

MORE EXPANSION FORMULAS FOR q, ω -APOSTOL BERNOULLI AND EULER POLYNOMIALS

THOMAS ERNST

ABSTRACT. The purpose of this article is to continue the study of q, ω -special functions in the spirit of Wolfgang Hahn from the previous papers by Annaby et al. and Varma et al., with emphasis on q, ω -Apostol Bernoulli and Euler polynomials, Ward- ω numbers and multiple q, ω -power sums. Like before, the q, ω -module for the alphabet of q, ω -real numbers plays a crucial role, as well as the q, ω -rational numbers and the Ward- ω numbers. There are many more formulas of this type, and the deep symmetric structure of these formulas is described in detail.

1. Introduction

Based on our previous papers on pure q -calculus [3] and [4], this paper is part of a series of five papers on q, ω -calculus. In each paper we start with many similar definitions, since the subject is quite new. Let $\omega \in \mathbb{R}$, $0 < \omega < 1$. Put $\omega_0 \equiv \frac{\omega}{1-q}$, $0 < q < 1$. We introduce a new calculus, which will be very similar to the well-known q -calculus, where many functions and operators appear again, with a similar name. The reason is that the q, ω -Appell polynomials form a ring, which is proved in one of these papers [7]. The convergence region in ω will always be a small interval above 0, depending on q . The subtle properties of absolute maximum for the two q, ω -additions are exemplified in [6].

The paper is organized as follows: In Section 1 we present preliminary definitions and theorems for q, ω -analogues, like the q, ω -difference operator. In Section 2 we define the four q, ω -additions, natural generalizations of the four q -additions, and point out that they obey identical laws. It is an intriguing fact that the q, ω -difference operator of the q, ω -addition is identical with the corresponding q -analogue, a formula that seems to be new. The q, ω -addition formulas and q, ω -differences for the q, ω -exponential functions are also identical. In Section 3 we investigate general q, ω -Appell polynomials in the spirit of [5].

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In Section 4 the two multiple alternating q, ω -power sums are defined together with some special cases. In Subsection 4.1 formulas containing q, ω -power sums in one dimension, q, ω -analogues of Wang and Wang, [15] are proved. Then in Subsection 4.2, mixed formulas of the same kind are proved.

Definition. The automorphism ϵ on the vector space of polynomials is defined by

$$(1) \quad \epsilon f(x) \equiv f(qx + \omega).$$

This automorphism is a generalization of the operator with the same name in q -calculus [2]. In [1, p. 136] it is proved that

$$(2) \quad \epsilon^k f(x) = f(q^k x + \omega\{k\}_q).$$

Definition. A q, ω -analogue of the mathematical object G is a mathematical function $F(q, \omega)$, with the property $\lim_{\omega \rightarrow 0} F(q, \omega) = G_q$, the q -analogue of G . Both F and G can depend on more, common variables. They can also be operators.

Definition. Let φ be a continuous real function of x . Then we define the q, ω -difference operator $D_{q, \omega}$ as follows:

$$(3) \quad D_{q, \omega}(\varphi)(x) \equiv \begin{cases} \frac{\varphi(qx + \omega) - \varphi(x)}{(q-1)x + \omega}, & \text{if } x \neq \omega_0; \\ \frac{d\varphi}{dx}(x) & \text{if } x = \omega_0. \end{cases}$$

We say that a function $f(x)$ is n times q, ω -differentiable if $D_{q, \omega}^n f(x)$ exists. If we want to point out that this operator operates on the variable x , we write $D_{q, \omega, x}$ for the operator. Furthermore, $D_{q, \omega}(K) = 0$, like for the derivative.

Furthermore, we need the following chain rule:

Definition.

$$(4) \quad D_{q, \omega}(\epsilon^k \varphi)(x) \equiv q^k \frac{\epsilon^{k+1} \varphi(x) - \epsilon^k \varphi(x)}{(q-1)x + \omega}.$$

The motivation for formula (4) is that it is identical with the q -calculus case and enables smooth proofs of the following formulas, like the Leibniz formula. It also follows from the chain rule (11).

Theorem 1.1. *The q, ω -difference operator is linear*

$$(5) \quad D_{q, \omega} \sum_{k=0}^{\infty} a_k f_k(x) = \sum_{k=0}^{\infty} a_k D_{q, \omega} f_k(x).$$

Theorem 1.2 ([1, (16), p. 137]). *The q, ω -difference operator for a product of functions.*

$$(6) \quad D_{q, \omega}(fg)(x) = D_{q, \omega}(f(x))g(x) + f(qx + \omega)D_{q, \omega}(g(x)).$$

We now introduce two basic sequences, which generalize the Ciglerian polynomials in [2, 5.5].

Definition.

$$(7) \quad (x)_{q,\omega}^k \equiv \prod_{m=0}^{k-1} (x - \omega\{m\}_q). \quad [14, (16)]$$

$$(8) \quad [x]_{q,\omega}^k \equiv \prod_{m=0}^{k-1} (q^m x + \omega\{m\}_q). \quad [14, (15)]$$

The following names will be used for the ensuing q, ω -trigonometric and hyperbolic functions [6].

Definition. A function f of two variables x, ω is called x, ω -even if $f(-x, -\omega) = f(x, \omega)$. A function f of two variables x, ω is called x, ω -odd if $f(-x, -\omega) = -f(x, \omega)$.

Lemma 1.3. *Products and sums of any number of x, ω -even functions are x, ω -even. The product and quotient of an x, ω -even function and an x, ω -odd function are x, ω -odd.*

Lemma 1.4. *The two functions $(x)_{q,\omega}^{2k}$ and $[x]_{q,\omega}^{2k}$ are x, ω -even. The two functions $(x)_{q,\omega}^{2k+1}$ and $[x]_{q,\omega}^{2k+1}$ are x, ω -odd.*

The two following formulas correspond to the formula $Dx^n = nx^{n-1}$:

$$(9) \quad D_{q,\omega}(x)_{q,\omega}^n = \{n\}_q (x)_{q,\omega}^{n-1}. \quad [9, 2.5], [14, (17)]$$

$$(10) \quad D_{q,\omega}[x]_{q,\omega}^n = \{n\}_q [qx + \omega]_{q,\omega}^{n-1}. \quad [14, (18)]$$

Theorem 1.5. *The chain rule for the q, ω -difference operator.*

$$(11) \quad D_{q,\omega}((ax)_{q,a\omega}^n) = a\{n\}_q (ax)_{q,a\omega}^{n-1}.$$

$$(12) \quad D_{q,\omega}([ax]_{q,a\omega}^n) = a\{n\}_q [aqx + a\omega]_{q,a\omega}^{n-1}.$$

Proof. We prove (11) by induction. The formula (11) is true for $n = 1, 2$. Assume that it is true for $n - 1$. Then we have

$$(13) \quad \begin{aligned} & D_{q,\omega} [(ax)_{q,a\omega}^{n-1} (ax - \{n-1\}_q a\omega)] \\ & \stackrel{\text{by(6)}}{=} a(ax)_{q,a\omega}^{n-1} + a^2 [qx + \omega - \{n-1\}_q] \{n-1\}_q (ax)_{q,a\omega}^{n-2} \\ & = a(ax)_{q,a\omega}^{n-1} [1 + q\{n-1\}_q] = \text{RHS}. \end{aligned}$$

Formula (12) is proved in a similar style. \square

We next introduce two q, ω -analogues of the exponential function:

Definition. The q, ω -exponential function $E_{q,\omega}(z)$ [14, (21)] is defined by

$$(14) \quad E_{q,\omega}(z) \equiv \sum_{k=0}^{\infty} \frac{(z)_{q,\omega}^k}{\{k\}_q!}, \quad |(1-q)z - \omega| < 1.$$

The complementary q, ω -exponential function $E_{\frac{1}{q}, \omega}(z)$ [14, (26)] is defined by

$$(15) \quad E_{\frac{1}{q}, \omega}(z) \equiv \sum_{k=0}^{\infty} \frac{[z]_{q, \omega}^k}{\{k\}_q!}, \quad |\omega| < 1.$$

We have changed the name to $E_{\frac{1}{q}, \omega}(z)$ since $E_{\frac{1}{q}, 0}(z) = E_{\frac{1}{q}}(z)$ [2].

