, Re(c_1-a)>0,\right),$$

(3.8)
$$X_{81}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_3, a_4, a_5; c_1, c_2, c_1, c_1; x, y, z, u\right)$$
$$= \frac{\Gamma(a_1+a_5)\Gamma(a_2+a_3)(1+M_1)^{a_1}(1+M_2)^{a_2}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_5)} \int_0^1\int_0^1 \frac{\alpha^{a_1-1}(1-\alpha)^{a_5-1}}{(1+M_1\alpha)^{a_1+a_5}}$$
$$\times \frac{\beta^{a_2-1}(1-\beta)^{a_3-1}}{(1+M_2\beta)^{a_2+a_3}} X_{23}^{(4)}\left(a_1+a_5, a_1+a_5, a_2+a_3, a_1+a_5, a_1+a_5,\right.$$
$$a_2+a_3, a_4, a_2+a_3; c_1, c_1, c_1, c_2; \tfrac{(1+M_1)^2\alpha^2 x}{(1+M_1\alpha)^2}, \tfrac{(1+M_2)(1-\alpha)\beta u}{(1+M_1\alpha)(1+M_2\beta)},$$
$$\left.\tfrac{(1+M_2)\beta z}{(1+M_2\beta)}, \tfrac{(1+M_1)\alpha(1-\beta)y}{(1+M_1\alpha)(1+M_2\beta)}\right) d\alpha d\beta,$$
$$\left(Re(a_i)>0, (i=1,2,3,5), M_1>-1, M_2>-1\right),$$

(3.9)
$$X_{81}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_3, a_4, a_5; c_1, c_2, c_1, c_1; x, y, z, u\right)$$
$$= \frac{\Gamma(a_1+a_2)\Gamma(c_1)}{2^{a_1+a_2+c_1-2}\Gamma(a_1)\Gamma(a_2)\Gamma(a)\Gamma(c_1-a)} \int_{-1}^{1}\int_{-1}^{1} (1+\alpha)^{a_1-1}(1-\alpha)^{a_2-1}$$
$$\times (1+\beta)^{a-1}(1-\beta)^{c_1-a-1} X_{28}^{(4)}\left(a_1+a_2, a_1+a_2, a_1+a_2, a_1+a_2,\right.$$
$$a_1+a_2, a_4, a_3, a_5; c_1-a, c_1-a, c_2, a; \tfrac{(1+\alpha)^2(1-\beta)x}{8}, \tfrac{(1-\alpha)(1-\beta)z}{4},$$
$$\left.\tfrac{(1+\alpha)y}{2}, \tfrac{(1-\alpha)(1+\beta)u}{4}\right) d\alpha d\beta,$$
$$\left(Re(a_1)>0, Re(a_2)>0, Re(a)>0, Re(c_1-a)>0\right),$$

Theorem 3.3. *Each of the following integral representations holds*:

$$
\begin{aligned}
(3.10)\qquad & X_{82}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_4,a_5;c_2,c_1,c_1,c_1;x,y,z,u\right)\\
&=\frac{\Gamma(a_2+a_3)\Gamma(c_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(c_2-a_1)}\\
&\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\frac{1}{2}+\alpha\right)^{a_1-1}\left(\frac{1}{2}+\beta\right)^{a_2-1}\left(\frac{1}{2}-\beta\right)^{a_3-1}\\
&\times\left[\left(\frac{1}{2}-\alpha\right)+\left(\frac{1}{2}+\alpha\right)^2x\right]^{c_2-a_1-1}F_D^{(3)}\Big(a_2+a_3,1+a_1-c_2,\\
&a_4,a_5;c_1;-\frac{\left(\frac{1}{2}+\alpha\right)\left(\frac{1}{2}-\beta\right)y}{\left[\left(\frac{1}{2}-\alpha\right)+\left(\frac{1}{2}+\alpha\right)^2x\right]},\left(\tfrac{1}{2}+\beta\right)z,\left(\tfrac{1}{2}+\beta\right)u\Big)\,d\alpha d\beta\\
&\left(Re(a_i)>0,(i=1,2,),Re(c_2-a_1)>0\right),
\end{aligned}
$$

