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GENERIC LIGHTLIKE SUBMANIFOLDS OF AN INDEFINITE TRANS-SASAKIAN MANIFOLD WITH A NON-SYMMETRIC NON-METRIC CONNECTION OF TYPE (ℓ, m)

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Abstract. Jin [7] defined a new connection on semi-Riemannian manifolds, which is a non-symmetric and non-metric connection. He said that this connection is an $(\ell,\, m)$*-type connection*. Jin also studied lightlike hypersurfaces of an indefinite trans-Sasakian manifold with an (ℓ, m)-type connection in [7]. We study further the geometry of this subject. In this paper, we study generic lightlike submanifolds of an indefinite trans-Sasakian manifold endowed with an (ℓ, m)-type connection.

1. Introduction

The notion of (ℓ, m)-type connection on indefinite almost contact manifolds $\bar{M}$ was introduced by Jin [7]. Here we quote Jin's definition as follows:

A linear connection $\bar{\nabla}$ on $\bar{M}$ is called a *non-symmetric non-metric connection of type* $(\ell,\, m)$, and abbreviate it to $(\ell,\, m)$*-type connection*, if there exist smooth functions ℓ and m on $\bar{M}$ such that $\bar{\nabla}$ and its torsion tensor $\bar{T}$ satisfy

$$(\bar{\nabla}_{\bar{X}}\bar{g})(\bar{Y},\bar{Z}) = -\,\ell\{\theta(\bar{Y})\bar{g}(\bar{X},\bar{Z}) + \theta(\bar{Z})\bar{g}(\bar{X},\bar{Y})\} - m\{\theta(\bar{Y})\bar{g}(J\bar{X},\bar{Z}) + \theta(\bar{Z})\bar{g}(J\bar{X},\bar{Y})\}, \tag{1}$$

$$\bar{T}(\bar{X},\bar{Y}) = \ell\{\theta(\bar{Y})\bar{X} - \theta(\bar{X})\bar{Y}\} + m\{\theta(\bar{Y})J\bar{X} - \theta(\bar{X})J\bar{Y}\}, \tag{2}$$

where J is a $(1,1)$-type tensor field and θ is a 1-form associated with a smooth vector field ζ by $\theta(\bar{X}) = \bar{g}(\bar{X},\zeta)$. We set $(\ell,m) \neq (0,0)$ and we denote by $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ the smooth vector fields on $\bar{M}$. Semi-symmetric non-metric connection and non-metric ϕ-symmetric connection are important examples of this connection such that (1) $(\ell,\, m) = (1,\, 0)$ and (2) $(\ell,\, m) = (0,\, 1)$, respectively.

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Especially, in cases: (3) $(\ell, m) = (1, 0)$ in (1) and $(\ell, m) = (0, 1)$ in (2) (see [10] in details); (4) $(\ell, m) = (0, 0)$ in (1) and $(\ell, m) = (0, 1)$ in (2) and (5) $(\ell, m) = (0, 0)$ in (1) and $(\ell, m) = (1, 0)$ in (2), this connection $\bar{\nabla}$ reduce to quarter-symmetric non-metric connection, quarter-symmetric metric connection and semi-symmetric metric connection, respectively.

Remark 1.1. Denote by $\widetilde{\nabla}$ the Levi-Civita connection of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ with respect to the metric $\bar{g}$. It is known [7] that *a linear connection $\bar{\nabla}$ on $\bar{M}$ is an (ℓ, m)-type connection if and only if $\bar{\nabla}$ satisfies*

$$\bar{\nabla}_{\bar{X}}\bar{Y} = \widetilde{\nabla}_{\bar{X}}\bar{Y} + \theta(\bar{Y})\{\ell\bar{X} + mJ\bar{X}\}. \tag{3}$$

A lightlike submanifold M of an indefinite almost contact manifold $\bar{M}$ is said to be *generic* if there exists a screen distribution $S(TM)$ on M such that

$$J(S(TM)^{\perp}) \subset S(TM), \tag{4}$$

where $S(TM)^{\perp}$ is the orthogonal complement of $S(TM)$ in the tangent bundle $T\bar{M}$ on $\bar{M}$, *i.e.*, $T\bar{M} = S(TM) \oplus_{orth} S(TM)^{\perp}$. The notion of generic lightlike submanifolds was introduced by Jin-Lee [8] and later, studied by several geometers [3, 5, 6, 9]. Its geometry is an extension of that of lightlike hypersurfaces and half lightlike submanifolds. Much of its theory will be immediately generalized in a formal way to general lightlike submanifolds.

The subject of study in this paper is generic lightlike submanifolds of an indefinite trans-Sasakian manifold $M = (\bar{M}, \zeta, \theta, J, \bar{g})$ endowed with an (ℓ, m)-type connection subject to the following two conditions that (1) the tensor field J and the 1-form θ, defined by (1) and (2), are identical with the indefinite trans-Sasakian structure tensor J and the structure 1-form θ of $\bar{M}$, respectively, and (2) the structure vector field ζ of $\bar{M}$ is tangent to M.

2. (ℓ, m)-type connections

The notion of trans-Sasakian manifold $\bar{M}$, of type (α, β), was introduced by Oubina [11]. Sasakian, Kenmotsu and cosymplectic manifolds are important kinds of trans-Sasakian manifolds such that

$$\alpha = 1, \;\; \beta = 0; \qquad \alpha = 0, \;\; \beta = 1; \qquad \alpha = \beta = 0,$$

respectively. If $\bar{M}$ is semi-Riemannian manifold with a trans-Sasakian structure of type (α, β), then $\bar{M}$ is called indefinite trans-Sasakian manifold as follows:

An odd-dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called an *indefinite trans-Sasakian manifold* if there exist (1) a structure set $\{J, \zeta, \theta, \bar{g}\}$, where J is a $(1, 1)$-type tensor field, ζ is a vector field and θ is a 1-form such that

$$\begin{aligned} J^2\bar{X} = -\bar{X} + \theta(\bar{X})\zeta, \quad \theta(\zeta) = 1, \quad \theta(\bar{X}) = \epsilon\,\bar{g}(\bar{X}, \zeta), \\ \theta \circ J = 0, \quad \bar{g}(J\bar{X}, J\bar{Y}) = \bar{g}(\bar{X}, \bar{Y}) - \epsilon\,\theta(\bar{X})\theta(\bar{Y}), \end{aligned} \tag{5}$$

(2) a Levi-Civita connection $\widetilde{\nabla}$ and two smooth functions α and β such that

$$(\widetilde{\nabla}_{\bar{X}}J)\bar{Y} = \alpha\{\bar{g}(\bar{X}, \bar{Y})\zeta - \epsilon\,\theta(\bar{Y})\bar{X}\} + \beta\{\bar{g}(J\bar{X}, \bar{Y})\zeta - \epsilon\,\theta(\bar{Y})J\bar{X}\},$$

where ϵ denotes $\epsilon = 1$ or -1 according as ζ is spacelike or timelike respectively. $\{J, \zeta, \theta, \bar{g}\}$ is called an *indefinite trans-Sasakian structure of type* (α, β).

In the entire discussion of this article, we shall assume that the vector field ζ is a spacelike one, *i.e.*, $\epsilon = 1$, without loss of generality.

