

SOME RESULTS ON p -DISTANCE AND SEQUENCE OF COMPLEX UNCERTAIN VARIABLES

SANTANU ROY, SANGEETA SAHA, AND BINOD CHANDRA TRIPATHY

ABSTRACT. In this paper, we introduce the notion of p -distance in a complex uncertain sequence space. By using the concepts of p -distance, we give some theorems of convergence. Also, in a complex uncertain sequence space, we develop some properties on convergence in measure.

1. Introduction

In real scenario, we often unexpectedly be faced the case that there are lack of data about the events. This is not only for economic or technical difficulties but also influenced by unexpected events. The unavailability of data makes it difficult to use probability distribution of events and so a domain expert is to be consulted to give the belief degree so for each event to happen while making decision. Liu [3] first solved those problem through uncertainty theory. The uncertainty theory has successfully been applied for investigations in different areas by You [11], Liu and Ha [4], You and Yan ([12], [13]) and many other. Later, Peng [7] developed a new idea on uncertainty theory which known as complex uncertainty theory. Chen et al. [1] introduced the convergence of complex uncertain sequence and it has been applied for investigation of different sequence space by many researcher like Tripathy and Nath [10], Tripathy and Dowari [9], Nath and Tripathy ([5], [6]), Datta and Tripathy [2], Saha et al. [8]. We have introduced the idea of p -distance in complex uncertain sequence and also p -distance convergence with some results. In addition, we have developed some properties on convergence in measure.

2. Preliminaries

In this section, we procure some basic definitions and theorems of uncertainty theory, those will be used in the paper.

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Definition 2.1 ([3]). Let \mathcal{L} be a σ -algebra on a nonempty set Γ . A set function \mathcal{M} is called an uncertain measure if it satisfies the following axioms:

Axiom 1. (Normality Axiom) $\mathcal{M}\{\Gamma\} = 1$;

Axiom 2. (Duality Axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any $\Lambda \in \mathcal{L}$;

Axiom 3. (Subadditivity Axiom) For every countable sequence of $\{\Lambda_j\} \in \mathcal{L}$, we have

$$\mathcal{M}\left\{\bigcup_{j=1}^{\infty} \Lambda_j\right\} \leq \sum_{j=1}^{\infty} \mathcal{M}\{\Lambda_j\}.$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space, and each element Λ in \mathcal{L} is called an event. In order to obtain uncertainty measure of compound event, a product uncertain measure is defined by Liu [3] as follows:

Axiom 4. (Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$, $k = 1, 2, \dots$. The product uncertainty measure \mathcal{M} is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \prod_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\},$$

where Λ_k are arbitrarily chosen events from \mathcal{L}_k for $k = 1, 2, \dots$, respectively.

Definition 2.2 ([3]). An uncertain variable ξ is a measurable function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set B of real numbers, the set $\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\}$ is an event.

Definition 2.3 ([1]). The complex uncertain sequence (ζ_n) is said to be convergent almost surely to ζ if there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that

$$\lim_{n \rightarrow \infty} \|\zeta_n - \zeta\| = 0$$

for every $\gamma \in \Lambda$.

Definition 2.4 ([1]). The complex uncertain sequence (ζ_n) is said to be convergent in measure to ζ if

$$\lim_{n \rightarrow \infty} \mathcal{M}\{\|\zeta_n - \zeta\| \geq \varepsilon\} = 0$$

for every $\varepsilon > 0$.

Definition 2.5 ([1]). Let $\zeta, \zeta_1, \zeta_2, \dots$ be complex uncertain variables with finite expected values. Then the complex uncertain sequence (ζ_n) is said to be convergent in mean to ζ if

$$\lim_{n \rightarrow \infty} E[\|\zeta_n - \zeta\|] = 0.$$

Now, we introduce some new definitions on complex uncertain variables by using the notion of p -distance, the p -distance metric space, p -distance convergence and p -Cauchy variable sequence.

Definition 2.6. Let ζ and τ be complex uncertain variables. Then the p -distance between ζ and τ is defined by

$$D_p(\zeta, \tau) = (E[\|\zeta - \tau\|^p])^{\frac{1}{p+1}}, \quad p > 0.$$

Example 2.1. Consider the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ with $\mathcal{M}\{\gamma_1\} = \frac{1}{2}$ and $\mathcal{M}\{\gamma_2\} = \frac{1}{2}$.