$$
\begin{aligned}
(3.11)\qquad & X_{82}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_4,a_5;c_2,c_1,c_1,c_1;x,y,z,u\right)\\
&=\frac{4\Gamma(a_1+a_3)\Gamma(a_4+a_5)}{\Gamma(a_1)\Gamma(a_3)\Gamma(a_4)\Gamma(a_5)}\\
&\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\left(\sin^2\alpha\right)^{a_1-\frac{1}{2}}\left(\cos^2\alpha\right)^{a_3-\frac{1}{2}}\left(\sin^2\beta\right)^{a_4-\frac{1}{2}}\\
&\times\left(\cos^2\beta\right)^{a_5-\frac{1}{2}}F_R\left(\tfrac{a_1+a_3}{2},a_2,\tfrac{a_1+a_3}{2},\tfrac{a_1+a_3+1}{2},a_4+a_5,\tfrac{a_1+a_3+1}{2};\right.\\
&\left.c_2,c_1,c_1;4x\sin^4\alpha,z\sin^2\beta+u\cos^2\beta,y\sin^2 2\alpha\right)d\alpha d\beta,\\
&\left(Re(a_i)>0,(i=1,3,4,5)\right),
\end{aligned}
$$

$$
\begin{aligned}
(3.12)\qquad & X_{82}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_4,a_5;c_2,c_1,c_1,c_1;x,y,z,u\right)\\
&=\frac{2\Gamma(c_2)\left(1+M\right)}{\Gamma(a_1)\Gamma(c_2-a_1)}\int_0^{\frac{\pi}{2}}\frac{\left(\sin^2\alpha\right)^{\frac{1}{2}}\left(\cos^2\alpha\right)^{a_1-\frac{1}{2}}}{\left(1+M\sin^2\alpha\right)^{2c_2-a_1-1}}\\
&\times\left[(1+M)\sin^2\alpha\left(1+M\sin^2\alpha\right)+x\cos^4\alpha\right]^{c_2-a_1-1}F_S\left(1+a_1-c_2,\right.\\
&\left.a_2,a_2,a_3,a_4,a_5;c_1,c_1,c_1;\tfrac{-\cos^2\alpha\left(1+M\sin^2\alpha\right)y}{\left[(1+M)\sin^2\alpha(1+M\sin^2\alpha)+x\cos^4\alpha\right]},z,u\right)d\alpha,\\
&\left(Re(a_1)>0,Re(c_2-a_1)>0,M>-1\right),
\end{aligned}
$$

$$
\begin{aligned}
(3.13)\qquad & X_{82}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_4,a_5;c_2,c_1,c_1,c_1;x,y,z,u\right)\\
&=\frac{4\Gamma(a_3+a_4)\Gamma(c_2)M_1^{a_3}M_2^{a_1}}{\Gamma(a_1)\Gamma(a_3)\Gamma(a_4)\Gamma(c_2-a_1)}
\end{aligned}
$$