Let $\bar{\nabla}$ be an (ℓ, m)-type connection on $(\bar{M}, \bar{g})$. By directed calculation from (3), (5) and the fact that $\theta(JY) = 0$, we obtain

$$
\begin{aligned}
(\bar{\nabla}_{\bar{X}} J)\bar{Y} = {} & \alpha\{\bar{g}(\bar{X}, \bar{Y})\zeta - \theta(\bar{Y})\bar{X}\} + \beta\{\bar{g}(J\bar{X}, \bar{Y})\zeta - \theta(\bar{Y})J\bar{X}\} \\
& - \theta(\bar{Y})\{\ell J\bar{X} - m\bar{X} + m\theta(\bar{X})\zeta\}.
\end{aligned} \tag{6}
$$

Replacing $\bar{Y}$ by ζ to (6) and using $J\zeta = 0$ and $\theta(\bar{\nabla}_X \zeta) = \ell\theta(X)$, we obtain

$$
\bar{\nabla}_{\bar{X}}\zeta = (m - \alpha)J\bar{X} + (\ell + \beta)\bar{X} - \beta\theta(\bar{X})\zeta. \tag{7}
$$

Let (M, g) be an m-dimensional lightlike submanifold of an indefinite trans-Sasakian manifold $(\bar{M}, \bar{g})$ of dimension $(m + n)$. Then the radical distribution $Rad(TM) = TM \cap TM^{\perp}$ on M is a subbundle of the tangent bundle TM and the normal bundle $TM^{\perp}$, of rank $r\,(1 \leq r \leq \min\{m, n\})$. In case $1 < r < \min\{m, n\}$, we say that M is an *r-lightlike submanifold* [3] of $\bar{M}$. In this case, there exist two complementary non-degenerate distributions $S(TM)$ and $S(TM^{\perp})$ of $Rad(TM)$ in TM and $TM^{\perp}$, respectively, which are called the *screen distribution* and *co-screen distribution* of M such that

$$
TM = Rad(TM) \oplus_{orth} S(TM), \;\; TM^{\perp} = Rad(TM) \oplus_{orth} S(TM^{\perp}),
$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M. Also denote by $(5)_i$ the i-th equation of (5). We use the same notations for any others. Let X, Y, Z and W be the vector fields on M, unless otherwise specified. We use the following range of indices:

$$
i, j, k, \ldots \in \{1, \ldots, r\}, \qquad a, b, c, \ldots \in \{r+1, \ldots, n\}.
$$

Let $tr(TM)$ and $ltr(TM)$ be complementary vector bundles to TM in $T\bar{M}_{|M}$ and $TM^{\perp}$ in $S(TM)^{\perp}$ respectively and let $\{N_1, \ldots, N_r\}$ be a lightlike basis of $ltr(TM)_{|\mathcal{U}}$, where $\mathcal{U}$ is a coordinate neighborhood of M, such that

$$
\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0,
$$

where $\{\xi_1, \ldots, \xi_r\}$ is a lightlike basis of $Rad(TM)_{|\mathcal{U}}$. In this case,

$$
\begin{aligned}
T\bar{M} = {} & TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM) \\
= {} & \{Rad(TM) \oplus ltr(TM)\} \oplus_{orth} S(TM) \oplus_{orth} S(TM^{\perp}),
\end{aligned}
$$

$$
\{\xi_1, \ldots, \xi_r, \; N_1, \ldots, N_r, \; F_{r+1}, \ldots, F_m, \; E_{r+1}, \ldots, E_n\}
$$

is a quasi-orthonormal field of frames of $\bar{M}$, where $\{F_{r+1}, \ldots, F_m\}$ is an orthonormal basis of $S(TM)$ and $\{E_{r+1}, \ldots, E_n\}$ is an orthonormal basis of $S(TM^{\perp})$. Denote $\epsilon_a = \bar{g}(E_a, E_a)$. Then $\epsilon_a\delta_{ab} = \bar{g}(E_a, E_b)$.

Let P be the projection morphism of TM on $S(TM)$. Then the local Gauss-Weingarten formulae of M and $S(TM)$ are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + \sum_{i=1}^{r} h_i^{\ell}(X,Y)N_i + \sum_{a=r+1}^{n} h_a^s(X,Y)E_a, \tag{8}$$

$$\bar{\nabla}_X N_i = -A_{N_i} X + \sum_{j=1}^{r} \tau_{ij}(X)N_j + \sum_{a=r+1}^{n} \rho_{ia}(X)E_a, \tag{9}$$

$$\bar{\nabla}_X E_a = -A_{E_a} X + \sum_{i=1}^{r} \lambda_{ai}(X)N_i + \sum_{b=r+1}^{n} \sigma_{ab}(X)E_b; \tag{10}$$

$$\nabla_X PY = \nabla^*_X PY + \sum_{i=1}^{r} h_i^*(X,PY)\xi_i, \tag{11}$$

$$\nabla_X \xi_i = -A^*_{\xi_i} X - \sum_{j=1}^{r} \tau_{ji}(X)\xi_j, \tag{12}$$

where ∇ and ∇^* are induced linear connections on M and $S(TM)$ respectively, h_i^{ℓ} and h_a^s are called the *local second fundamental forms* on M, h_i^* are called the *local second fundamental forms* on $S(TM)$. A_{N_i}, A_{E_a} and $A^*_{\xi_i}$ are called the *shape operators*, and τ_{ij}, ρ_{ia}, λ_{ai} and σ_{ab} are 1-forms.

Let M be a generic lightlike submanifold of $\bar{M}$. From (4) we show that $J(Rad(TM))$, $J(ltr(TM))$ and $J(S(TM^{\perp}))$ are vector subbundles of $S(TM)$. Thus there exist two non-degenerate almost complex distributions H_o and H with respect to J, *i.e.*, $J(H_o) = H_o$ and $J(H) = H$, such that

$$S(TM) = \{J(Rad(TM)) \oplus J(ltr(TM))\} \oplus_{orth} J(S(TM^{\perp})) \oplus_{orth} H_o,$$
$$H = Rad(TM) \oplus_{orth} J(Rad(TM)) \oplus_{orth} H_o.$$

In this case, the tangent bundle TM on M is decomposed as follows:

$$TM = H \oplus J(ltr(TM)) \oplus_{orth} J(S(TM^{\perp})). \tag{13}$$

Consider local null vector fields U_i and V_i for each i, local non-null unit vector fields W_a for each a, and their 1-forms u_i, v_i and w_a defined by

$$U_i = -JN_i, \qquad V_i = -J\xi_i, \qquad W_a = -JE_a, \tag{14}$$

$$u_i(X) = g(X,V_i), \quad v_i(X) = g(X,U_i), \quad w_a(X) = \epsilon_a g(X,W_a). \tag{15}$$

Denote by S the projection morphism of TM on H and by F the tensor field of type $(1,1)$ globally defined on M by $F = J \circ S$. Then JX is expressed as

$$JX = FX + \sum_{i=1}^{r} u_i(X)N_i + \sum_{a=r+1}^{n} w_a(X)E_a. \tag{16}$$

Applying J to (16) and using $(5)_1$ and (14), we have

$$F^2X = -X + \theta(X)\zeta + \sum_{i=1}^{r} u_i(X)U_i + \sum_{a=r+1}^{n} w_a(X)W_a. \tag{17}$$

3. Structure equations

Let $\bar{M}$ be an indefinite trans-Sasakian manifold with an (ℓ, m)-type connection $\bar{\nabla}$. In the following, we shall assume that ζ is tangent to M. Călin [2] proved that if ζ is tangent to M, then it belongs to $S(TM)$ which we assumed in this paper. Using (1), (2), (8) and (16), we see that

$$\begin{aligned}(\nabla_X g)(Y,Z) = &\sum_{i=1}^{r}\{h_i^\ell(X,Y)\eta_i(Z) + h_i^\ell(X,Z)\eta_i(Y)\} \\ &- \ell\{\theta(Y)g(X,Z) + \theta(Z)g(X,Y)\} \\ &- m\{\theta(Y)\bar{g}(JX,Z) + \theta(Z)\bar{g}(JX,Y)\},\end{aligned} \tag{18}$$

$$T(X,Y) = \ell\{\theta(Y)X - \theta(X)Y\} + m\{\theta(Y)FX - \theta(X)FY\}, \tag{19}$$

$$h_i^\ell(X,Y) - h_i^\ell(Y,X) = m\{\theta(Y)u_i(X) - \theta(X)u_i(Y)\}, \tag{20}$$

$$h_a^s(X,Y) - h_a^s(Y,X) = m\{\theta(Y)w_a(X) - \theta(X)w_a(Y)\}, \tag{21}$$

for all i and a, where η_i's are 1-forms such that $\eta_i(X) = \bar{g}(X, N_i)$.