Define two complex uncertain variables as follows:

$$\zeta(\gamma) = \begin{cases} 4i, & \text{if } \gamma = \gamma_1 \\ 1, & \text{if } \gamma = \gamma_2, \end{cases}$$

$$\tau(\gamma) = \begin{cases} i, & \text{if } \gamma = \gamma_1 \\ 0, & \text{if } \gamma = \gamma_2. \end{cases}$$

Then

$$\|\zeta - \tau\|^p(\gamma) = \begin{cases} 3^p, & \text{if } \gamma = \gamma_1 \\ 1, & \text{if } \gamma = \gamma_2. \end{cases}$$

Thus we have,

$$\begin{aligned} E[\|\zeta_n - \tau\|^p] &= \int_0^1 1dx + \int_1^{3^p} \frac{1}{2}dx \\ &= 1 + \frac{1}{2}(3^p - 1). \end{aligned}$$

Therefore, $D_p(\zeta, \tau) = (E[\|\zeta - \tau\|^p])^{\frac{1}{p+1}} = (1 + \frac{1}{2}(3^p - 1))^{\frac{1}{p+1}}$.

Definition 2.7. Let \mathcal{U} be the set of all complex uncertain variables having finite expected values. Then the set \mathcal{U} with the p -distance \mathcal{D}_p is called a metric space of complex uncertain variables and is denoted by $(\mathcal{U}, \mathcal{D}_p)$.

Definition 2.8. Let $\zeta, \zeta_1, \zeta_2, \dots$ be complex uncertain variables defined on metric space $(\mathcal{U}, \mathcal{D}_p)$. Then the sequence of a complex uncertain variable (ζ_n) is said to be p -distance convergent to ζ if

$$\lim_{n \rightarrow \infty} \mathcal{D}_p(\zeta_n, \zeta) = 0.$$

Definition 2.9. Let (ζ_n) be a sequence of complex uncertain variables. Then the sequence of a complex uncertain variable (ζ_n) is called a p -Cauchy variable sequence if

$$\lim_{n, k \rightarrow \infty} \mathcal{D}_p(\zeta_n, \zeta_k) = 0$$

3. Main results

In this section, we establish some results on p -distance and p -distance convergence of complex uncertain sequence. Also, we develop some properties on convergence in measure on a complex uncertain sequence.

Theorem 3.1. Let ζ, τ, θ be complex uncertain variables and let $D_p(*, *)$ be the p -distance. Then

- (a) $D_p(\zeta, \tau) \geq 0$; (non-negativity)
- (b) $D_p(\zeta, \tau) = 0$; (identification)
- (c) $D_p(\zeta, \tau) = D_p(\tau, \zeta)$; (symmetry)
- (d) $D_p(\zeta, \theta) \leq D_p(\zeta, \tau) + D_p(\tau, \theta)$. (triangle inequality)

Proof. From Definition 2.6, Parts (a) and (c) follow immediately.

(b) If $\zeta = \tau$, then $D_p(\zeta, \tau) = 0$.

If $\zeta \neq \tau$, then there exists γ_k such that $\mathcal{M}(\gamma_k) > 0$ and $\zeta(\gamma_k) \neq \tau(\gamma_k)$. Then

$$\begin{aligned} D_p^{p+1}(\zeta, \tau) &= E[\|\zeta - \tau\|^p] \\ &= \int_0^\infty \mathcal{M}\{\|\zeta - \tau\|^p \geq x\} dx \\ &\geq \mathcal{M}\{\gamma_k\} \|\zeta(\gamma_k) - \tau(\gamma_k)\|^p > 0. \end{aligned}$$

Thus, $D_p(\zeta, \tau) > 0$.

(d) If either $D_p(\zeta, \tau)$ or $D_p(\tau, \theta)$ is zero, then the inequality holds.

Let $D_p(\zeta, \tau) > 0$ and $D_p(\tau, \theta) > 0$.