Theorem 1.6 ([14, (19)]). *The q, ω -exponential function is the unique solution of the first order initial value problem*

$$(16) \quad D_{q, \omega} f(z) = f(z), \quad f(0) = 1. \quad [14, (24)]$$

The complementary q, ω -exponential function is the unique solution of the first order initial value problem

$$(17) \quad D_{q, \omega} f(z) = f(qz + \omega), \quad f(0) = 1.$$

Theorem 1.7 ([14, (21)]). *The meromorphic continuation of the q, ω -exponential function $E_{q, \omega}(z)$ is given by*

$$(18) \quad E_{q, \omega}(z) = \frac{(-\omega; q)_{\infty}}{((1-q)z - \omega; q)_{\infty}}. \quad [14, (26)]$$

The meromorphic continuation of the complementary q, ω -exponential function $E_{\frac{1}{q}, \omega}(z)$ is given by

$$(19) \quad E_{\frac{1}{q}, \omega}(z) = \frac{((q-1)z + \omega; q)_{\infty}}{(\omega; q)_{\infty}}.$$

2. On the q, ω -addition with applications to q, ω -special functions

In order to use these functions, we need to generalize the q -addition. The ordinary q -addition is the special case $\omega = 0$. Just like for the q -addition, we use letters in an alphabet for the q, ω -additions. Equality between letters is denoted by \sim . In the following, beware of the fact that whenever we multiply the function argument x in $(x)_{q, \omega}^{\nu}$ or in $[x]_{q, \omega}^{\nu}$ by the constant a , we must also multiply ω by a .

Definition. Let $\{f_{\nu}\}_{\nu=0}^{\infty}$ denote an arbitrary sequence of real numbers. The q, ω -addition for the sequences $(x)_{q, \omega}^k$ is defined by

$$(20) \quad (f \oplus_q (x)_{q, \omega})^{\nu} \equiv \sum_{k=0}^{\nu} \binom{\nu}{k}_q f_{\nu-k} (x)_{q, \omega}^k.$$

The NWA q, ω -addition is defined as follows:

$$(21) \quad (x \oplus_{q, \omega} y)^n \equiv \sum_{k=0}^n \binom{n}{k}_q (x)_{q, \omega}^{n-k} (y)_{q, \omega}^k.$$

The NWA q, ω -subtraction is defined as follows:

$$(22) \quad (x \ominus_{q, \omega} y)^n \equiv \sum_{k=0}^n \binom{n}{k}_q (x)_{q, \omega}^{n-k} (-y)_{q, -\omega}^k.$$

The JHC q, ω -addition is defined as follows:

$$(23) \quad (x \boxplus_{q, \omega} y)^n \equiv \sum_{k=0}^n \binom{n}{k}_q (x)_{q, \omega}^{n-k} [y]_{q, \omega}^k.$$

The JHC q, ω -subtraction is defined as follows:

$$(24) \quad (x \boxminus_{q, \omega} y)^n \equiv \sum_{k=0}^n \binom{n}{k}_q (x)_{q, \omega}^{n-k} [-y]_{q, -\omega}^k.$$

Theorem 2.1. *The NWA q, ω -addition is commutative and associative.*

Proof. Similar to the proof for NWA q -addition. \square

Corollary 2.2. *Four q, ω -additions for the q, ω -exponential function.*

$$(25) \quad E_{q, \omega}(x \oplus_{q, \omega} y) \equiv E_{q, \omega}(x)E_{q, \omega}(y).$$

$$(26) \quad E_{q, \omega}(x \ominus_{q, \omega} y) \equiv E_{q, \omega}(x)E_{q, -\omega}(-y).$$

$$(27) \quad E_{q, \omega}(x \boxplus_{q, \omega} y) \equiv E_{q, \omega}(x)E_{\frac{1}{q}, \omega}(y).$$

$$(28) \quad E_{q, \omega}(x \boxminus_{q, \omega} y) \equiv E_{q, \omega}(x)E_{\frac{1}{q}, -\omega}(-y).$$

Definition. A q, ω -groupoid $(G_{q, \omega}, \oplus_{q, \omega}, \sim)$ is a set of letters with an associative and commutative mapping $\oplus_{q, \omega} : G_{q, \omega} \times G_{q, \omega} \mapsto G_{q, \omega}$. The associativity can be expressed as follows:

$$(29) \quad (a \oplus_{q, \omega} b) \oplus_{q, \omega} c \sim a \oplus_{q, \omega} (b \oplus_{q, \omega} c), a, b, c \in G_{q, \omega}.$$

Theorem 2.3 (Compare with [13, p. 39]). *In a q, ω -groupoid, all composite operations represent the same element. It is denoted by $\oplus_{q, \omega, l=0}^{j-1} a_l$, $\{a_l\}_{l=0}^{j-1} \in G_{q, \omega}$.*

Definition. A q, ω -module is a generalization of the vector space over a field, where the corresponding scalars belong to \mathbb{R} . In a q, ω -module we can multiply letters $\alpha \in \mathbb{Q}_{\oplus_{q, \omega}} \vee \mathbb{R}_{q, \omega}$ with scalars $b \in \mathbb{R}$ to form the letters $\gamma \in \mathbb{R}_{q, \omega}$:

$$(30) \quad \gamma \sim b\alpha.$$

This operation is distributive over the q, ω -addition:

$$(31) \quad b(\alpha \oplus_{q, \omega} \beta) \sim b\alpha \oplus_{q, \omega} b\beta, b \in \mathbb{R}, \alpha, \beta \in \mathbb{R}_{q, \omega}.$$

The operations (30) and (31), as well as similar formulas for q, ω -modules are used in the rest of the article without further explanation.

Theorem 2.4. *The q, ω -differences for the q, ω -exponential functions are:*

$$(32) \quad D_{q,\omega} E_{q,a\omega}(ax) = a E_{q,a\omega}(ax),$$

$$(33) \quad D_{q,\omega} E_{\frac{1}{q},a\omega}(ax) = a E_{\frac{1}{q},a\omega}(aqx + a\omega),$$

Proof. This follows from the chain rule (11) and (12). \square

In our second book [5] we introduced several new q -deformed number systems, semiring, biring etc., each with an extra index q . We can extend these number systems by adding another index ω . The proofs will be very similar, and we just state the definitions and corresponding theorems.