$$\times \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\left(\sin^2\alpha\right)^{a_3-\frac{1}{2}} \left(\cos^2\alpha\right)^{a_4-\frac{1}{2}}}{\left(\cos^2\alpha + M_1\sin^2\alpha\right)^{a_3+a_4}} \frac{\left(\sin^2\beta\right)^{a_1-\frac{1}{2}} \left(\cos^2\beta\right)^{c_2-a_1-\frac{1}{2}}}{\left(\cos^2\beta + M_2\sin^2\beta\right)^{2c_2-a_1-1}}$$
$$\times \left[\left(\cos^2\beta + M_2\sin^2\beta\right) + M_2^2 x \sin^2\beta\tan^2\beta\right]^{c_2-a_1-1}$$
$$F_T\left(1+a_1-c_1, a_2, a_2, a_3+a_4, a_5, a_3+a_4; c_1, c_1, c_1;\right.$$
$$\frac{-M_1M_2\sin^2\alpha\tan^2\beta\left(\cos^2\beta+M_2\sin^2\beta\right)y}{(\cos^2\alpha+M_1\sin^2\alpha)\left[(\cos^2\beta+M_2\sin^2\beta)+M_2^2x\sin^2\beta\tan^2\beta\right]},$$
$$\left. u, \frac{z\cos^2\alpha}{(\cos^2\alpha+M_1\sin^2\alpha)}\right) d\alpha d\beta,$$
$$(Re(a_i) > 0, (i = 1, 3, 4), Re(c_2 - a_1) > 0, M_1 > 0, M_2 > 0).$$

Corollary 3.2. *By considering equation* (3.13) *we obtain, when* $u = 0$.

$$X_{19}\left(a_1, a_3, a_2, a_4; c_2, c_1; x, y, z\right) \tag{3.14}$$
$$= \frac{4\Gamma(a_3+a_4)\Gamma(c_2)M_1^{a_3}M_2^{a_1}}{\Gamma(a_1)\Gamma(a_3)\Gamma(a_4)\Gamma(c_2-a_1)}$$
$$\times \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\left(\sin^2\alpha\right)^{a_3-\frac{1}{2}} \left(\cos^2\alpha\right)^{a_4-\frac{1}{2}}}{\left(\cos^2\alpha + M_1\sin^2\alpha\right)^{a_3+a_4}} \frac{\left(\sin^2\beta\right)^{a_1-\frac{1}{2}} \left(\cos^2\beta\right)^{c_2-a_1-\frac{1}{2}}}{\left(\cos^2\beta + M_2\sin^2\beta\right)^{2c_2-a_1-1}}$$
$$\times \left[\left(\cos^2\beta + M_2\sin^2\beta\right) + M_2^2 x \sin^2\beta\tan^2\beta\right]^{c_2-a_1-1}$$
$$F_1\left(a_3+a_4, 1+a_1-c_2, a_2; c_1;\right.$$
$$\frac{-M_1M_2\sin^2\alpha\tan^2\beta\left(\cos^2\beta+M_2\sin^2\beta\right)y}{(\cos^2\alpha+M_1\sin^2\alpha)\left[(\cos^2\beta+M_2\sin^2\beta)+M_2^2x\sin^2\beta\tan^2\beta\right]},$$
$$\left. \frac{z\cos^2\alpha}{(\cos^2\alpha+M_1\sin^2\alpha)}\right) d\alpha d\beta,$$
$$(Re(a_i) > 0, (i = 1, 3, 4), Re(c_2 - a_1) > 0, M_1 > 0, M_2 > 0).$$

Theorem 3.4. *Each of the following integral representations holds true*:

$$X_{83}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_3, a_4, a_5; c, c, c, c; x, y, z, u\right) \tag{3.15}$$
$$= \frac{\Gamma(a_1+a_2)\Gamma(a_4+a_5)\Gamma(c)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_4)\Gamma(a_5)\Gamma(a)\Gamma(c-a)\left(S_1-R_1\right)^{a_1+a_2-1}\left(S_2-R_2\right)^{a_4+a_5-1}\left(S_3-R_3\right)^{c-1}}$$
$$\times \int_{R_1}^{S_1}\int_{R_2}^{S_2}\int_{R_3}^{S_3} (\alpha-R_1)^{a_1-1}(S_1-\alpha)^{a_2-1}(\beta-R_2)^{a_4-1}(S_2-\beta)^{a_5-1}(\gamma-R_3)^{a-1}$$
$$\times (S_3-\gamma)^{c-a-1} X_6\left(a_1+a_2, a_3, a_4+a_5; c-a, a; \frac{(\alpha-R_1)^2(S_3-\gamma)x}{(S_1-R_1)^2(S_3-R_3)},\right.$$
$$\left.\frac{(\alpha-R_1)(S_3-\gamma)y}{(S_1-R_1)(S_3-R_3)}, \frac{(S_1-\alpha)(\beta-R_2)(\gamma-R_3)z+(S_1-\alpha)(S_2-\beta)(\gamma-R_3)u}{(S_1-R_1)(S_2-R_2)(S_3-R_3)}\right) d\alpha d\beta d\gamma,$$
$$(Re(a_i) > 0, (i = 1, 2, 4, 5), Re(a) > 0, Re(c-a) > 0, R_j < S_j, (j = 1, 2, 3)),$$