Theorem 3.1. *Let M be a lightlike hypersurface of an indefinite trans-Sasakian manifold $\bar{M}$ with an (ℓ, m)-type connection subject such that ζ is tangent to M. Then either h_i^ℓ or h_a^s is symmetric if and only if $m = 0$.*

Proof. (1) If $m = 0$, then h_i^ℓ are symmetric by (20). Conversely, if h_i^ℓ is symmetric, then, taking $X = \zeta$ and $Y = U_i$ to (20), we have $m = 0$.

(2) If $m = 0$, then h_a^s are symmetric by (21). Conversely, if h_a^s is symmetric, then, taking $X = \zeta$ and $Y = W_a$ to (21), we have $m = 0$. □

From the facts that $h_i^\ell(X,Y) = \bar{g}(\bar{\nabla}_X Y, \xi_i)$ and $\epsilon_a h_a^s(X,Y) = \bar{g}(\bar{\nabla}_X Y, E_a)$, we know that h_i^ℓ and h_a^s are independent of the choice of $S(TM)$. Applying $\bar{\nabla}_X$ to $g(\xi_i, \xi_j) = 0$, $\bar{g}(\xi_i, E_a) = 0$, $\bar{g}(N_i, N_j) = 0$, $\bar{g}(N_i, E_a) = 0$ and $\bar{g}(E_a, E_b) = \epsilon\delta_{ab}$ by turns and using (1) and (8) ∼ (10), we obtain

$$\begin{aligned}&h_i^\ell(X,\xi_j) + h_j^\ell(X,\xi_i) = 0, \qquad h_a^s(X,\xi_i) = -\epsilon_a\lambda_{ai}(X), \\ &\eta_j(A_{N_i}X) + \eta_i(A_{N_j}X) = 0, \quad \eta_i(A_{E_a}X) = \epsilon_a\rho_{ia}(X), \\ &\epsilon_b\sigma_{ab} + \epsilon_a\sigma_{ba} = 0; \quad h_i^\ell(X,\xi_i) = 0,\ h_i^\ell(\xi_j,\xi_k) = 0,\ A^*_{\xi_i}\xi_i = 0.\end{aligned} \tag{22}$$

The local second fundamental forms are related to their shape operators by

$$h_i^\ell(X,Y) = g(A^*_{\xi_i}X, Y) + m\theta(Y)u_i(X) - \sum_{k=1}^{r} h_k^\ell(X,\xi_i)\eta_k(Y), \tag{23}$$

$$\epsilon_a h_a^s(X,Y) = g(A_{E_a}X, Y) + \epsilon_a m\theta(Y)w_a(X) - \sum_{k=1}^{r}\lambda_{ak}(X)\eta_k(Y), \tag{24}$$

$$h^*_i(X, PY) = g(A_{N_i}X, PY) + \{\ell\eta_i(X) + mv_i(X)\}\theta(PY). \tag{25}$$

Replacing Y by ζ to (8) and using (7), (16), (23) and (24), we have

$$\nabla_X\zeta = (m-\alpha)FX + (\ell+\beta)X - \beta\theta(X)\zeta, \tag{26}$$

$$\theta(A^*_{\xi_i}X) = -\alpha u_i(X), \qquad h^\ell_i(X,\zeta) = (m-\alpha)u_i(X), \tag{27}$$

$$\theta(A_{E_a}X) = -\epsilon_a\alpha w_a(X), \quad h^s_a(X,\zeta) = (m-\alpha)w_a(X). \tag{28}$$

Applying $\bar\nabla_X$ to $\bar g(\zeta, N_i) = 0$ and using (7), (9) and (25), we have

$$\begin{aligned}\theta(A_{N_i}X) &= -\alpha v_i(X) + \beta\eta_i(X),\\ h^*_i(X,\zeta) &= (\ell+\beta)\eta_i(X) + (m-\alpha)v_i(X).\end{aligned} \tag{29}$$

Applying $\bar\nabla_X$ to $(14)_{1,2,3}$ and (16) by turns and using (6), (8) $\sim$ (12), (14) $\sim$ (16) and (23) $\sim$ (25), we have

$$\begin{aligned}&h^\ell_j(X,U_i) = h^*_i(X,V_j), \qquad \epsilon_a h^*_i(X,W_a) = h^s_a(X,U_i),\\ &h^\ell_j(X,V_i) = h^\ell_i(X,V_j), \qquad \epsilon_a h^\ell_i(X,W_a) = h^s_a(X,V_i),\\ &\epsilon_b h^s_b(X,W_a) = \epsilon_a h^s_a(X,W_b),\end{aligned} \tag{30}$$

$$\begin{aligned}\nabla_X U_i \;=\; &F(A_{N_i}X) + \sum_{j=1}^r \tau_{ij}(X)U_j + \sum_{a=r+1}^n \rho_{ia}(X)W_a\\ &- \{\alpha\eta_i(X) + \beta v_i(X)\}\zeta,\end{aligned} \tag{31}$$

$$\begin{aligned}\nabla_X V_i \;=\; &F(A^*_{\xi_i}X) - \sum_{j=1}^r \tau_{ji}(X)V_j + \sum_{j=1}^r h^\ell_j(X,\xi_i)U_j\\ &- \sum_{a=r+1}^n \epsilon_a\lambda_{ai}(X)W_a - \beta u_i(X)\zeta,\end{aligned} \tag{32}$$

$$\begin{aligned}\nabla_X W_a \;=\; &F(A_{E_a}X) + \sum_{i=1}^r \lambda_{ai}(X)U_i + \sum_{b=r+1}^n \sigma_{ab}(X)W_b\\ &- \epsilon_a\beta w_a(X)\zeta,\end{aligned} \tag{33}$$

$$\begin{aligned}(\nabla_X F)(Y) \;=\; &\sum_{i=1}^r u_i(Y)A_{N_i}X + \sum_{a=r+1}^n w_a(Y)A_{E_a}X\\ &- \sum_{i=1}^r h^\ell_i(X,Y)U_i - \sum_{a=r+1}^n h^s_a(X,Y)W_a\\ &+ \{\alpha g(X,Y) + \beta\bar g(JX,Y) - \theta(X)\theta(Y)\}\zeta\\ &+ (m-\alpha)\theta(Y)X - (\ell+\beta)\theta(Y)FX,\end{aligned} \tag{34}$$

$$\begin{aligned}(\nabla_X u_i)(Y) = &-\sum_{j=1}^r u_j(Y)\tau_{ji}(X) - \sum_{a=r+1}^n w_a(Y)\lambda_{ai}(X)\\ &- h^\ell_i(X,FY) - (\ell+\beta)\theta(Y)u_i(X),\end{aligned} \tag{35}$$

$$(\nabla_X v_i)(Y) = \sum_{j=1}^{r} v_j(Y)\tau_{ij}(X) + \sum_{a=r+1}^{n} \epsilon_a w_a(Y)\rho_{ia}(X) + \sum_{j=r+1}^{r} u_j(Y)\eta_i(A_{N_j}X) - g(A_{N_i}X, FY) + (m-\alpha)\theta(Y)\eta_i(X) - (\ell+\beta)\theta(Y)v_i(X). \tag{36}$$