Definition. The Ward- ω number $\bar{n}_{q,\omega}$ is defined by

$$(34) \quad \bar{n}_{q,\omega} \sim 1 \oplus_{q,\omega} 1 \oplus_{q,\omega} \cdots \oplus_{q,\omega} 1,$$

where the number of 1 on the RHS is n .

Definition (An extension of [2, 4.70]).

$$(35) \quad (\bar{n}_{q,\omega})^k \equiv \sum_{m_1 + \cdots + m_n = k} \binom{k}{m_1, \dots, m_n}_q \prod_{i=1}^n (1)_{q,\omega}^{m_i},$$

where each partition of k is multiplied with its number of permutations. We have the following special cases:

$$(36) \quad (\bar{0}_{q,\omega})^k = \delta_{k,0}; \quad (\bar{n}_{q,\omega})^0 = 1; \quad (\bar{n}_{q,\omega})^1 = n.$$

Let $(\mathbb{N}_{\oplus_{q,\omega}}, \oplus_{q,\omega}, \odot_{q,\omega})$ denote the semiring of Ward- ω numbers $\bar{k}_{q,\omega}$, $k \geq 0$ together with two binary operations: $\oplus_{q,\omega}$ is the usual q, ω -addition. The multiplication $\odot_{q,\omega}$ is defined as follows:

$$(37) \quad \bar{n}_{q,\omega} \odot_{q,\omega} \bar{m}_{q,\omega} \sim \bar{n}\bar{m}_{q,\omega},$$

where \sim denotes the equivalence in the alphabet. In long formulas, the q, ω -multiplication is abbreviated by juxtaposition.

Theorem 2.5. *Functional equations for Ward- ω numbers operating on the q, ω -exponential function. First assume that the letters $\bar{m}_{q,\omega}$ and $\bar{n}_{q,\omega}$ are independent, i.e., come from two different functions, when operating with the functional. Furthermore, $m, n < \frac{1+\omega}{1-q}$. Then we have*

$$(38) \quad E_{q,\omega}(\bar{m}_{q,\omega} \bar{n}_{q,\omega} t) = E_{q,\omega}(\bar{m}\bar{n}_{q,\omega} t).$$

Furthermore,

$$(39) \quad E_{q,\omega}(\bar{j}\bar{m}_{q,\omega}) = E_{q,\omega}(\bar{j}_{q,\omega})^m = E_{q,\omega}(\bar{m}_{q,\omega})^j = E_{q,\omega}(\bar{j}_{q,\omega} \odot_{q,\omega} \bar{m}_{q,\omega}).$$

Definition. Let the q, ω -rational numbers $\mathbb{Q}_{q,\omega}$ be defined as follows:

$$(40) \quad \mathbb{Q}_{q,\omega} \equiv \left\{ \frac{\bar{m}_{q,\omega}}{\bar{n}_{q,\omega}}, m \in \mathbb{N} \cup \{0\}, n \in \mathbb{N}, m \neq n, \frac{\bar{0}_{q,\omega}}{\bar{n}_{q,\omega}} \sim \theta, \frac{\bar{n}_{q,\omega}}{\bar{n}_{q,\omega}} \sim 1 \right\},$$

together with a linear functional

$$(41) \quad v, \mathbb{R}[x] \times \mathbb{Q}_{\oplus_{q,\omega}} \rightarrow \mathbb{R},$$

called the evaluation. If $v(x) = \sum_{k=0}^n a_k x^k$, then

$$(42) \quad v\left(\frac{\overline{m}_{q,\omega}}{\overline{n}_{q,\omega}}\right) \equiv \sum_{k=0}^n a_k \frac{(\overline{m}_{q,\omega})^k}{(\overline{n}_{q,\omega})^k}.$$

3. q, ω -Appell polynomials

The most general form of polynomial in this article is the Hahn–Appell polynomial, which we will now define.

Definition. Let $\mathcal{A}_{q,\omega}$ denote the set of real sequences $\{u_{\nu,q}\}_{\nu=0}^{\infty}$ such that

$$(43) \quad \sum_{\nu=0}^{\infty} |u_{\nu,q}| \frac{r^{\nu}}{\{\nu\}_q!} < \infty$$

for some q, ω -dependent convergence radius $r = r(q) > 0$, where $0 < q < 1$.

The q, ω -Appell number sequence is denoted by $\{\Phi_{\nu,q,\omega}^{(n)}\}_{\nu=0}^{\infty}$.

Definition. Assume that $h(t, q, \omega), h(t, q, \omega)^{-1} \in \mathbb{R}[[t]]$. For $f_n(t, q, \omega) = h(t, q, \omega)^n$, the multiplicative q, ω -Appell numbers of degree ν and order n $\Phi_{\nu,q,\omega} \in \mathcal{A}_{q,\omega}$ are given by the generating function

$$(44) \quad f_n(t, q, \omega) = \sum_{\nu=0}^{\infty} \frac{t^{\nu}}{\{\nu\}_q!} \Phi_{\nu,q,\omega}^{(n)}, \quad \Phi_{0,q,\omega}^{(n)} = 1.$$

It will be convenient to fix the value for $n = 0$ and $n = 1$:

$$(45) \quad \Phi_{\nu,q,\omega}^{(0)} \equiv \delta_{0,\nu}, \quad \Phi_{\nu,q,\omega}^{(1)} \equiv \Phi_{\nu,q,\omega}.$$

Definition (Compare with [14, (31)]). For $f_n(t, q, \omega) \in \mathbb{R}[[t]]$ as above, the multiplicative q, ω -Appell polynomial sequence $\{\Phi_{\nu;q,\omega}^{(n)}(x)\}_{\nu=0}^{\infty}$ of degree ν and order n is defined by the generating function

$$(46) \quad f_n(t, q, \omega) \mathbb{E}_{q,\omega t}(xt) = \sum_{\nu=0}^{\infty} \frac{t^{\nu}}{\{\nu\}_q!} \Phi_{\nu;q,\omega}^{(n)}(x).$$

It will be convenient to fix the value for $n = 0$ and $n = 1$:

Theorem 3.1.

$$(47) \quad \Phi_{\nu,q,\omega}^{(0)}(x) = (x)_{q,\omega}^{\nu}, \quad \Phi_{\nu,q,\omega}^{(1)}(x) \equiv \Phi_{\nu,q,\omega}(x).$$

We prove the first equation.

Proof. By using the linearity of $D_{q,\omega}$, (5), and the q, ω Taylor formula [7], it would suffice to prove that

$$D_{q,\omega,x}^k (xt)_{q,\omega t}^k = t^k \{k\}_q!.$$

But this is obvious by the chain rule (11). \square

Definition. For $f_n(t, q, \omega) \in \mathbb{R}[[t]]$ as above, the complementary, multiplicative q, ω -Appell polynomial sequence $\{\Phi_{\nu; \frac{1}{q}, \omega}^{(n)}(x)\}_{\nu=0}^{\infty}$ of degree ν and order n is defined by the generating function

$$(48) \quad f_n(t, q, \omega) E_{\frac{1}{q}, \omega t}(xt) = \sum_{\nu=0}^{\infty} \frac{t^{\nu}}{\{\nu\}_q!} \Phi_{\nu; \frac{1}{q}, \omega}^{(n)}(x).$$

It will be convenient to fix the value for $n = 0$ and $n = 1$:

Definition.

$$(49) \quad \Phi_{\nu; \frac{1}{q}, \omega}^{(0)}(x) \equiv [x]_{q, \omega}^{\nu}, \quad \Phi_{\nu; \frac{1}{q}, \omega}^{(1)}(x) \equiv \Phi_{\nu; \frac{1}{q}, \omega}(x).$$

We next present generalizations of the three formulas [2, 4.107, 4.108, 4.111].