So for any $0 < \alpha < 1$, we have

$$\begin{aligned} D_p^{p+1}(\zeta, \theta) &= \int_0^\infty \mathcal{M}\{\|\zeta - \theta\|^p \geq x\} dx \\ &= \int_0^\infty \mathcal{M}\{\|\zeta - \theta\| \geq x^{\frac{1}{p}}\} dx \\ &\leq \int_0^\infty \mathcal{M}\{\|\zeta - \tau\| + \|\tau - \theta\| \geq x^{\frac{1}{p}}\} dx \\ &\leq \int_0^\infty \mathcal{M}\{\{\|\zeta - \tau\| \geq \alpha x^{\frac{1}{p}}\} \cup \{\|\tau - \theta\| \geq (1-\alpha)x^{\frac{1}{p}}\}\} dx \\ &\leq \int_0^\infty \mathcal{M}\{\{\|\zeta - \tau\| \geq \alpha x^{\frac{1}{p}}\}\} dx + \int_0^\infty \mathcal{M}\{\{\|\tau - \theta\| \geq (1-\alpha)x^{\frac{1}{p}}\}\} dx \\ &\leq \int_0^\infty \mathcal{M}\{\{\|\zeta - \tau\|^p \geq \alpha^p x\}\} dx + \int_0^\infty \mathcal{M}\{\{\|\tau - \theta\|^p \geq (1-\alpha)^p x\}\} dx \\ &= \frac{E[\|\zeta - \tau\|^p]}{\alpha^p} + \frac{E[\|\tau - \theta\|^p]}{(1-\alpha)^p} \\ &= \frac{D_p^{p+1}(\zeta, \tau)}{\alpha^p} + \frac{D_p^{p+1}(\tau, \theta)}{(1-\alpha)^p}. \end{aligned}$$

Setting

$$\begin{aligned} \alpha &= \frac{D_p(\zeta, \tau)}{D_p(\zeta, \tau) + D_p(\tau, \theta)}, \\ 1 - \alpha &= \frac{D_p(\tau, \theta)}{D_p(\zeta, \tau) + D_p(\tau, \theta)}. \end{aligned}$$

That is, $D_p(\zeta, \theta) \leq D_p(\zeta, \tau) + D_p(\tau, \theta)$. \square

Theorem 3.2. Let $(\mathcal{U}, \mathcal{D}_p)$ be a metric space of complex uncertain variables.

For any complex uncertain variables $\zeta, \tau, \theta \in \mathcal{U}$ and $\lambda \in \mathcal{R}$,

(i) $D_p(\zeta + \tau, \theta + \tau) = D_p(\zeta, \theta)$; (translation invariant)

(ii) $D_p(\lambda\zeta, \lambda\theta) = |\lambda|^{\frac{p}{p+1}} D_p(\zeta, \theta)$.

Proof. (i) $D_p(\zeta + \tau, \theta + \tau) = (E[\|(\zeta + \tau) - (\theta + \tau)\|^p])^{\frac{1}{p+1}}$

$$= (E[\|\zeta - \theta\|^p])^{\frac{1}{p+1}}$$

$$= D_p(\zeta, \theta)$$

(ii) $D_p(\lambda\zeta, \lambda\theta) = (E[\|\lambda\zeta - \lambda\theta\|^p])^{\frac{1}{p+1}}$

$$= (E[|\lambda|^p \|\zeta - \theta\|^p])^{\frac{1}{p+1}}$$

$$= |\lambda|^{\frac{p}{p+1}} (E[\|\zeta - \theta\|^p])^{\frac{1}{p+1}}$$

$$= |\lambda|^{\frac{p}{p+1}} D_p(\zeta, \theta). \quad \square$$

Theorem 3.3. Let (ζ_n) be a sequence of complex uncertain variables. Then the sequence (ζ_n) of complex uncertain variables is a p -Cauchy sequence if and only if $\lim_{n \rightarrow \infty} \mathcal{D}_p(\zeta_n, \zeta) = 0$,

Proof. Since

$$\begin{aligned} \lim_{n, k \rightarrow \infty} (\mathcal{D}_p(\zeta_n, \zeta) + \mathcal{D}_p(\zeta_k, \zeta)) &= \lim_{n \rightarrow \infty} \mathcal{D}_p(\zeta_n, \zeta) + \lim_{k \rightarrow \infty} \mathcal{D}_p(\zeta_k, \zeta) \\ &= 0 \end{aligned}$$

for any n, k , it follows from the triangle inequality that,

$$\begin{aligned} \lim_{n, k \rightarrow \infty} \mathcal{D}_p(\zeta_n, \zeta_k) &\leq \lim_{n \rightarrow \infty} \mathcal{D}_p(\zeta_n, \zeta) + \lim_{k \rightarrow \infty} \mathcal{D}_p(\zeta_k, \zeta) \\ &= 0. \end{aligned}$$

Therefore, $\lim_{n, k \rightarrow \infty} \mathcal{D}_p(\zeta_n, \zeta_k) = 0$. So, (ζ_n) is a p -Cauchy sequence.