Theorem 3.2. *There exist no generic lightlike submanifolds of an indefinite trans-Sasakian manifold with an (ℓ, m)-type connection subject such that ζ is tangent to M and F satisfies the following equation:*

$$(\nabla_X F)Y = (\nabla_Y F)X, \qquad \forall\, X,\, Y \in \Gamma(TM).$$

Proof. Let $(\nabla_X F)Y = (\nabla_Y F)X$. Using (20), (21) and (34), we obtain

$$\sum_{i=1}^{r}\{u_i(Y)A_{N_i}X - u_i(X)A_{N_i}Y\} + \sum_{a=r+1}^{n}\{w_a(Y)A_{E_a}X - w_a(X)A_{E_a}Y\} - 2\beta\bar{g}(X, JY)\zeta + \{\theta(X)u_i(Y) - \theta(Y)u_i(X)\}U_i + \{\theta(X)w_a(Y) - \theta(Y)w_a(X)\}W_a + (m-\alpha)\{\theta(Y)X - \theta(X)Y\} - (\ell+\beta)\{\theta(Y)FX - \theta(X)FY\} = 0. \tag{37}$$

Taking the scalar product with ζ and using $(28)_1$ and $(29)_1$, we have

$$\alpha\sum_{i=1}^{r}\{u_i(Y)v_i(X) - u_i(X)v_i(Y)\} = \beta\sum_{i=1}^{r}\{u_i(Y)\eta_i(X) - u_i(X)\eta_i(Y)\} - 2\beta\bar{g}(X, JY).$$

Taking $X = V_j$, $Y = U_j$ and $X = \xi_j$, $Y = U_j$ to this equation by turns, we obtain $\alpha = 0$ and $\beta = 0$, respectively. Taking $X = \xi_j$ to (37), we have

$$\theta(X)\{m\xi_i + \ell V_i\} + \sum_{j=1}^{r} u_j(X)A_{N_j}\xi_i + \sum_{a=r+1}^{n} w_a(X)A_{E_a}\xi_i = 0.$$

Taking $X = \zeta$ to this, we have $m\xi_i + \ell V_i = 0$. It follows that $\ell = m = 0$. It is a contradiction to $(\ell, m) \neq (0, 0)$. Thus we have our theorem. □

Corollary 3.3. *There exist no generic lightlike submanifolds of an indefinite trans-Sasakian manifold with an (ℓ, m)-type connection subject such that ζ is tangent to M and F is parallel with respect to the connection ∇.*

Theorem 3.4. *Let M be a generic lightlike submanifold of an indefinite trans-Sasakian manifold $\bar{M}$ with an (ℓ, m)-type connection such that ζ is tangent to M. If U_i's or V_i's are parallel with respect to ∇, then $\tau_{ij} = 0$ and $\alpha = \beta = 0$, i.e., $\bar{M}$ is an indefinite cosymplectic manifold.*

Proof. (1) If U_i is parallel with respect to ∇, then, taking the scalar product with ζ, V_j, W_a, U_j and N_j to (31) such that $\nabla_X U_i = 0$ respectively, we get

$$\alpha = \beta = 0, \quad \tau_{ij} = 0, \quad \rho_{ia} = 0, \quad \eta_j(A_{N_i} X) = 0, \quad h_i^*(X, U_j) = 0. \tag{38}$$

As $\alpha = \beta = 0$, $\bar{M}$ is an indefinite cosymplectic manifold.

(2) If V_i is parallel with respect to ∇, then, taking the scalar product with ζ, U_j, V_j, W_a and N_j to (32) with $\nabla_X V_i = 0$ respectively, we get

$$\beta = 0, \quad \tau_{ji} = 0, \quad h_j^\ell(X, \xi_i) = 0, \quad \lambda_{ai} = 0, \quad h_i^\ell(X, U_j) = 0. \tag{39}$$

As $h_i^\ell(X, U_j) = 0$, we get $h_i^\ell(\zeta, U_j) = 0$. Taking $X = U_j$ and $Y = \zeta$ to (20), we get $h_i^\ell(U_j, \zeta) = m\delta_{ij}$. On the other hand, replacing X by U to $(27)_2$, we have $h_i^\ell(U_j, \zeta) = (m - \alpha)\delta_{ij}$. It follows that $\alpha = 0$. Since $\alpha = \beta = 0$, $\bar{M}$ is an indefinite cosymplectic manifold. □

4. Indefinite generalized Sasakian space forms

Definition. An indefinite trans-Sasakian manifold $\bar{M}$ is said to be an *indefinite generalized Sasakian space form* and denote it by $\bar{M}(f_1, f_2, f_3)$ if there exist three smooth functions f_1, f_2 and f_3 on $\bar{M}$ such that

$$\begin{aligned} \widetilde{R}(\bar{X}, \bar{Y})\bar{Z} = \; & f_1\{\bar{g}(\bar{Y}, \bar{Z})\bar{X} - \bar{g}(\bar{X}, \bar{Z})\bar{Y}\} \\ & + f_2\{\bar{g}(\bar{X}, J\bar{Z})J\bar{Y} - \bar{g}(\bar{Y}, J\bar{Z})J\bar{X} + 2\bar{g}(\bar{X}, J\bar{Y})J\bar{Z}\} \\ & + f_3\{\theta(\bar{X})\theta(\bar{Z})\bar{Y} - \theta(\bar{Y})\theta(\bar{Z})\bar{X} \\ & \quad + \bar{g}(\bar{X}, \bar{Z})\theta(\bar{Y})\zeta - \bar{g}(\bar{Y}, \bar{Z})\theta(\bar{X})\zeta\}, \end{aligned} \tag{40}$$

where $\widetilde{R}$ is the curvature tensor of the Levi-Civita connection $\bar{\nabla}$.

The notion of generalized Sasakian space form was introduced by Alegre et al. [1], while the indefinite generalized Sasakian space forms were introduced by Jin [4]. Sasakian space form, Kenmotsu space form and cosymplectic space form are important kinds of generalized Sasakian space forms such that

$$f_1 = \tfrac{c+3}{4}, \; f_2 = f_3 = \tfrac{c-1}{4}; \quad f_1 = \tfrac{c-3}{4}, \; f_2 = f_3 = \tfrac{c+1}{4}; \quad f_1 = f_2 = f_3 = \tfrac{c}{4}$$

respectively, where c is a constant J-sectional curvature of each space forms.