Theorem 3.2.

$$(50) \quad D_{q, \omega} \Phi_{\nu; q, \omega}(x) = \{\nu\}_q \Phi_{\nu-1; q, \omega}(x).$$

[14, (30)] in umbral form:

$$(51) \quad \Phi_{\nu; q, \omega}(x) \doteq (\Phi_{q, \omega} \oplus_{q, \omega} x)^{\nu}.$$

Theorem 3.3. Assume that M and K are the x -order and y -order, respectively. Then we have:

$$(52) \quad \Phi_{\nu; q, \omega}^{(M+K)}(x \oplus_{q, \omega} y) = \sum_{k=0}^{\nu} \binom{\nu}{k}_q \Phi_{k; q, \omega}^{(M)}(x) \Phi_{\nu-k; q, \omega}^{(K)}(y).$$

Proof. We show that both sides of (52) have the same generating function.

$$(53) \quad \begin{aligned} f_{M+K}(t, q, \omega) E_{q, \omega t}((x \oplus_{q, \omega} y)t) &\stackrel{\text{by(25)}}{=} f_M(t, q, \omega) E_{q, \omega t}(xt), \\ &\stackrel{\text{by(46)}}{=} \sum_{k=0}^{\infty} \frac{t^k}{\{k\}_q!} \Phi_{k; q, \omega}^{(M)}(x) \sum_{l=0}^{\infty} \frac{t^l}{\{l\}_q!} \Phi_{l; q, \omega}^{(K)}(y) \\ &= \sum_{\nu=0}^{\infty} \frac{t^{\nu}}{\{\nu\}_q!} \sum_{k=0}^{\nu} \binom{\nu}{k}_q \Phi_{k; q, \omega}^{(M)}(x) \Phi_{\nu-k; q, \omega}^{(K)}(y). \quad \square \end{aligned}$$

Remark 3.4. Formula (52) defines $\Phi_{\nu; q, \omega}^{(M+K)}(x \oplus_{q, \omega} y)$ as the right hand side of the formula. There is no other definition of this function.

Theorem 3.5. Assume that M and K are the x -order and y -order, respectively. Then we have:

$$(54) \quad \Phi_{\nu; q, \omega}^{(M+K)}(x \boxplus_{q, \omega} y) = \sum_{k=0}^{\nu} \binom{\nu}{k}_q \Phi_{k; q, \omega}^{(M)}(x) \Phi_{\nu-k; \frac{1}{q}, \omega}^{(K)}(y).$$

Proof. We show that both sides of (54) have the same generating function.

$$\begin{aligned}
f_{M+K}(t, q, \omega) E_{q, \omega t}((x \boxplus_{q, \omega} y)t) &\stackrel{\text{by(27)}}{=} f_M(t, q, \omega) E_{q, \omega t}(xt) f_K(t, \frac{1}{q}, \omega), \\
E_{\frac{1}{q}, \omega t}(yt) &\stackrel{\text{by(46), (48)}}{=} \sum_{k=0}^{\infty} \frac{t^k}{\{k\}_q!} \Phi_{k; q, \omega}^{(M)}(x) \sum_{l=0}^{\infty} \frac{t^l}{\{l\}_q!} \Phi_{l; \frac{1}{q}, \omega}^{(K)}(y) \\
(55) \qquad &= \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \sum_{k=0}^{\nu} \binom{\nu}{k}_q \Phi_{k; q, \omega}^{(M)}(x) \Phi_{\nu-k; \frac{1}{q}, \omega}^{(K)}(y). \quad \square
\end{aligned}$$

Theorem 3.6 (A q, ω -analogue of [12, p. 125]).

$$(56) \quad (E_{q, \omega}(t) - 1) f_n(t, q, \omega) E_{q, \omega t}(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \Delta_{\text{NWA}, q, \omega} \Phi_{\nu, q, \omega}^{(n)}(x).$$

Proof. Operate on (46) with $\Delta_{\text{NWA}, q, \omega}$. □

Theorem 3.7 (A q, ω -analogue of [12, p. 125]).

$$(57) \quad \frac{(E_{q, \omega}(t) + 1)}{2} f_n(t, q, \omega) E_{q, \omega t}(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \nabla_{\text{NWA}, q, \omega} \Phi_{\nu, q, \omega}^{(n)}(x).$$

Proof. Operate on (46) with $\nabla_{\text{NWA}, q, \omega t}$. □

Theorem 3.8 (A q, ω -analogue of [12, p. 125]).

$$(58) \quad (E_{\frac{1}{q}, \omega}(t) - 1) f_n(t, q, \omega) E_{q, \omega t}(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \Delta_{\text{JHC}, q, \omega} \Phi_{\nu, q, \omega}^{(n)}(x).$$

Proof. Operate on (46) with $\Delta_{\text{JHC}, q, \omega t}$. □

Theorem 3.9 (A q, ω -analogue of [12, p. 125]).

$$(59) \quad \frac{(E_{\frac{1}{q}, \omega}(t) + 1)}{2} f_n(t, q, \omega) E_{q, \omega t}(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \nabla_{\text{JHC}, q, \omega} \Phi_{\nu, q, \omega}^{(n)}(x).$$

Proof. Operate on (46) with $\nabla_{\text{JHC}, q, \omega t}$. □

4. Multiple q, ω -power sums

Definition (A q, ω -analogue of [11, (20) p. 381]). The multiple q, ω -power sum is defined by

$$(60) \quad s_{\text{NWA}, \lambda, m, q, \omega}^{(l)}(n) \equiv \sum_{|\vec{j}|=l} \binom{l}{\vec{j}} \lambda^k (\bar{k}_{q, \omega})^m,$$

where $k \equiv j_1 + 2j_2 + \cdots + (n-1)j_{n-1}$, $\forall j_i \geq 0$.

Definition (A q, ω -analogue of [11, (46) p. 386]). The multiple alternating q, ω -power sum is defined by

$$(61) \quad \sigma_{\text{NWA}, \lambda, m, q, \omega}^{(l)}(n) \equiv (-1)^l \sum_{|\vec{j}|=l} \binom{l}{\vec{j}} (-\lambda)^k (\bar{k}_{q, \omega})^m,$$

where $k \equiv j_1 + 2j_2 + \cdots + (n-1)j_{n-1}$, $\forall j_i \geq 0$.

For $l = 1$, formulas (60) and (61) reduce to single sums. In order to keep the same notation as in [2], we make a slight change from [15, p. 309]. The following definitions are special cases of the q, ω -power sums in [8].