Again, for any $j > 0$, there exists an integer $n_j > 0$ such that $E[\|\zeta_n - \zeta_k\|^p] \leq \frac{1}{4^j}$, $n, k \geq n_j$. We see by Axiom 3 that

$$\begin{aligned} E[\|\xi_{n_j} - \xi\|^p] &= E\left[\left\|\sum_{l \geq j} (\xi_{n_l} - \xi_{n_{l+1}})\right\|^p\right] \\ &= \int_0^{+\infty} \mathcal{M}\left\{\left\|\sum_{l \geq j} (\xi_{n_l} - \xi_{n_{l+1}})\right\| \geq x^{1/p}\right\} dx \\ &\leq \int_0^{+\infty} \mathcal{M}\left\{\left\|\bigcup_{l \geq j} (\xi_{n_l} - \xi_{n_{l+1}})\right\| \geq \frac{x^{1/p}}{2^l}\right\} dx \end{aligned}$$

$$\begin{aligned}
&\leq \int_0^{+\infty} \sum_{l \geq j} \mathcal{M} \left\{ \|\xi_{n_l} - \xi_{n_{l+1}}\| \geq \frac{x^{1/p}}{2^l} \right\} dx \\
&= \sum_{l \geq j} 2^{lp} \int_0^{+\infty} \mathcal{M} \{ \|\xi_{n_l} - \xi_{n_{l+1}}\|^p \geq x \} dx \\
&= \sum_{l \geq j} 2^{lp} E [\|\xi_{n_l} - \xi_{n_{l+1}}\|^p] \\
&\leq \sum_{l \geq j} \frac{2^p}{2^l} = \frac{2^p}{2^{(j-1)}}.
\end{aligned}$$

That is, $\lim_{j \rightarrow \infty} \mathcal{D}_p(\zeta_{n_j}, \zeta) = \lim_{j \rightarrow \infty} (E[\|\zeta_{n_j} - \zeta\|^p])^{\frac{1}{p+1}} = 0$.

Let $n_j, n \rightarrow \infty$, we have

$$\mathcal{D}_p(\zeta_n, \zeta) \leq \mathcal{D}(\zeta_n, \zeta_{n_j}) + \mathcal{D}(\zeta_{n_j}, \zeta) \rightarrow 0.$$

The proof is complete. \square

Theorem 3.4. Let (ζ_n) and (τ_n) be two sequences of complex uncertain variables defined on a metric space $(\mathcal{U}, \mathcal{D}_p)$. Then

- (i) If $\lim_{n \rightarrow \infty} \mathcal{D}_p(\zeta_n, \zeta) = 0$, then $\lim_{n \rightarrow \infty} \mathcal{D}_p(\|\zeta_n\|, \|\zeta\|) = 0$.
- (ii) The \mathcal{D}_p -limit of a sequence is unique.
- (iii) If $\lim_{n \rightarrow \infty} \mathcal{D}_p(\zeta_n, \zeta) = 0$ and $\lim_{n \rightarrow \infty} \mathcal{D}_p(\zeta_n, \tau) = 0$, then $\lim_{n \rightarrow \infty} \mathcal{D}_p(a\zeta_n + b\tau_n, a\zeta + b\tau) = 0$ for any real numbers a, b .

Proof. (i) It follows from the p -distance convergence that

$$\begin{aligned}
\lim_{n \rightarrow \infty} \mathcal{D}_p(\zeta_n, \zeta) &= \lim_{n \rightarrow \infty} (E[\|\zeta_n - \zeta\|^p])^{\frac{1}{p+1}} \\
&\geq \lim_{n \rightarrow \infty} (E[\|\|\zeta_n\| - \|\zeta\|\|^p])^{\frac{1}{p+1}} \\
&= \lim_{n \rightarrow \infty} \mathcal{D}_p(\|\zeta_n\|, \|\zeta\|).
\end{aligned}$$

Therefore, $\lim_{n \rightarrow \infty} \mathcal{D}_p(\|\zeta_n\|, \|\zeta\|) = 0$.