$$X_{83}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_3, a_4, a_5; c, c, c, c; x, y, z, u\right) \tag{3.16}$$
$$= \frac{\Gamma(a_3+a_4+a_5)\left(S_1-T_1\right)^{a_5}\left(R_1-T_1\right)^{a_3+a_4}\left(S_2-T_2\right)^{a_3}\left(R_2-T_2\right)^{a_4}}{\Gamma(a_3)\Gamma(a_4)\Gamma(a_5)\left(S_1-R_1\right)^{a_3+a_4+a_5-1}\left(S_2-R_2\right)^{a_3+a_4-1}}$$

$$\times \int_{R_1}^{S_1}\int_{R_2}^{S_2} \frac{(\alpha-R_1)^{a_5-1}(S_1-\alpha)^{a_3+a_4-1}}{(\alpha-T_1)^{a_3+a_4+a_5}} \frac{(\beta-R_2)^{a_3-1}(S_2-\beta)^{a_4-1}}{(\beta-T_2)^{a_3+a_4}}$$
$$\times X_{13}\Big(a_1, a_3+a_4+a_5, a_2; c; x, \tfrac{(R_1-T_1)(S_2-T_2)(S_1-\alpha)(\beta-R_2)y}{(S_1-R_1)(S_2-R_2)(\alpha-T_1)(\beta-T_2)},$$
$$\tfrac{(R_1-T_1)(R_2-T_2)(S_1-\alpha)(S_2-\beta)z+(S_1-T_1)(S_2-R_2)(\alpha-R_1)(\beta-T_2)u}{(S_1-R_1)(S_2-R_2)(\alpha-T_1)(\beta-T_2)}\Big)\, d\alpha d\beta,$$
$$(Re(a_i)>0, (i=3,4,5), T_1<R_1<S_1, T_2<R_2<S_2),$$

(3.17)
$$X_{83}^{(4)}(a_1,a_1,a_2,a_2,a_1,a_3,a_4,a_5;c,c,c,c;x,y,z,u)$$
$$= \frac{2M^{a_4}\Gamma(a_4+a_5)}{\Gamma(a_4)\Gamma(a_5)} \int_0^\infty \frac{\cosh\alpha\left(\sinh^2\alpha\right)^{a_4-\frac{1}{2}}}{\left(1+M\sinh^2\alpha\right)^{a_4+a_5}}$$
$$\times X_{18}\left(a_1,a_3,a_2,a_4+a_5;c;x,y, \tfrac{Mz\sinh^2\alpha+u}{(1+M\sinh^2\alpha)}\right) d\alpha,$$
$$(Re(a_4)>0, Re(a_5)>0, M>0),$$