Definition (Almost a q, ω -analogue of [15, p. 309]). The q, ω -power sum and the alternate q, ω -power sum (with respect to λ) are defined by

$$(62) \quad s_{\text{NWA}, \lambda, m, q, \omega}(n) \equiv \sum_{k=0}^{n-1} \lambda^k (\bar{k}_{q, \omega})^m,$$

$$(63) \quad \sigma_{\text{NWA}, \lambda, m, q, \omega}(n) \equiv \sum_{k=0}^{n-1} (-1)^k \lambda^k (\bar{k}_{q, \omega})^m.$$

Their respective generating functions are

$$(64) \quad \sum_{m=0}^{\infty} s_{\text{NWA}, \lambda, m, q, \omega}(n) \frac{t^m}{\{m\}_q!} = \frac{\lambda^n \mathbf{E}_{q, \omega}(\bar{n}_{q, \omega} t) - 1}{\lambda \mathbf{E}_{q, \omega}(t) - 1}$$

and

$$(65) \quad \sum_{m=0}^{\infty} \sigma_{\text{NWA}, \lambda, m, q, \omega}(n) \frac{t^m}{\{m\}_q!} = \frac{(-1)^{n+1} \lambda^n \mathbf{E}_{q, \omega}(\bar{n}_{q, \omega} t) + 1}{\lambda \mathbf{E}_{q, \omega}(t) + 1}.$$

Proof. Let us prove (64). We have

$$(66) \quad \begin{aligned} \sum_{m=0}^{\infty} s_{\text{NWA}, \lambda, m, q, \omega}(n) \frac{t^m}{\{m\}_q!} &= \sum_{m=0}^{\infty} \sum_{k=0}^{n-1} \lambda^k \frac{(\bar{k}_{q, \omega} t)^m}{\{m\}_q!} \\ &= \sum_{k=0}^{n-1} \lambda^k (\mathbf{E}_{q, \omega}(t))^k = \text{RHS}. \end{aligned}$$

□

We have the following special cases:

$$(67) \quad s_{\text{NWA}, \lambda, m, q, \omega}(1) = \sigma_{\text{NWA}, \lambda, m, q, \omega}(1) = \delta_{0, m},$$

$$(68) \quad s_{\text{NWA}, \lambda, m, q, \omega}(2) = \delta_{0, m} + \lambda, \quad \sigma_{\text{NWA}, \lambda, m, q, \omega}(2) = \delta_{0, m} - \lambda.$$

4.1. Single formulas for q, ω -power sums

Theorem 4.1 (A q, ω -analogue of [15, p. 310], and extensions of [2, pp. 121, 131]).

$$(69) \quad s_{\text{NWA}, \lambda, m, q, \omega}(n) = \frac{\lambda^n \mathcal{B}_{\text{NWA}, \lambda, m+1, q, \omega}(\bar{n}_{q, \omega}) - \mathcal{B}_{\text{NWA}, \lambda, m+1, q, \omega}}{\{m+1\}_q}.$$

$$(70) \quad \sigma_{\text{NWA}, \lambda, m, q, \omega}(n) = \frac{(-1)^{n+1} \lambda^n \mathcal{F}_{\text{NWA}, \lambda, m, q, \omega}(\bar{n}_{q, \omega}) - \mathcal{F}_{\text{NWA}, \lambda, m, q, \omega}}{2}.$$

Theorem 4.2 (A q, ω -analogue of [15, (18), p. 311]).

$$(71) \quad \begin{aligned} & \sum_{k=0}^n \binom{n}{k}_q \frac{(\bar{i}_{q, \omega})^k}{i} (\bar{j}_{q, \omega})^{n-k} \mathcal{B}_{\text{NWA}, \lambda^i, k, q, \omega}(\bar{j}_{q, \omega} x) s_{\text{NWA}, \lambda^j, n-k, q, \omega}(i) \\ &= \sum_{k=0}^n \binom{n}{k}_q \frac{(\bar{j}_{q, \omega})^k}{j} (\bar{i}_{q, \omega})^{n-k} \mathcal{B}_{\text{NWA}, \lambda^j, k, q, \omega}(\bar{i}_{q, \omega} x) s_{\text{NWA}, \lambda^i, n-k, q, \omega}(j) \\ &= \frac{(\bar{i}_{q, \omega})^n}{i} \sum_{m=0}^{i-1} \lambda^{jm} \mathcal{B}_{\text{NWA}, \lambda^i, n, q, \omega} \left(\bar{j}_{q, \omega} x \oplus_{q, \omega} \frac{\bar{j}m_{q, \omega}}{\bar{i}_{q, \omega}} \right) \\ &= \frac{(\bar{j}_{q, \omega})^n}{j} \sum_{m=0}^{j-1} \lambda^{im} \mathcal{B}_{\text{NWA}, \lambda^j, n, q, \omega} \left(\bar{i}_{q, \omega} x \oplus_{q, \omega} \frac{\bar{i}m_{q, \omega}}{\bar{j}_{q, \omega}} \right). \end{aligned}$$

Proof. The following function is symmetric in i and j .

$$(72) \quad \begin{aligned} f_{q, \omega}(t) &\equiv \frac{t \mathbb{E}_{q, \omega}(\bar{i} \bar{j}_{q, \omega} x t) (\lambda^{ij} \mathbb{E}_{q, \omega}(\bar{i} \bar{j}_{q, \omega} t) - 1)}{(\lambda^i \mathbb{E}_{q, \omega}(\bar{i}_{q, \omega} t) - 1) (\lambda^j \mathbb{E}_{q, \omega}(\bar{j}_{q, \omega} t) - 1)} \\ &= \left(\frac{(\bar{i}_{q, \omega} t)^1 \mathbb{E}_{q, \omega}(\bar{i} \bar{j}_{q, \omega} x t)}{\lambda^i \mathbb{E}_{q, \omega}(\bar{i}_{q, \omega} t) - 1} \right) \left(\frac{\lambda^{ij} \mathbb{E}_{q, \omega}(\bar{i} \bar{j}_{q, \omega} t) - 1}{\lambda^j \mathbb{E}_{q, \omega}(\bar{j}_{q, \omega} t) - 1} \right) \frac{1}{i}. \end{aligned}$$

We can expand $f_{q, \omega}(t)$ in two ways by using the formula for a geometric sequence.

$$(73) \quad \begin{aligned} & f_{q, \omega}(t) \\ &= \left(\sum_{\nu=0}^{\infty} \mathcal{B}_{\text{NWA}, \lambda^i, \nu, q, \omega}(\bar{j}_{q, \omega} x) \frac{(\bar{i}_{q, \omega} t)^\nu}{\{\nu\}_q!} \right) \left(\sum_{m=0}^{\infty} s_{\text{NWA}, \lambda^j, m, q, \omega}(i) \frac{(\bar{j}_{q, \omega} t)^m}{\{m\}_q!} \right) \frac{1}{i} \\ &= \frac{(\bar{i}_{q, \omega})^1 t}{\lambda^i \mathbb{E}_{q, \omega}(\bar{i}_{q, \omega} t) - 1} \sum_{m=0}^{i-1} \lambda^{jm} \left(\mathbb{E}_{q, \omega} \left(\bar{j}_{q, \omega} x \oplus_{q, \omega} \frac{\bar{j}m_{q, \omega}}{\bar{i}_{q, \omega}} \right) \bar{i}_{q, \omega} t \right) \frac{1}{i} \\ &= \sum_{\nu=0}^{\infty} \left(\frac{(\bar{i}_{q, \omega})^\nu}{i} \sum_{m=0}^{i-1} \lambda^{jm} \mathcal{B}_{\text{NWA}, \lambda^i, \nu, q, \omega} \left(\bar{j}_{q, \omega} x \oplus_{q, \omega} \frac{\bar{j}m_{q, \omega}}{\bar{i}_{q, \omega}} \right) \right) \frac{t^\nu}{\{\nu\}_q!}. \end{aligned}$$