(ii) Suppose (ζ_n) be p -distance convergent to ζ and τ . Let $\varepsilon > 0$. Since (ζ_n) is p -distance converges to ζ , so there exists $n_1 \in \mathbb{N}$ such that $\mathcal{D}_p(\zeta_n, \zeta) < \frac{\varepsilon}{2}$ for all $n \geq n_1$.

Again (ζ_n) is p -distance converges to τ , so there exists $n_2 \in \mathbb{N}$ such that $\mathcal{D}_p(\zeta_n, \tau) < \frac{\varepsilon}{2}$ for all $n \geq n_2$. Taking $n_0 = \max\{n_1, n_2\}$.

It follows from the triangle inequality and for all $n \geq n_0$, we have

$$\begin{aligned}
\mathcal{D}_p(\zeta, \tau) &\leq \mathcal{D}_p(\zeta_n, \zeta) + \mathcal{D}_p(\zeta_n, \tau) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Hence the \mathcal{D}_p -limit is unique.

(iii) We have from the triangle inequality,

$$\begin{aligned} 0 &\leq \lim_{n \rightarrow \infty} \mathcal{D}_p(a\zeta_n + b\tau_n, a\zeta + b\tau) \\ &\leq \lim_{n \rightarrow \infty} \mathcal{D}_p(a\zeta_n + b\tau_n, a\zeta_n + b\tau) + \lim_{n \rightarrow \infty} \mathcal{D}_p(a\zeta_n + b\tau, a\zeta + b\tau) = 0. \end{aligned}$$

Therefore, $\lim_{n \rightarrow \infty} \mathcal{D}_p(a\zeta_n + b\tau_n, a\zeta + b\tau) = 0$. \square

Theorem 3.5. *Let (ζ_n) and (τ_n) be two sequences of complex uncertain variables convergent in p -distance to ζ, τ respectively. If there exist positive numbers M_1, M, N_1 and N such that $M_1 \leq \|\zeta_n\| \leq M$ and $N_1 \leq \|\tau_n\| \leq N$ for any n , then*

- (i) $\lim_{n \rightarrow \infty} \mathcal{D}_p(\zeta_n - \tau_n, \zeta - \tau) = 0$,
- (ii) $\lim_{n \rightarrow \infty} \mathcal{D}_p\left(\frac{\zeta_n}{\tau_n}, \frac{\zeta}{\tau}\right)$.

Proof. (i) It follows from the triangle inequality that

$$\begin{aligned} 0 &\leq \lim_{n \rightarrow \infty} \mathcal{D}_p(\zeta_n - \tau_n, \zeta - \tau) \\ &\leq \lim_{n \rightarrow \infty} \mathcal{D}_p(\zeta_n - \tau_n, \zeta_n - \tau) + \lim_{n \rightarrow \infty} \mathcal{D}_p(\zeta_n - \tau, \zeta - \tau) \\ &= \lim_{n \rightarrow \infty} (E[\|(\zeta_n - \tau_n) - (\zeta_n - \tau)\|^p])^{\frac{1}{p+1}} + \lim_{n \rightarrow \infty} (E[\|(\zeta_n - \tau) - (\zeta - \tau)\|^p])^{\frac{1}{p+1}} \\ &= \lim_{n \rightarrow \infty} (E[\|\tau_n - \tau\|^p])^{\frac{1}{p+1}} + \lim_{n \rightarrow \infty} (E[\|\zeta_n - \zeta\|^p])^{\frac{1}{p+1}} \\ &= \lim_{n \rightarrow \infty} \mathcal{D}_p(\tau_n, \tau) + \lim_{n \rightarrow \infty} \mathcal{D}_p(\zeta_n, \zeta) = 0. \end{aligned}$$

Therefore, $\lim_{n \rightarrow \infty} \mathcal{D}_p(\zeta_n - \tau_n, \zeta - \tau) = 0$.