Denote by $\bar{R}$ the curvature tensors of the non-metric ϕ-symmetric connection $\bar{\nabla}$ on $\bar{M}$. By directed calculations from (2), (3) and (5), we see that

$$\begin{aligned} \bar{R}(\bar{X}, \bar{Y})\bar{Z} = \; & \widetilde{R}(\bar{X}, \bar{Y})\bar{Z} \\ & + (\bar{\nabla}_{\bar{X}}\theta)(\bar{Z})\{\ell\bar{Y} + mJ\bar{Y}\} - (\bar{\nabla}_{\bar{Y}}\theta)(\bar{Z})\{\ell\bar{X} + mJ\bar{X}\} \\ & + \theta(\bar{Z})\{(\bar{X}\ell)\bar{Y} - (\bar{Y}\ell)\bar{X} + (\bar{X}m)J\bar{Y} - (\bar{Y}m)J\bar{X} \\ & \quad - m\alpha[\theta(\bar{Y})\bar{X} - \theta(\bar{X})\bar{Y}] - m\beta[\theta(\bar{Y})J\bar{X} - \theta(\bar{X})J\bar{Y}] \\ & \quad - 2m\beta\bar{g}(\bar{X}, J\bar{Y})\zeta\}. \end{aligned} \tag{41}$$

Taking the scalar product with ξ_i and N_i to (41) by turns and, then denote by R and R^* the curvature tensors of the induced linear connections ∇ and

∇^* on M and $S(TM)$ respectively. Using the Gauss-Weingarten formulae, we obtain Gauss equations for M and $S(TM)$, respectively:

$$
\begin{aligned}
\bar{R}(X,Y)Z &= R(X,Y)Z \\
&+ \sum_{i=1}^{r}\{h_i^\ell(X,Z)A_{N_i}Y - h_i^\ell(Y,Z)A_{N_i}X\} \\
&+ \sum_{a=r+1}^{n}\{h_a^s(X,Z)A_{E_a}Y - h_a^s(Y,Z)A_{E_a}X\} \\
&\quad + \sum_{i=1}^{r}\{(\nabla_X h_i^\ell)(Y,Z) - (\nabla_Y h_i^\ell)(X,Z) \\
&\quad + \sum_{j=1}^{r}[\tau_{ji}(X)h_j^\ell(Y,Z) - \tau_{ji}(Y)h_j^\ell(X,Z)] \\
&\quad + \sum_{a=r+1}^{n}[\lambda_{ai}(X)h_a^s(Y,Z) - \lambda_{ai}(Y)h_a^s(X,Z)] \\
&\quad - \ell[\theta(X)h_i^\ell(Y,Z) - \theta(Y)h_i^\ell(X,Z)] \\
&\quad - m[\theta(X)h_i^\ell(FY,Z) - \theta(Y)h_i^\ell(FX,Z)]\}N_i \\
&+ \sum_{a=r+1}^{n}\{(\nabla_X h_a^s)(Y,Z) - (\nabla_Y h_a^s)(X,Z) \\
&\quad + \sum_{i=1}^{r}[\rho_{ia}(X)h_i^\ell(Y,Z) - \rho_{ia}(Y)h_i^\ell(X,Z)] \\
&\quad + \sum_{b=r+1}^{n}[\sigma_{ba}(X)h_b^s(Y,Z) - \sigma_{ba}(Y)h_b^s(X,Z)] \\
&\quad - \ell[\theta(X)h_a^s(Y,Z) - \theta(Y)h_a^s(X,Z)] \\
&\quad - m[\theta(X)h_a^s(FY,Z) - \theta(Y)h_a^s(FX,Z)]\}E_a,
\end{aligned} \tag{42}
$$

$$
\begin{aligned}
R(X,Y)PZ &= R^*(X,Y)PZ \\
&+ \sum_{i=1}^{r}\{h_i^*(X,PZ)A^*_{\xi_i}Y - h_i^*(Y,PZ)A_{\xi_i}X\} \\
&+ \sum_{i=1}^{r}\{(\nabla_X h_i^*)(Y,PZ) - (\nabla_Y h_i^*)(X,PZ) \\
&\quad + \sum_{k=1}^{r}[\tau_{ik}(Y)h_k^*(X,PZ) - \tau_{ik}(X)h_k^*(Y,PZ)] \\
&\quad - \ell[\theta(X)h_i^*(Y,PZ) - \theta(Y)h_i^*(FX,PZ)] \\
&\quad - m[\theta(X)h_i^*(FY,PZ) - \theta(Y)h_i^*(FX,PZ)]\}\xi_i.
\end{aligned} \tag{43}
$$

substituting (42) and (40) and using $(22)_4$ and (43), we get

$$
\begin{aligned}
(44)\quad & (\nabla_X h_i^\ell)(Y,Z) - (\nabla_Y h_i^\ell)(X,Z) \\
& + \sum_{j=1}^{r}\{\tau_{ji}(X)h_j^\ell(Y,Z) - \tau_{ji}(Y)h_j^\ell(X,Z)\} \\
& + \sum_{a=r+1}^{n}\{\lambda_{ai}(X)h_a^s(Y,Z) - \lambda_{ai}(Y)h_a^s(X,Z)\} \\
& - \ell\{\theta(X)h_i^\ell(Y,Z) - \theta(Y)h_i^\ell(X,Z)\} \\
& - m\{\theta(X)h_i^\ell(FY,Z) - \theta(Y)h_i^\ell(FX,Z)\} \\
& - m\{(\bar\nabla_X\theta)(Z)u_i(Y) - (\bar\nabla_Y\theta)(Z)u_i(X)\} \\
& - \theta(Z)\{[Xm + m\beta\theta(X)]u_i(Y) - [Ym + m\beta\theta(Y)]u_i(X)\} \\
= {} & f_2\{u_i(Y)\bar g(X,JZ) - u_i(X)\bar g(Y,JZ) + 2u_i(Z)\bar g(X,JY)\},
\end{aligned}
$$

$$
\begin{aligned}
(45)\quad & (\nabla_X h_i^*)(Y, PZ) - (\nabla_Y h_i^*)(X, PZ) \\
& - \sum_{j=1}^{r}\{\tau_{ij}(X)h_j^*(Y, PZ) - \tau_{ij}(Y)h_j^*(X, PZ)\} \\
& - \sum_{a=r+1}^{n}\epsilon_a\{\rho_{ia}(X)h_a^s(Y, PZ) - \rho_{ia}(Y)h_a^s(X, PZ)\} \\
& + \sum_{j=1}^{r}\{h_j^\ell(X, PZ)\eta_i(A_{N_j}Y) - h_j^\ell(Y, PZ)\eta_i(A_{N_j}X)\} \\
& - \ell\{\theta(X)h_i^*(Y,PZ) - \theta(Y)h_i^*(X,PZ)\} \\
& - m\{\theta(X)h_i^*(FY,PZ) - \theta(Y)h_i^*(FX,PZ)\} \\
& - (\bar\nabla_X\theta)(PZ)\{\ell\eta_i(Y) + mv_i(Y)\} + (\bar\nabla_Y\theta)(PZ)\{\ell\eta_i(X) + mv_i(X)\} \\
& - \theta(PZ)\{[X\ell + m\alpha\theta(X)]\eta_i(Y) - [Y\ell + m\alpha\theta(Y)]\eta_i(X) \\
& + [Xm + m\beta\theta(X)]v_i(Y) - [Ym + m\beta\theta(Y)]v_i(X)\} \\
= {} & f_1\{g(Y,PZ)\eta_i(X) - g(X,PZ)\eta_i(Y)\} \\
& + f_2\{v_i(Y)\bar g(X,JPZ) - v_i(X)\bar g(Y,JPZ) + 2v_i(PZ)\bar g(X,JY)\} \\
& + f_3\{\theta(X)\eta_i(Y) - \theta(Y)\eta_i(X)\}\theta(PZ).
\end{aligned}
$$

Theorem 4.1. *Let M be a generic lightlike submanifold of an indefinite generalized Sasakian space form $\bar M(f_1, f_2, f_3)$ with an (ℓ, m)-type connection such that ζ is tangent to M. Then the functions α, β, f_1, f_2 and f_3 satisfy*