Finally, equate the coefficients of $\frac{t^\nu}{\{\nu\}_q!}$ and use the symmetry in i and j of $f_{q, \omega}(t)$. \square

Corollary 4.3 (A q, ω -analogue of [15, (19), p. 311]).

$$\begin{aligned}
\mathcal{B}_{\text{NWA}, \lambda, n, q, \omega}(\bar{i}_{q, \omega} x) &= \sum_{k=0}^n \binom{n}{k}_q \frac{(\bar{i}_{q, \omega})^k}{i} \mathcal{B}_{\text{NWA}, \lambda^i, k, q, \omega}(x) s_{\text{NWA}, \lambda, n-k, q, \omega}(i) \\
(74) \qquad \qquad \qquad &= \frac{(\bar{i}_{q, \omega})^n}{i} \sum_{m=0}^{i-1} \lambda^m \mathcal{B}_{\text{NWA}, \lambda^i, n, q, \omega} \left(x \oplus_{q, \omega} \frac{\bar{m}_{q, \omega}}{\bar{i}_{q, \omega}} \right).
\end{aligned}$$

Proof. Put $j = 1$ in (71) and use (68). \square

Corollary 4.4 (A q, ω -analogue of [15, (20), p. 311]).

$$\begin{aligned}
(75) \qquad \qquad \qquad &\sum_{m=0}^1 \lambda^{im} \mathcal{B}_{\text{NWA}, \lambda^2, n, q, \omega} \left(\bar{i}_{q, \omega} x \oplus_{q, \omega} \frac{\bar{im}_{q, \omega}}{\bar{2}_{q, \omega}} \right) \\
&= \frac{2}{(\bar{2}_{q, \omega})^n} \sum_{k=0}^n \binom{n}{k}_q \frac{(\bar{i}_{q, \omega})^k}{i} (\bar{2}_{q, \omega})^{n-k} \mathcal{B}_{\text{NWA}, \lambda^i, k, q, \omega}(\bar{2}_{q, \omega} x) s_{\text{NWA}, \lambda^2, n-k, q, \omega}(i) \\
&= \frac{2}{(\bar{2}_{q, \omega})^n} \frac{(\bar{i}_{q, \omega})^n}{i} \sum_{m=0}^{i-1} \lambda^{2m} \mathcal{B}_{\text{NWA}, \lambda^i, n, q, \omega} \left(\bar{2}_{q, \omega} x \oplus_{q, \omega} \frac{\bar{2m}_{q, \omega}}{\bar{i}_{q, \omega}} \right).
\end{aligned}$$

Proof. Put $j = 2$ in (71) and multiply by $\frac{2}{(\bar{2}_{q, \omega})^n}$. \square

Moreover, we have

$$(76) \quad \mathcal{B}_{\text{NWA}, \lambda, n, q, \omega}(x) = \frac{(\bar{2}_{q, \omega})^n}{2} \sum_{m=0}^1 \lambda^m \mathcal{B}_{\text{NWA}, \lambda^2, n, q, \omega} \left(\frac{x}{\bar{2}_{q, \omega}} \oplus_{q, \omega} \frac{\bar{m}_{q, \omega}}{\bar{2}_{q, \omega}} \right).$$

Proof. Put $i = 2$ in (74) and replace x by $\frac{x}{\bar{2}_{q, \omega}}$. \square

Theorem 4.5 (A q, ω -analogue of [15, (22) p. 312]). *For i and j either both odd, or both even, we have*

$$\begin{aligned}
&\sum_{k=0}^n \binom{n}{k}_q (\bar{i}_{q, \omega})^k (\bar{j}_{q, \omega})^{n-k} \mathcal{F}_{\text{NWA}, \lambda^i, k, q, \omega}(\bar{j}_{q, \omega} x) \sigma_{\text{NWA}, \lambda^j, n-k, q, \omega}(i) \\
&= \sum_{k=0}^n \binom{n}{k}_q (\bar{j}_{q, \omega})^k (\bar{i}_{q, \omega})^{n-k} \mathcal{F}_{\text{NWA}, \lambda^j, k, q, \omega}(\bar{i}_{q, \omega} x) \sigma_{\text{NWA}, \lambda^i, n-k, q, \omega}(j) \\
(77) \qquad \qquad \qquad &= (\bar{i}_{q, \omega})^n \sum_{m=0}^{i-1} \lambda^{jm} (-1)^m \mathcal{F}_{\text{NWA}, \lambda^i, n, q, \omega} \left(\bar{j}_{q, \omega} x \oplus_{q, \omega} \frac{\bar{jm}_{q, \omega}}{\bar{i}_{q, \omega}} \right) \\
&= (\bar{j}_{q, \omega})^n \sum_{m=0}^{j-1} \lambda^{im} (-1)^m \mathcal{F}_{\text{NWA}, \lambda^j, n, q, \omega} \left(\bar{i}_{q, \omega} x \oplus_{q, \omega} \frac{\bar{im}_{q, \omega}}{\bar{j}_{q, \omega}} \right).
\end{aligned}$$

Proof. Let us define the following symmetric function

$$(78) \quad \begin{aligned} f_{q,\omega}(t) &\equiv \frac{\mathbb{E}_{q,\omega t}(\bar{i}\bar{j}_{q,\omega}xt)((-1)^{i+1}\lambda^{ij}\mathbb{E}_{q,\omega}(\bar{i}\bar{j}_{q,\omega}t) + 1)}{(\lambda^i\mathbb{E}_{q,\omega}(\bar{i}_{q,\omega}t) + 1)(\lambda^j\mathbb{E}_{q,\omega}(\bar{j}_{q,\omega}t) + 1)} \\ &= \frac{1}{2} \left(\frac{2\mathbb{E}_{q,\omega t}(\bar{i}\bar{j}_{q,\omega}xt)}{\lambda^i\mathbb{E}_{q,\omega}(\bar{i}_{q,\omega}t) + 1} \right) \left(\frac{(-1)^{i+1}\lambda^{ij}\mathbb{E}_{q,\omega}(\bar{i}\bar{j}_{q,\omega}t) + 1}{\lambda^j\mathbb{E}_{q,\omega}(\bar{j}_{q,\omega}t) + 1} \right). \end{aligned}$$

By using the formula for a geometric sequence, we can expand $f_{q,\omega}(t)$ in two ways:

$$(79) \quad \begin{aligned} &f_{q,\omega}(t) \\ &= \frac{1}{2} \left(\sum_{\nu=0}^{\infty} \mathcal{F}_{\text{NWA},\lambda^i,\nu,q,\omega}(\bar{j}_{q,\omega}x) \frac{(\bar{i}_{q,\omega}t)^\nu}{\{\nu\}_q!} \right) \left(\sum_{m=0}^{\infty} \sigma_{\text{NWA},\lambda^j,m,q,\omega}(i) \frac{(\bar{j}_{q,\omega}t)^m}{\{m\}_q!} \right) \\ &= \frac{1}{\lambda^i\mathbb{E}_{q,\omega}(\bar{i}_{q,\omega}t) + 1} \sum_{m=0}^{i-1} (-1)^m \lambda^{jm} \mathbb{E}_{q,\omega t} \left(\left(\bar{j}_{q,\omega}x \oplus_{q,\omega} \frac{\bar{j}m_{q,\omega}}{\bar{i}_{q,\omega}} \right) \bar{i}_{q,\omega}t \right) \\ &= \frac{1}{2} \sum_{\nu=0}^{\infty} \left((\bar{i}_{q,\omega})^\nu \sum_{m=0}^{i-1} (-1)^m \lambda^{jm} \mathcal{F}_{\text{NWA},\lambda^i,\nu,q,\omega} \left(\bar{j}_{q,\omega}x \oplus_{q,\omega} \frac{\bar{j}m_{q,\omega}}{\bar{i}_{q,\omega}} \right) \right) \frac{t^\nu}{\{\nu\}_q!}. \end{aligned}$$