(ii) Since $0 < M_1 \leq \|\zeta_n\| \leq M$, so we have $M_1 \leq \|\zeta\| \leq M$.

In a similar way, if $N_1 \leq \|\tau_n\| \leq N$, then $N_1 \leq \|\tau\| \leq N$.

It follows from the triangle inequality that,

$$\begin{aligned} 0 &\leq \lim_{n \rightarrow \infty} \mathcal{D}_p\left(\frac{\zeta_n}{\tau_n}, \frac{\zeta}{\tau}\right) \\ &\leq \lim_{n \rightarrow \infty} \mathcal{D}_p\left(\frac{\zeta_n}{\tau_n}, \frac{\zeta_n}{\tau}\right) + \lim_{n \rightarrow \infty} \mathcal{D}_p\left(\frac{\zeta_n}{\tau}, \frac{\zeta}{\tau}\right) \\ &= \lim_{i \rightarrow \infty} \left(E\left[\left\|\frac{\zeta_n}{\tau_n} - \frac{\zeta_n}{\tau}\right\|^p\right]\right)^{1/(p+1)} + \lim_{i \rightarrow \infty} \left(E\left[\left\|\frac{\zeta_n}{\tau} - \frac{\zeta}{\tau}\right\|^p\right]\right)^{1/(p+1)} \\ &= \lim_{i \rightarrow \infty} \left(E\left[\left\|\frac{\zeta_n(\tau - \tau_n)}{\tau\tau_n}\right\|^p\right]\right)^{1/(p+1)} + \lim_{i \rightarrow \infty} \left(E\left[\left\|\frac{(\zeta_n - \zeta)}{\tau}\right\|^p\right]\right)^{1/(p+1)} \\ &\leq \left(\frac{M}{N_1^2}\right)^{1/p+1} \lim_{n \rightarrow \infty} \mathcal{D}_p(\tau_n, \tau) + \left(\frac{1}{N_1}\right)^{1/(p+1)} \lim_{n \rightarrow \infty} \mathcal{D}_p(\zeta_n, \zeta) = 0. \end{aligned}$$

Thus $\lim_{n \rightarrow \infty} \mathcal{D}_p\left(\frac{\zeta_n}{\tau_n}, \frac{\zeta}{\tau}\right)$. \square

Theorem 3.6. *Suppose that (ζ_n) and (τ_n) be two sequences of complex uncertain variables converge in measure to ζ, τ , respectively. If there exist positive numbers M_1, M, N_1 and N such that $M_1 \leq \|\zeta_n\| \leq M$ and $N_1 \leq \|\tau_n\| \leq N$ for any n , then*

- (i) $\zeta_n + \tau_n$ converges in measure to $\zeta + \tau$.
- (ii) $\zeta_n - \tau_n$ converges in measure to $\zeta - \tau$.
- (iii) $\zeta_n \tau_n$ converges in measure to $\zeta \tau$.
- (iv) $\left\{ \frac{\zeta_n}{\tau_n} \right\}$ converges in measure to $\left\{ \frac{\zeta}{\tau} \right\}$.

Proof. (i) Since the complex uncertain sequence (ζ_n) converges in measure to ζ , it follows that there exists a subsequence (ζ_{i_k}) such that $\lim_{n \rightarrow \infty} \|\zeta_{i_k} - \zeta\| = 0$ almost surely. Since $M_1 \leq \|\zeta_n\| \leq M$, we have $M_1 \leq \|\zeta\| \leq M$. In a similar way, if $N_1 \leq \|\tau_n\| \leq N$, then $N_1 \leq \|\tau\| \leq N$.

We see by Axiom 3 that

$$\begin{aligned} \mathcal{M}\{\|(\zeta_n + \tau_n) - (\zeta + \tau)\| \geq \varepsilon\} &= \mathcal{M}\{\|(\zeta_n - \zeta) + (\tau_n - \tau)\| \geq \varepsilon\} \\ &\leq \mathcal{M}\left\{\|\zeta_n - \zeta\| \geq \frac{\varepsilon}{2}\right\} + \mathcal{M}\left\{\|\tau_n - \tau\| \geq \frac{\varepsilon}{2}\right\} \\ &\rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. Hence, (i) is proved.