(3.18)
$$X_{83}^{(4)}(a_1,a_1,a_2,a_2,a_1,a_3,a_4,a_5;c,c,c,c;x,y,z,u)$$
$$= \frac{\Gamma(a_4+a_5)\Gamma(c)}{\Gamma(a_4)\Gamma(a_5)\Gamma(a)\Gamma(c-a)} \int_0^\infty\int_0^\infty \frac{\alpha^{a-1}}{(1+\alpha)^c}\frac{\beta^{a_5-1}}{(1+\beta)^{a_4+a_5}}$$
$$\times X_{20}(a_1,a_3,a_2,a_4+a_5;$$
$$a, c-a; \tfrac{\alpha x}{(1+\alpha)}, \tfrac{y}{(1+\alpha)}, \tfrac{\alpha z+\alpha\beta u}{(1+\alpha)(1+\beta)}\Big)\, d\alpha d\beta,$$
$$(\Re(a_4)>0, \Re(a_5)>0, \Re(a)>0, \Re(c-a)>0).$$

Proof. It is noted that each of the integral representations of Euler-type in (3.1)-(3.18) can be proved mainly by expressing the series definition of the involved special function in each integrand and changing the order of the integral sign and the summation, and finally using the following integral representations of the Beta function (see [12, 20]):

$$B(a,b) = \begin{cases} \displaystyle\int_0^1 \alpha^{a-1}(1-\alpha)^{b-1}\, dt & (\Re(a)>0, \Re(b)>0), \\ \dfrac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} & (a,b\in\mathbb{C}\setminus\mathbb{Z}_0^-), \end{cases}$$
$$B(a,b) = \int_0^1 \alpha^{a-1}(1-\alpha)^{b-1}\, d\alpha$$
$$= (S-R)^{1-a-b}\int_R^S (\alpha-R)^{a-1}(S-\alpha)^{b-1}\, d\alpha$$
$$(Re(a)>0, Re(b)>0, R<S),$$
$$B(a,b) = \frac{(S-T)^a(R-T)^b}{(S-R)^{a+b-1}}\int_R^S \frac{(\alpha-R)^{a-1}(S-\alpha)^{b-1}}{(\alpha-T)^{a+b}}\, d\alpha$$

$$= (M+1)^a \int_0^1 \frac{\alpha^{a-1}(1-\alpha)^{b-1}}{(1+M\alpha)} \, d\alpha$$
$$(T < R < S, M > -1, Re(a) > 0, Re(b) > 0)\,,$$
$$B(a,b) = 2\int_0^{\frac{\pi}{2}} (\sin\alpha)^{2a-1}(\cos\alpha)^{2b-1} \, d\alpha = \int_0^\infty \frac{\alpha^{a-1}}{(1+\alpha)^{a+b}} \, d\alpha$$
$$(Re(a) > 0, Re(b) > 0)\,,$$
$$B(a,b) = 2^{1-a-b} \int_{-1}^1 (1+\alpha)^{a-1}(1-\alpha)^{b-1} \, d\alpha$$
$$= 2M^a \int_0^\infty \frac{\cosh\alpha\,(\sinh\alpha)^{2a-1}}{\left(1+M\sinh^2\alpha\right)^{a+b}} \, d\alpha$$
$$(Re(a) > 0, Re(b) > 0, M > 0)\,.$$ □

4. Laplace-type integrals

The confluent hypergeometric functions ${}_0F_1, {}_1F_1, \Phi_2, \Phi_3, \Phi_2^{(3)}$ and $\Phi_3^{(3)}$ are defined, respectively, by (see [22])

$$ {}_0F_1\left(-;c;x\right) = \sum_{m=0}^{\infty} \frac{1}{(c)_m} \frac{x^m}{m!}, \tag{4.1}$$

$$ {}_1F_1\left(a;c;x\right) = \sum_{m=0}^{\infty} \frac{(a)_m}{(c)_m} \frac{x^m}{m!}, \tag{4.2}$$

$$ \Phi_2\left(a,b;c;x,y\right) = \sum_{m,n=0}^{\infty} \frac{(a)_m(b)_n}{(c)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}, \tag{4.3}$$

$$ \Phi_3\left(a;c;x,y\right) = \sum_{m,n=0}^{\infty} \frac{(a)_m}{(c)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}, \tag{4.4}$$

$$ \Phi_2^{(3)}\left(a,b,c;d;x,y,z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a)_m(b)_n(c)_p}{(d)_{m+n+p}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \tag{4.5}$$

and

$$ \Phi_3^{(3)}\left(a,b;c;x,y,z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a)_m(b)_n}{(c)_{m+n+p}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}. \tag{4.6}$$