The theorem follows by equating the coefficients of $\frac{t^\nu}{\{\nu\}_q!}$ and using the symmetry in i and j of $f_{q,\omega}(t)$. \square

Theorem 4.6 (A q,ω -analogue of [15, (24) p. 313]). *For i odd we have*

$$(80) \quad \begin{aligned} \mathcal{F}_{\text{NWA},\lambda,n,q,\omega}(\bar{i}_{q,\omega}x) &= \sum_{k=0}^n \binom{n}{k}_q (\bar{i}_{q,\omega})^k \mathcal{F}_{\text{NWA},\lambda^i,k,q,\omega}(x) \sigma_{\text{NWA},\lambda,n-k,q,\omega}(i) \\ &= (\bar{i}_{q,\omega})^n \sum_{m=0}^{i-1} (-\lambda)^m \mathcal{F}_{\text{NWA},\lambda^i,n,q,\omega} \left(x \oplus_{q,\omega} \frac{\bar{m}_{q,\omega}}{\bar{i}_{q,\omega}} \right). \end{aligned}$$

(A q,ω -analogue of [15, (25) p. 313]) *For i even,*

$$(81) \quad \begin{aligned} &\sum_{m=0}^1 \lambda^{im} (-1)^m \mathcal{F}_{\text{NWA},\lambda^2,n,q,\omega} \left(\bar{i}_{q,\omega}x \oplus_{q,\omega} \frac{\bar{im}_{q,\omega}}{\bar{2}_{q,\omega}} \right) \\ &= \frac{1}{(\bar{2}_{q,\omega})^n} \sum_{k=0}^n \binom{n}{k}_q (\bar{i}_{q,\omega})^k (\bar{2}_{q,\omega})^{n-k} \mathcal{F}_{\text{NWA},\lambda^i,k,q,\omega}(\bar{2}_{q,\omega}x) \\ &\quad \times \sigma_{\text{NWA},\lambda^2,n-k,q,\omega}(i) \\ &= \frac{(\bar{i}_{q,\omega})^n}{(\bar{2}_{q,\omega})^n} \sum_{m=0}^{i-1} (-1)^m \lambda^{2m} \mathcal{F}_{\text{NWA},\lambda^i,n,q,\omega} \left(\bar{2}_{q,\omega}x \oplus_{q,\omega} \frac{\bar{2m}_{q,\omega}}{\bar{i}_{q,\omega}} \right). \end{aligned}$$

Proof. Put $j = 1$ or 2 in (77), and divide by $(\bar{2}_{q,\omega})^n$. \square

4.2. q, ω -power sums, mixed formulas

We now turn to mixed formulas, which contain polynomials of both kinds.

Theorem 4.7 (A q, ω -analogue of [15, (26) p. 313]). *If i is even, then*

$$\begin{aligned}
& \sum_{k=0}^n \binom{n}{k}_q \frac{(\bar{i}_{q,\omega})^k}{i} (\bar{j}_{q,\omega})^{n-k} \mathcal{B}_{\text{NWA},\lambda^i,k,q,\omega}(\bar{j}_{q,\omega}x) \sigma_{\text{NWA},\lambda^j,n-k,q,\omega}(i) \\
&= -\frac{\{n\}_q}{2} \sum_{k=0}^{n-1} \binom{n-1}{k}_q (\bar{j}_{q,\omega})^k (\bar{i}_{q,\omega})^{n-k-1} \\
(82) \quad & \times \mathcal{F}_{\text{NWA},\lambda^j,k,q,\omega}(\bar{i}_{q,\omega}x) s_{\text{NWA},\lambda^i,n-k-1,q,\omega}(j) \\
&= \frac{(\bar{i}_{q,\omega})^n}{i} \sum_{m=0}^{i-1} (-1)^m \lambda^{jm} \mathcal{B}_{\text{NWA},\lambda^i,n,q,\omega} \left(\bar{j}_{q,\omega}x \oplus_{q,\omega} \frac{\bar{j}m_{q,\omega}}{\bar{i}_{q,\omega}} \right) \\
&= -\frac{\{n\}_q}{2} (\bar{j}_{q,\omega})^{n-1} \sum_{m=0}^{j-1} \lambda^{im} \mathcal{F}_{\text{NWA},\lambda^j,n-1,q,\omega} \left(\bar{i}_{q,\omega}x \oplus_{q,\omega} \frac{\bar{i}m_{q,\omega}}{\bar{j}_{q,\omega}} \right).
\end{aligned}$$

Proof. Let us define the following function

$$\begin{aligned}
(83) \quad f_{q,\omega}(t) &\equiv \frac{t \mathbf{E}_{q,\omega}(\bar{i}j_{q,\omega}xt) ((-1)^{i+1} \lambda^{ij} \mathbf{E}_{q,\omega}(\bar{i}j_{q,\omega}t) + 1)}{(\lambda^i \mathbf{E}_{q,\omega}(\bar{i}_{q,\omega}t) - 1)(\lambda^j \mathbf{E}_{q,\omega}(\bar{j}_{q,\omega}t) + 1)} \\
&= \left(\frac{(\bar{i}_{q,\omega}t)^1 \mathbf{E}_{q,\omega}(\bar{i}j_{q,\omega}xt)}{\lambda^i \mathbf{E}_{q,\omega}(\bar{i}_{q,\omega}t) - 1} \right) \left(\frac{((-1)^{i+1} \lambda^{ij} \mathbf{E}_{q,\omega}(\bar{i}j_{q,\omega}t) + 1)}{\lambda^j \mathbf{E}_{q,\omega}(\bar{j}_{q,\omega}t) + 1} \right) \frac{1}{i}.
\end{aligned}$$

By using the formula for a geometric sequence, we can expand $f_{q,\omega}(t)$ in two ways:

$$\begin{aligned}
(84) \quad f_{q,\omega}(t) &= \left(\sum_{\nu=0}^{\infty} \mathcal{B}_{\text{NWA},\lambda^i,\nu,q,\omega}(\bar{j}_{q,\omega}x) \frac{(\bar{i}_{q,\omega}t)^\nu}{\{\nu\}_q!} \right) \left(\sum_{m=0}^{\infty} \sigma_{\text{NWA},\lambda^j,m,q,\omega}(i) \frac{(\bar{j}_{q,\omega}t)^m}{\{m\}_q!} \right) \frac{1}{i} \\
&= \frac{(\bar{i}_{q,\omega}t)^1}{\lambda^i \mathbf{E}_{q,\omega}(\bar{i}_{q,\omega}t) - 1} \sum_{m=0}^{i-1} (-1)^m \lambda^{jm} \mathbf{E}_{q,\omega}t \left(\left(\bar{j}_{q,\omega}x \oplus_{q,\omega} \frac{\bar{j}m_{q,\omega}}{\bar{i}_{q,\omega}} \right) \bar{i}_{q,\omega}t \right) \frac{1}{i} \\
&= \sum_{\nu=0}^{\infty} \left(\frac{(\bar{i}_{q,\omega})^\nu}{i} \sum_{m=0}^{i-1} (-1)^m \lambda^{jm} \mathcal{B}_{\text{NWA},\lambda^i,\nu,q,\omega} \left(\bar{j}_{q,\omega}x \oplus_{q,\omega} \frac{\bar{j}m_{q,\omega}}{\bar{i}_{q,\omega}} \right) \right) \frac{t^\nu}{\{\nu\}_q!}.
\end{aligned}$$

By equating the coefficients of $\frac{t^\nu}{\{\nu\}_q!}$, we obtain rows 1 and 3 of formula (82).