(1) *α is a constant on M,*
(2) *$\alpha\beta = 0$, and*
(3) *$f_1 - f_2 = \alpha^2 - \beta^2$ and $f_1 - f_3 = \alpha^2 - \beta^2 - \zeta\beta$.*

Proof. Applying $\bar{\nabla}_X$ to $\theta(U_i)=0$ and $\theta(V_i)=0$ by turns and using (8), (31), (32) and the facts that $g(FX,\zeta)=0$ and $\zeta\in\Gamma(S(TM))$, we get

$$(\bar{\nabla}_X\theta)(U_i)=\alpha\eta_i(X)+\beta v_i(X), \qquad (\bar{\nabla}_X\theta)(V_i)=\beta u_i(X). \tag{46}$$

Applying ∇_X to $(30)_1$: $h^\ell_j(Y,U_i)=h^*_i(Y,V_j)$ and using (5), (16), (23), (25), $(27)_2$, $(29)_2$, $(30)_{1,2,4}$, (31) and (32), we obtain

$$\begin{aligned}
(\nabla_X h^\ell_j)(Y,U_i) = (\nabla_X h^*_i)(Y,V_j)&\\
&-\sum_{k=1}^{r}\{\tau_{kj}(X)h^\ell_k(Y,U_i)+\tau_{ik}(X)h^*_k(Y,V_j)\}\\
&-\sum_{a=r+1}^{n}\{\lambda_{aj}(X)h^s_a(Y,U_i)+\epsilon_a\rho_{ia}(X)h^s_a(Y,V_j)\}\\
&+\sum_{k=1}^{r}\{h^*_i(Y,U_k)h^\ell_k(X,\xi_j)+h^*_i(X,U_k)h^\ell_k(Y,\xi_j)\}\\
&-g(A^*_{\xi_j}X,F(A_{N_i}Y))-g(A^*_{\xi_j}Y,F(A_{N_i}X))\\
&-\sum_{k=1}^{r}h^\ell_j(X,V_k)\eta_k(A_{N_i}Y)\\
&+\beta(m-\alpha)\{u_j(Y)v_i(X)-u_j(X)v_i(Y)\}\\
&+\alpha(m-\alpha)u_j(Y)\eta_i(X)-\beta(\ell+\beta)u_j(X)\eta_i(Y).
\end{aligned}$$

Substituting this and (29) into the modified equation (44) which is change i with j and Z with U_i, and using $(22)_3$, $(30)_3$ and $(46)_1$, we have

$$\begin{aligned}
&(\nabla_X h^*_i)(Y,V_j)-(\nabla_Y h^*_i)(X,V_j)\\
&-\sum_{k=1}^{r}\{\tau_{ik}(X)h^*_k(Y,V_j)-\tau_{ik}(Y)h^*_k(X,V_j)\}\\
&-\sum_{a=r+1}^{n}\epsilon_a\{\rho_{ia}(X)h^s_a(Y,V_j)-\rho_{ia}(Y)h^s_a(X,V_j)\}\\
&+\sum_{k=1}^{r}\{h^\ell_k(X,V_j)\eta_i(A_{N_k}Y)-h^\ell_k(Y,V_j)\eta_i(A_{N_k}X)\}\\
&-\theta(X)h^*_i(FY,V_j)+\theta(Y)h^*_i(FX,V_j)\\
&+\beta(m-2\alpha)\{u_j(Y)v_i(X)-u_j(X)v_i(Y)\}\\
&+(\ell\beta-\alpha^2+\beta^2)\{u_j(Y)\eta_i(X)-u_j(X)\eta_i(Y)\}\\
=\;& f_2\{u_j(Y)\eta_i(X)-u_j(X)\eta_i(Y)+2\delta_{ij}\bar{g}(X,JY)\}.
\end{aligned}$$

Comparing this with (45) such that $PZ=V_j$ and using $(46)_2$, we obtain

$$\begin{aligned}
&\{f_1-f_2-\alpha^2+\beta^2\}\{u_j(Y)\eta_i(X)-u_j(X)\eta_i(Y)\}\\
=\;&2\alpha\beta\{u_j(Y)v_i(X)-u(_jX)v_i(Y)\}.
\end{aligned}$$

Taking $Y = U_j$, $X = \xi_i$ and $Y = U_j$, $X = V_i$ to this by turns, we have

$$f_1 - f_2 = \alpha^2 - \beta^2, \qquad \alpha\beta = 0.$$

Applying $\bar{\nabla}_X$ to $\theta(\zeta) = 1$ and using (7) and the fact: $\theta \circ J = 0$, we get

$$(\bar{\nabla}_X\theta)(\zeta) = -\ell\theta(X). \tag{47}$$

Applying $\bar{\nabla}_X$ to $\eta_i(Y) = \bar{g}(Y, N_i)$ and using (1) and (9), we have

$$\begin{aligned}(\nabla_X\eta_i)(Y) = & -g(A_{N_i}X, Y) + \sum_{j=1}^{r}\tau_{ij}(X)\eta_j(Y) \\ & - \{\ell\eta_i(X) + mv_i(X)\}\theta(Y).\end{aligned} \tag{48}$$

Applying ∇_X to $h_i^*(Y,\zeta) = (\ell+\beta)\eta_i(Y) + (m-\alpha)v_i(Y)$ and using (25), (26), (36), (48) and the fact that $\alpha\beta = 0$, we get

$$\begin{aligned}(\nabla_X h_i^*)(Y,\zeta) = & X(\ell+\beta)\eta_i(Y) + X(m-\alpha)v_i(Y) \\ & + (\ell+\beta)\{-g(A_{N_i}X, Y) - g(A_{N_i}Y, X) + \sum_{j=1}^{r}\tau_{ij}(X)\eta_j(Y) \\ & + \beta\theta(X)\eta_i(Y) - \ell[\theta(Y)\eta_i(X) + \theta(X)\eta_i(Y)] \\ & - m[\theta(Y)v_i(X) + \theta(X)v_i(Y)]\} \\ & + (m-\alpha)\{-g(A_{N_i}X, FY) - g(A_{N_i}Y, FX) \\ & + \sum_{j=1}^{r}v_j(Y)\tau_{ij}(X) + \sum_{a=r+1}^{n}\epsilon_a w_a(Y)\rho_{ia}(X) \\ & + \sum_{j=1}^{r}u_j(Y)\eta_i(A_{N_j}X) + (m-\alpha)\theta(Y)\eta_i(X) \\ & + \beta\theta(X)v_i(Y) - (\ell+\beta)\theta(Y)v_i(X)\}.\end{aligned}$$

Substituting this and $(29)_2$ into (45) with $PZ = \zeta$ and using (47), we get

$$\begin{aligned}& \{X\beta + (f_1 - f_3 - \alpha^2 + \beta^2)\theta(X)\}\eta_i(Y) \\ & - \{Y\beta + (f_1 - f_3 - \alpha^2 + \beta^2)\theta(Y)\}\eta_i(X) \\ & = (X\alpha)v_i(Y) - (Y\alpha)v_i(X).\end{aligned}$$