Theorem 4.1. *For $X_{80}^{(4)}, X_{81}^{(4)}, X_{82}^{(4)}, X_{83}^{(4)}$, we have the following integral representations*:

$$ X_{80}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_4,a_5;c_1,c_2,c_1,c_2;x,y,z,u\right) \tag{4.7}$$

$$= \frac{1}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty \int_0^\infty e^{-(s+t)} s^{a_1-1} t^{a_2-1} \times \Phi_3\left(a_4; c_1; tz, s^2x\right) \Phi_2\left(a_3, a_5; c_2; sy, tu\right) dsdt, \quad \left(Re(a_1) > 0, Re(a_2) > 0\right),$$

(4.8)
$$X_{81}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_3, a_4, a_5; c_1, c_2, c_1, c_1; x, y, z, u\right) = \frac{1}{\Gamma(a_1)\Gamma(a_4)\Gamma(a_5)} \int_0^\infty \int_0^\infty \int_0^\infty e^{-(s+t+v)} s^{a_1-1} t^{a_4-1} v^{a_5-1} \times \Phi_3\left(a_2; c_1; tz+uv, s^2x\right) {}_1F_1\left(a_3; c_2; sy\right) dsdtdv, \quad \left(Re(a_1) > 0, Re(a_4) > 0, Re(a_5) > 0\right),$$

(4.9)
$$X_{82}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_3, a_4, a_5; c_2, c_1, c_1, c_1; x, y, z, u\right) = \frac{1}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty \int_0^\infty e^{-(s+t)} s^{a_1-1} t^{a_2-1} \times {}_0F_1\left(-; c_2; s^2x\right) \Phi_2^{(3)}\left(a_3, a_4, a_5; c_1; sy, tz, tu\right) dsdt, \quad \left(Re(a_1) > 0, Re(a_2) > 0\right),$$

(4.10)
$$X_{83}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_3, a_4, a_5; c, c, c, c; x, y, z, u\right) = \frac{1}{\Gamma(a_1)\Gamma(a_4)\Gamma(a_5)} \int_0^\infty \int_0^\infty \int_0^\infty e^{-(s+t+v)} s^{a_1-1} t^{a_4-1} v^{a_5-1} \times \Phi_3^{(3)}\left(a_2, a_3; c; tz+uv, sy, s^2x\right) dsdtdv, \quad \left(Re(a_1) > 0, Re(a_4) > 0, Re(a_5) > 0\right).$$

Proof. To prove the equality (4.7), denote by Δ the right side of the equality (4.7). Then, by substituting the expression of the confluent hypergeometric functions (4.3) and (4.4) into the right hand side of (4.7), we obtain

$$\Delta = \sum_{m,n,p,q=0}^{\infty} \frac{(a_3)_n (a_4)_p (a_5)_q}{(c_1)_{m+p} (c_2)_{n+q} \Gamma(a_1)\Gamma(a_2)} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!} \int_0^\infty \int_0^\infty e^{-s} s^{a_1+2m+n-1} \times e^{-t} t^{a_2+p+q-1} dsdt.$$

Using the known result (see [12])

$$\Gamma(a) = \int_0^\infty e^{-s}\ s^{a-1}\ ds, (\Re(a) > 0),$$

after a little simplification, we easily arrive at the left-hand side of (4.7). Then, we easily can obtain the equalities (4.8) to (4.10) in similar way. □