(ii) We see by Axiom 3 that

$$\begin{aligned} \mathcal{M}\{\|(\zeta_n - \tau_n) - (\zeta - \tau)\| \geq \varepsilon\} &= \mathcal{M}\{\|(\zeta_n - \zeta) + (\tau - \tau_n)\| \geq \varepsilon\} \\ &\leq \mathcal{M}\left\{\|\zeta_n - \zeta\| \geq \frac{\varepsilon}{2}\right\} + \mathcal{M}\left\{\|\tau - \tau_n\| \geq \frac{\varepsilon}{2}\right\} \\ &\rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. Hence, (ii) is proved.

(iii) We see by Axiom 3 that

$$\begin{aligned} \mathcal{M}\{\|\zeta_n \tau_n - \zeta \tau\| \geq \varepsilon\} &= \mathcal{M}\{\|\zeta_n \tau_n - \zeta_n \tau + \zeta_n \tau - \zeta \tau\| \geq \varepsilon\} \\ &\leq \mathcal{M}\left\{\|\zeta_n \tau_n - \zeta_n \tau\| \geq \frac{\varepsilon}{2}\right\} + \mathcal{M}\left\{\|\zeta_n \tau - \zeta \tau\| \geq \frac{\varepsilon}{2}\right\} \\ &\leq \mathcal{M}\left\{M\|\tau_n - \tau\| \geq \frac{\varepsilon}{2}\right\} + \mathcal{M}\left\{N\|\zeta_n - \zeta\| \geq \frac{\varepsilon}{2}\right\} \\ &\leq \mathcal{M}\left\{\|\tau_n - \tau\| \geq \frac{\varepsilon}{2M}\right\} + \mathcal{M}\left\{N\|\zeta_n - \zeta\| \geq \frac{\varepsilon}{2N}\right\} \\ &\rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. Hence, (iii) is proved.

$$\begin{aligned} \mathcal{M}\left\{\left\|\frac{\zeta_n}{\tau_n} - \frac{\zeta}{\tau}\right\| \geq \varepsilon\right\} &= \mathcal{M}\left\{\left\|\frac{\zeta_n}{\tau_n} - \frac{\zeta_n}{\tau} + \frac{\zeta_n}{\tau} - \frac{\zeta}{\tau}\right\| \geq \varepsilon\right\} \\ &= \mathcal{M}\left\{\left\|\frac{\zeta_n(\tau - \tau_n)}{\tau \tau_n} + \frac{\zeta_n - \zeta}{\tau}\right\| \geq \varepsilon\right\} \end{aligned}$$

$$\begin{aligned} &\leq \mathcal{M} \left\{ \left\| \frac{\zeta_n(\tau - \tau_n)}{\tau\tau_n} \right\| \geq \frac{\varepsilon}{2} \right\} + \mathcal{M} \left\{ \left\| \frac{\zeta_n - \zeta}{\tau} \right\| \geq \frac{\varepsilon}{2} \right\} \\ &\leq \mathcal{M} \left\{ \frac{M}{N_1^2} \|\tau_n - \tau\| \geq \frac{\varepsilon}{2} \right\} + \mathcal{M} \left\{ \frac{1}{N_1} \|\zeta_n - \zeta\| \geq \frac{\varepsilon}{2} \right\} \\ &\leq \mathcal{M} \left\{ \|\tau_n - \tau\| \geq \frac{N_1^2 \varepsilon}{2M} \right\} + \mathcal{M} \left\{ \|\zeta_n - \zeta\| \geq \frac{N_1 \varepsilon}{2} \right\} \\ &\rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$.

Hence, (iv) is proved. \square

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SANTANU ROY
 DEPARTMENT OF MATHEMATICS
 NATIONAL INSTITUTE OF TECHNOLOGY SILCHAR
 788010, INDIA
 Email address: santanuroy79@yahoo.in

SANGEETA SAHA
DEPARTMENT OF MATHEMATICS
NATIONAL INSTITUTE OF TECHNOLOGY SILCHAR
788010, INDIA
Email address: sahasangeeta99@gmail.com

BINOD CHANDRA TRIPATHY
DEPARTMENT OF MATHEMATICS
TRIPURA UNIVERSITY
799022, INDIA
Email address: tripathybc@yahoo.com, tripathybc@rediffmail.com