Taking $X = \zeta$, $Y = \xi_i$ and $X = U_j$, $Y = V_i$ to this by turns, we have

$$f_1 - f_3 = \alpha^2 - \beta^2 - \zeta\beta, \qquad U_j\alpha = 0.$$

Applying ∇_X to $h_i^\ell(Y,\zeta) = (m-\alpha)u_i(Y)$ and using (26) and (35), we get

$$\begin{aligned}(\nabla_X h_i^\ell)(Y,\zeta) = & X(m-\alpha)u_i(Y) - (\ell+\beta)h_i^\ell(Y, X) \\ & - (m-\alpha)\{\sum_{j=1}^{r}u_j(Y)\tau_{ji}(X) + \sum_{a=r+1}^{n}\epsilon_a w_a(Y)\lambda_{ai}(X) \\ & + h_i^\ell(X, FY) + h_i^\ell(Y, FX) + \ell\theta(Y)u_i(X)\end{aligned}$$

$$+ \ \beta[\theta(Y)u_i(X) - \theta(X)u_i(Y)]\}.$$

Substituting this into (44) with $Z = \zeta$ and using (20) and (47), we have

$$(X\alpha)u_i(Y) = (Y\alpha)u_i(X).$$

Taking $Y = U_i$ to this result and using the fact that $U_i\alpha = 0$, we have $X\alpha = 0$. Therefore α is a constant. This completes the proof of the theorem. □

Definition. (1) A screen distribution $S(TM)$ is said to be *totally umbilical* [3] in M if there exist smooth functions γ_i on a neighborhood $\mathcal{U}$ such that

$$h_i^*(X, PY) = \gamma_i g(X, PY).$$

In case $\gamma_i = 0$, we say that $S(TM)$ is *totally geodesic* in M.

(2) An r-lightlike submanifold M is said to be *screen conformal* [4] if there exist non-vanishing smooth functions φ_i on $\mathcal{U}$ such that

$$h_i^*(X, PY) = \varphi_i h_i^\ell(X, PY). \tag{49}$$

Theorem 4.2. *Let M be a generic lightlike submanifold of an indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ with an (ℓ, m)-type connection subject such that ζ is tangent to M. If*

(1) $S(TM)$ is totally umbilical, or

(2) M is screen conformal,

then $\bar{M}(f_1, f_2, f_3)$ is an indefinite β-Kenmotsu manifold with a semi-symmetric non-metric connection such that

$$\alpha = m = 0, \quad \beta = -\ell \neq 0, \qquad f_1 = -\beta^2, \quad f_2 = 0, \quad f_3 = -\zeta\beta.$$

Proof. (1) If $S(TM)$ is totally umbilical, then $(29)_2$ is reduced to

$$\gamma_i\theta(X) = (\ell + \beta)\eta_i(X) + (m - \alpha)v_i(X).$$

Taking $X = \zeta$, $X = \xi_i$ and $X = V_i$ to this equation by turns, we have

$$\gamma_i = 0, \qquad \ell = -\beta, \qquad m = \alpha, \tag{50}$$

respectively. As $\gamma_i = 0$, we obtain $h_i^* = 0$. Thus. from $(30)_{1,\,2}$, we have

$$h_j^\ell(X, U_i) = 0, \qquad h_a^s(X, U_i) = 0. \tag{51}$$

Replacing Y by U_j to (20) and using $(50)_1$ and the result: $m = \alpha$, we get

$$h_i^\ell(U_j, X) = \alpha\theta(X)\delta_{ij}.$$

Taking $X = \zeta$ to this and using $(27)_2$ such that $m = \alpha$, we have $\alpha = 0$.

As $\alpha = m = 0$ and $\beta = -\ell \neq 0$, $\bar{M}$ is an indefinite β-Kenmotsu manifold with a semi-symmetric non-metric connection and $f_1 + \beta^2 = f_2$ by Theorem 4.1. Taking $PZ = U_j$ to (45) and using $(46)_1$, (50) and (51), we have

$$f_2\{[v_j(Y)\eta_i(X) - v_j(X)\eta_i(Y)] + [v_i(Y)\eta_j(X) - v_i(X)\eta_j(Y)]\} = 0.$$

Taking $X = \xi_i$ and $Y = U_j$, we get $f_2 = 0$. Thus $f_1 = -\beta^2$ and $f_3 = -\zeta\beta$.

(2) If M is screen conformal, then, from $(28)_2$, $(29)_2$ and (49), we have

$$(\ell+\beta)\eta_i(X)+(m-\alpha)v_i(X)=\varphi_i(m-\alpha)u_i(X)\}.$$

Taking $X=\xi_i$ and $X=V_i$ to this equation by turns, we see that $\ell=-\beta$ and $m=\alpha$, respectively. As $\alpha\beta=0$, it follows that

$$\ell m=\ell\alpha=m\beta=0, \qquad \ell\beta=-\beta^2, \qquad m\alpha=\alpha^2. \tag{52}$$

Denote by μ_i the r-th vector fields on $S(TM)$ such that $\mu_i=U_i-\varphi_iV_i$. Using $(30)_{1,2,3,4}$ and (49), we see that

$$\begin{aligned} &h^\ell_j(X,\mu_i)=0, && h^s_a(X,\mu_i)=0,\\ &g(\mu_i,\mu_j)=-(\varphi_j+\varphi_i)\delta_{ij}, && J\mu_i=N_i-\varphi_i\xi_i. \end{aligned} \tag{53}$$

Applying ∇_X to $\mu_i=U_i-\varphi_iV_i$ and then, taking the scalar product with ζ to the resulting equation and using (31) and (32), we obtain

$$g(\nabla_X\mu_i,\zeta)=-\{\alpha\eta_i(X)+\beta v_i(X)-\varphi_i\beta u_i(X)\}.$$

Applying $\bar\nabla_X$ to $\theta(\mu_i)=0$ and using (8) and the last equation, we get

$$(\bar\nabla_X\theta)(\mu_i)=\alpha\eta_i(X)+\beta v_i(X)-\varphi_i\beta u_i(X). \tag{54}$$

Applying ∇_Y to (49), we have

$$(\nabla_X h^*_i)(Y,PZ)=(X\varphi_i)h^\ell_i(Y,PZ)+\varphi_i(\nabla_X h^\ell_i)(Y,PZ).$$

Substituting this and (49) into (45) and using (44), we have

$$\begin{aligned} &\sum_{j=1}^{r}\{(X\varphi_i)\delta_{ij}-\varphi_i\tau_{ji}(X)-\varphi_j\tau_{ij}(X)-\eta_i(A_{N_j}X)\}h^\ell_j(Y,PZ)\\ &-\sum_{j=1}^{r}\{(Y\varphi_i)\delta_{ij}-\varphi_i\tau_{ji}(Y)-\varphi_j\tau_{ij}(Y)-\eta_i(A_{N_j}Y)\}h^\ell_j(X,PZ)\\ &-\sum_{a=r+1}^{n}\{\epsilon_a\rho_{ia}(X)+\varphi_i\lambda_{ai}(X)\}h^s_a(Y,PZ)\\ &+\sum_{a=r+1}^{n}\{\epsilon_a\rho_{ia}(Y)+\varphi_i\lambda_{ai}(Y)\}h^s_a(X,PZ)\\ &-(\bar\nabla_X\theta)(PZ)\{\ell\eta_i(Y)+mv_i(Y)-\varphi_imu_i(Y)\}\\ &+(\bar\nabla_Y\theta)(PZ)\{\ell\eta_i(X)+mv_i(X)-\varphi_imu_i(X)\}\\ &-\theta(PZ)\{[X\ell+\alpha^2\theta(X)]\eta_i(Y)-[Y\ell+\alpha^2\theta(Y)]\eta_i(X)\\ &+(Xm)g(\mu_i,Y)-(Ym)g(\mu_i,X)\}\\ =\;&f_1\{g(Y,PZ)\eta_i(X)-g(X,PZ)\eta_i(Y)\}\\ &+f_2\{g(\mu_i,Y)\bar g(X,JPZ)-g(\mu_i,X)\bar g(Y,JPZ)+2g(\mu_i,PZ)\bar g(X,JY)\}\\ &+f_3\{\theta(X)\eta_i(Y)-\theta(Y)\eta_i(X)\}\theta(PZ). \end{aligned}$$