5. Operational formulas

Hasanov and Srivastava [14] define following operators:

$$\begin{aligned} H_{t_1,\ldots,t_i}(a,b) &:= \frac{\Gamma(b)\Gamma(a+\delta_1+\cdots+\delta_i)}{\Gamma(a)\Gamma(b+\delta_1+\cdots+\delta_i)} \\ &= \sum_{k_1,\ldots,k_i=0}^{\infty} \frac{(b-a)_{k_1+\cdots+k_i}(-\delta_1)_{k_1}\cdots(-\delta_i)_{k_i}}{(b)_{k_1+\cdots+k_i}k_1!\cdots k_i!} \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} \bar{H}_{t_1,\ldots,t_i}(a,b) &:= \frac{\Gamma(a)\Gamma(a+\delta_1+\cdots+\delta_i)}{\Gamma(b)\Gamma(a+\delta_1+\cdots+\delta_i)} \\ &= \sum_{k_1,\ldots,k_i=0}^{\infty} \frac{(b-a)_{k_1+\cdots+k_i}(-\delta_1)_{k_1}\cdots(-\delta_i)_{k_i}}{(1-a-\delta_1-\cdots-\delta_i)_{k_1+\cdots+k_i}k_1!\cdots k_i!}, \end{aligned} \tag{5.2}$$

where $\delta_j := t_j\frac{\partial}{\partial t_j}, j=1,\ldots,i; i\in\mathbb{N}:=\{1,2,3,\ldots\}$.

Now, by using the symbolic operators (5.1) and (5.2), we establish the following operational representations.

Theorem 5.1. *The following results hold*:

$$\begin{aligned} & X_{80}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_4,a_5;c_1,c_2,c_1,c_2;x,y,z,u\right) \\ &= H_{z,u}(a_2,a)X_{80}^{(4)}\left(a_1,a_1,a,a,a_1,a_3,a_4,a_5;c_1,c_2,c_1,c_2;x,y,z,u\right), \end{aligned} \tag{5.3}$$

$$\begin{aligned} & X_{80}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_4,a_5;c_1,c_2,c_1,c_2;x,y,z,u\right) \\ &= H_{y}(a_3,a)X_{80}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a,a_4,a_5;c_1,c_2,c_1,c_2;x,y,z,u\right), \end{aligned} \tag{5.4}$$

$$\begin{aligned} & X_{80}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_4,a_5;c_1,c_2,c_1,c_2;x,y,z,u\right) \\ &= H_{y}(a_3,a)H_z(a_4,\acute{a})X_{80}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a,\acute{a},a_5;c_1,c_2,c_1,c_2;x,y,z,u\right), \end{aligned} \tag{5.5}$$

$$\begin{aligned} & X_{80}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_4,a_5;c_1,c_2,c_1,c_2;x,y,z,u\right) \\ &= \bar{H}_{y}(a,a_3)\bar{H}_z(\acute{a},a_4)X_{80}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a,\acute{a},a_5;c_1,c_2,c_1,c_2;x,y,z,u\right). \end{aligned} \tag{5.6}$$

Theorem 5.2. *The following results hold*:

$$\begin{aligned} & X_{81}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_5,a_4,a_3;c_1,c_2,c_1,c_1;x,u,z,y\right) \\ &= H_u(a_5,c_2)\left(1-u\right)^{-a_1}F_S\left(\tfrac{a_1}{2},a_2,a_2,\tfrac{a_1+1}{2},a_3,a_4;c_1,c_1,c_1;\tfrac{4x}{(1-u)^2},y,z\right), \end{aligned} \tag{5.7}$$

$$\begin{aligned} & \left(1-u\right)^{-a_1}F_S\left(\tfrac{a_1}{2},a_2,a_2,\tfrac{a_1+1}{2},a_3,a_4;c_1,c_1,c_1;\tfrac{4x}{(1-u)^2},y,z\right) \\ &= \bar{H}_u(a_5,c_2)X_{81}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_5,a_4,a_3;c_1,c_2,c_1,c_1;x,u,z,y\right), \end{aligned} \tag{5.8}$$

$$X_{81}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_4,a_5;c_1,c_2,c_1,c_1;x,y,z,u\right) \tag{5.9}$$

$$= H_{x,z,u}(c, c_1) X_{81}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_3, a_4, a_5; c, c_2, c, c; x, y, z, u\right),$$