Replacing PZ by μ_j to this and using (46) and (52) ∼ (54), we obtain

$$\begin{aligned}(f_1+\beta^2)&\{v_j(Y)\eta_i(X)-v_j(X)\eta_i(Y)\}\\ -\ &\varphi_j(f_1+\beta^2)\{u_j(Y)\eta_i(X)-u_j(X)\eta_i(Y)\}\\ +\ &(f_2+\alpha^2)\{v_i(Y)\eta_j(X)-v_i(X)\eta_j(Y)\}\\ -\ &\varphi_i(f_2+\alpha^2)\{u_i(Y)\eta_j(X)-u_i(X)\eta_j(Y)\}\\ =\ &2f_2\delta_{ij}(\varphi_j+\varphi_i)\bar{g}(X,JY).\end{aligned}\tag{55}$$

Taking $X=\xi_i$ and $Y=V_j$ to this equation, we obtain

$$f_1+f_2=-(\alpha^2+\beta^2).$$

From this result and Theorem 5.1, we see that $\alpha=0$. As $\alpha=m=0$ and $\beta=-\ell\neq 0$, $\bar{M}(f_1,f_2,f_3)$ is an indefinite β-Kenmotsu manifold with a semi-symmetric non-metric connection. Taking $X=\xi_j$ and $Y=U_j$ to the modified equation (55) which is change j with i, we obtain $f_2\varphi_i=0$. As all φ_i are non-vanishing functions, we get $f_2=0$. Thus $f_1=-\beta^2$ and $f_3=-\zeta\beta$. □

Theorem 4.3. *Let M be a generic lightlike submanifold of an indefinite generalized Sasakian space form $\bar{M}(f_1,f_2,f_3)$ with an (ℓ,m)-type connection such that ζ is tangent to M. If U_i's or V_i's are parallel with respect to ∇, then $\bar{M}(f_1,f_2,f_3)$ is a flat manifold with an indefinite cosymplectic structure*;

$$\alpha=\beta=0,\qquad f_1=f_2=f_3=0.$$

Proof. (1) If U_i's are parallel with respect to the connection ∇, then we have the equations of (38). As $\alpha=\beta=0$, we get $f_1=f_2=f_3$ by Theorem 4.1. Applying ∇_Y to $(38)_5$ and using the fact that $\nabla_Y U_j=0$, we obtain

$$(\nabla_X h^*_i)(Y,U_j)=0.$$

Substituting this equation and (38) into (45) with $PZ=U_j$, we have

$$f_1\{v_j(Y)\eta_i(X)-v_j(X)\eta_i(Y)\}+f_2\{v_i(Y)\eta_j(X)-v_i(X)\eta_j(Y)\}=0.$$

Taking $X=\xi_i$ and $Y=V_j$ to this equation, we get $f_1+f_2=0$. Thus we see that $f_1=f_2=f_3=0$ and $\bar{M}$ is flat.

(2) If V_i's are parallel with respect to the connection ∇, then we have the equations in (39). As $\alpha=\beta=0$, $f_1=f_2=f_3$ by Theorem 4.1. Taking $Y=\xi_j$ and $Y=U_j$ to (20) by turns and using $(39)_{3,5}$, we have

$$h^\ell_i(\xi_j,X)=0,\qquad h^\ell_i(U_j,X)=m\theta(X)\delta_{ij}.$$

Using these two equations and (30), we see that

$$\begin{aligned}h^\ell_k(\xi_i,V_j)=0,\qquad h^s_a(\xi_i,V_j)=\epsilon_a h^\ell_j(\xi_i,W_a)=0,\\ h^\ell_k(U_j,V_j)=0,\qquad h^s_a(U_j,V_j)=\epsilon_a h^\ell_j(U_j,W_a)=0.\end{aligned}\tag{56}$$

From $(30)_1$ and $(39)_5$ and using the fact that $\nabla_X V_i=0$, we have

$$h^*_i(Y,V_j)=0.$$

Applying ∇_X to this equation and using the fact that $\nabla_X V_j = 0$, we have

$$(\nabla_X h_i^*)(Y, V_j) = 0.$$

Substituting the last two equations into (45) such that $PZ = V_j$, we get

$$\begin{aligned}
&\sum_{a=r+1}^{n} \epsilon_a\{\rho_{ia}(Y)h_a^s(X, V_j) - \rho_{ia}(X)h_a^s(Y, V_j)\} \\
&+ \sum_{k=1}^{r} \{h_k^\ell(X, V_j)\eta_i(A_{N_k} Y) - h_k^\ell(Y, V_j)\eta_i(A_{N_k} X)\} \\
= \; & f_1\{u_j(Y)\eta_i(X) - u_j(X)\eta_i(Y)\} + 2f_2\delta_{ij}\bar{g}(X, JY).
\end{aligned}$$

Taking $X = \xi_i$ and $Y = U_j$ to this equation and using (56), we obtain $f_1+2f_2 = 0$. It follows that $f_1 = f_2 = f_3 = 0$ and $\bar{M}$ is flat. □

References

[1] P. Alegre, D. E. Blair, and A. Carriazo, *Generalized Sasakian-space-forms*, Israel J. Math. **141** (2004), 157–183. https://doi.org/10.1007/BF02772217

[2] C. Călin, *Contributions to geometry of CR-submanifold*, Thesis, University of Iasi (Romania), 1998.

[3] K. L. Duggal and D. J. Jin, *Generic lightlike submanifolds of an indefinite Sasakian manifold*, Int. Electron. J. Geom. **5** (2012), no. 1, 108–119.

[4] D. H. Jin, *Indefinite trans-Sasakian manifold admitting an ascreen lightlike hypersurface*, Honam Math. J. **35** (2013), no. 4, 657–666. https://doi.org/10.5831/HMJ.2013.35.4.657

[5] ______, *Indefinite generalized Sasakian space form admitting a generic lightlike submanifold*, Bull. Korean Math. Soc. **51** (2014), no. 6, 1711–1726. https://doi.org/10.4134/BKMS.2014.51.6.1711

[6] ______, *Generic lightlike submanifolds of an indefinite trans-Sasakian manifold of a quasi-constant curvature*, Appl. Math. Sci., **9** (2015), no. 60, 2985–2997.

[7] ______, *Lightlike hypersurfaces of an indefinite trans-Sasakian manifold with an (ℓ, m)-type connection*, J. Korean Math. Soc. **55** (2018), no. 5, 1075–1089. https://doi.org/10.4134/JKMS.j170538

[8] D. H. Jin and J. W. Lee, *Generic lightlike submanifolds of an indefinite cosymplectic manifold*, Math. Probl. Eng. **2011** (2011), Art. ID 610986, 16 pp. https://doi.org/10.1155/2011/610986

[9] ______, *Generic lightlike submanifolds of an indefinite Kaehler manifold*, Inter. J. Pure and Appl. Math. **101** (2015), no. 4, 543–560.

[10] ______, *Generic lightlike submanifolds of an indefinite trans-Sasakian manifold with a non-metric ϕ-symmetric connection*, Ann. Math. Inform. **47** (2017), 85–107.

[11] J. A. Oubiña, *New classes of almost contact metric structures*, Publ. Math. Debrecen **32** (1985), no. 3-4, 187–193.

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