(5.10)
$$X_{81}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_3, a_4, a_5; c_1, c_2, c_1, c_1; x, y, z, u\right) = H_y(c, c_2) X_{81}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_3, a_4, a_5; c_1, c, c_1, c_1; x, y, z, u\right),$$

Corollary 5.1. *In formula (5.8), if we put $z = 0$, we have the following result.*

(5.11)
$$X_{20}\left(a_1, a_5, a_2, a_3; c_1, c_2; x, u, y\right) = H_u(a_5, c_2)\,(1-u)^{-a_1}\, F_3\left(\tfrac{a_1}{2}, a_2, \tfrac{a_1+1}{2}, a_3; c_1; \tfrac{4x}{(1-u)^2}, y\right).$$

Theorem 5.3. *The following results hold:*

(5.12)
$$X_{82}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_3, a_4, a_5; c_2, c_1, c_1, c_1; x, y, z, u\right) = H_y(a_3, a) H_z(a_4, \acute{a}) H_u(a_5, \acute{\acute{a}}) \times X_{82}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a, \acute{a}, \acute{\acute{a}}; c_2, c_1, c_1, c_1; x, y, z, u\right),$$

(5.13)
$$X_{82}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_3, a_4, a_5; c_2, c_1, c_1, c_1; x, y, z, u\right) = \bar{H}_y(a, a_3) \bar{H}_z(\acute{a}, a_4) \bar{H}_u(\acute{\acute{a}}, a_5) \times X_{82}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a, \acute{a}, \acute{\acute{a}}; c_2, c_1, c_1, c_1; x, y, z, u\right),$$

(5.14)
$$X_{82}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_3, a_4, a_5; c_2, c_1, c_1, c_1; x, y, z, u\right) = H_x(c, c_2) X_{82}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_3, a_4, a_5; c, c_1, c_1, c_1; x, y, z, u\right),$$

(5.15)
$$X_{82}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_3, a_4, a_5; c_2, c_1, c_1, c_1; x, y, z, u\right) = \bar{H}_x(c_2, c) X_{82}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_3, a_4, a_5; c, c_1, c_1, c_1; x, y, z, u\right).$$

Theorem 5.4. *The following results hold:*

(5.16)
$$X_{83}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_3, a_4, a_5; c, c, c, c; x, y, z, u\right) = H_{z,u}(a_2, a) H_y(a_3, \acute{a}) X_{83}^{(4)}\left(a_1, a_1, a, a, a_1, \acute{a}, a_4, a_5; c, c, c, c; x, y, z, u\right),$$

(5.17)
$$X_{83}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_3, a_4, a_5; c, c, c, c; x, y, z, u\right) = H_u(a_5, a) X_{83}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_3, a_4, a; c, c, c, c; x, y, z, u\right),$$

(5.18)
$$X_{83}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_3, a_4, a_5; c, c, c, c; x, y, z, u\right) = \bar{H}_u(a, a_5) X_{83}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_3, a_4, a; c, c, c, c; x, y, z, u\right),$$

(5.19)
$$X_{83}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_3, a_4, a_5; c, c, c, c; x, y, z, u\right) = \bar{H}_{x,y,z,u}(c, \acute{c}) X_{83}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_3, a_4, a; \acute{c}, \acute{c}, \acute{c}, \acute{c}; x, y, z, u\right).$$

Proof. The proof of the results (5.3) to (5.19) is based upon application of Mellin and Mellin-Barnes integral representation methods for hypergeometric functions (see [17]). We omit the details involved in these derivations. □

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