On the other hand, we can rewrite $f_{q,\omega}(t)$ in the following way:

$$(85) \quad \begin{aligned} f_{q,\omega}(t) &= -\frac{t}{2} \frac{2\mathbb{E}_{q,\omega t}(\bar{i}\bar{j}_{q,\omega}xt)(\lambda^{ij}\mathbb{E}_{q,\omega}(\bar{i}\bar{j}_{q,\omega}t) - 1)}{(\lambda^i\mathbb{E}_{q,\omega}(\bar{i}_{q,\omega}t) - 1)(\lambda^j\mathbb{E}_{q,\omega}(\bar{j}_{q,\omega}t) + 1)} \\ &= -\frac{t}{2} \left(\frac{2\mathbb{E}_{q,\omega t}(\bar{i}\bar{j}_{q,\omega}xt)}{\lambda^j\mathbb{E}_{q,\omega}(\bar{j}_{q,\omega}t) + 1} \right) \left(\frac{\lambda^{ij}\mathbb{E}_{q,\omega}(\bar{i}\bar{j}_{q,\omega}t) - 1}{\lambda^i\mathbb{E}_{q,\omega}(\bar{i}_{q,\omega}t) - 1} \right). \end{aligned}$$

By using the formula for a geometric sequence, we can expand (85) in two ways:

$$(86) \quad \begin{aligned} f_{q,\omega}(t) &= -\frac{t}{2} \left(\sum_{\nu=0}^{\infty} \mathcal{F}_{\text{NWA},\lambda^j,\nu,q,\omega}(\bar{i}_{q,\omega}x) \frac{(\bar{j}_{q,\omega}t)^\nu}{\{\nu\}_q!} \right) \left(\sum_{m=0}^{\infty} s_{\text{NWA},\lambda^i,m,q,\omega}(j) \frac{(\bar{i}_{q,\omega}t)^m}{\{m\}_q!} \right) \\ &= -\frac{t}{2} \sum_{m=0}^{j-1} \lambda^{im} \frac{2}{\lambda^j\mathbb{E}_{q,\omega}(\bar{j}_{q,\omega}t) + 1} \mathbb{E}_{q,\omega t} \left(\left(\bar{i}_{q,\omega}x \oplus_{q,\omega} \frac{\bar{i}\bar{m}_{q,\omega}}{\bar{j}_{q,\omega}} \right) \bar{j}_{q,\omega}t \right) \\ &= -\frac{t}{2} \sum_{\nu=0}^{\infty} \left((\bar{j}_{q,\omega})^\nu \sum_{m=0}^{j-1} \lambda^{im} \mathcal{F}_{\text{NWA},\lambda^j,\nu,q,\omega} \left(\bar{i}_{q,\omega}x \oplus_{q,\omega} \frac{\bar{i}\bar{m}_{q,\omega}}{\bar{j}_{q,\omega}} \right) \right) \frac{t^\nu}{\{\nu\}_q!}. \end{aligned}$$

By equating the coefficients of $\frac{t^\nu}{\{\nu\}_q!}$, we obtain rows 2 and 4 of formula (82). \square

Corollary 4.8 (A q, ω -analogue of [15, (28) p. 313]). *If i is even, then*

$$(87) \quad \begin{aligned} \mathcal{F}_{\text{NWA},\lambda,n-1,q,\omega}(\bar{i}_{q,\omega}x) &= -\frac{2}{\{n\}_q} \sum_{k=0}^n \binom{n}{k}_q \frac{(\bar{i}_{q,\omega})^k}{i} \mathcal{B}_{\text{NWA},\lambda^i,k,q,\omega}(x) \\ &\quad \times \sigma_{\text{NWA},\lambda,n-k,q,\omega}(i) \\ &= -\frac{2(\bar{i}_{q,\omega})^n}{i\{n\}_q} \sum_{m=0}^{i-1} (-\lambda)^m \mathcal{B}_{\text{NWA},\lambda^i,n,q,\omega} \left(x \oplus_{q,\omega} \frac{\bar{m}_{q,\omega}}{\bar{i}_{q,\omega}} \right). \end{aligned}$$

Proof. Put $j = 1$ in formula (82) and multiply by $-\frac{2}{\{n\}_q}$. \square

Corollary 4.9 (A q, ω -analogue of [15, (29) p. 313]).

$$(88) \quad \begin{aligned} \mathcal{F}_{\text{NWA},\lambda,n-1,q,\omega}(x) &= -\frac{2}{\{n\}_q} \sum_{k=0}^n \binom{n}{k}_q \frac{(\bar{2}_{q,\omega})^k}{2} \mathcal{B}_{\text{NWA},\lambda^i,k,q,\omega} \left(\frac{x}{\bar{2}_{q,\omega}} \right) \sigma_{\text{NWA},\lambda,n-k,q,\omega}(2) \\ &= -\frac{(\bar{2}_{q,\omega})^n}{\{n\}_q} \sum_{m=0}^1 (-\lambda)^m \mathcal{B}_{\text{NWA},\lambda^2,n,q,\omega} \left(\frac{x}{\bar{2}_{q,\omega}} \oplus_{q,\omega} \frac{\bar{m}_{q,\omega}}{\bar{2}_{q,\omega}} \right). \end{aligned}$$

Proof. Put $i = 2$ in formula (87), and replace x by $\frac{x}{\bar{2}_{q,\omega}}$. \square

Corollary 4.10 (A q, ω -analogue of [15, (31) p. 314]). *If i is even, then*

$$\begin{aligned}
(89) \quad & \sum_{m=0}^1 \lambda^{im} \mathcal{F}_{\text{NWA}, \lambda^2, n-1, q, \omega} \left(\bar{i}_{q, \omega} x \oplus_{q, \omega} \frac{\bar{i} \bar{m}_{q, \omega}}{\bar{2}_{q, \omega}} \right) \\
&= - \frac{2}{\{n\}_q (\bar{2}_{q, \omega})^{n-1}} \sum_{k=0}^n \binom{n}{k}_q \frac{(\bar{i}_{q, \omega})^k}{i} (\bar{2}_{q, \omega})^{n-k} \\
&\quad \times \mathcal{B}_{\text{NWA}, \lambda^i, k, q, \omega} (\bar{2}_{q, \omega} x) \sigma_{\text{NWA}, \lambda^2, n-k, q, \omega}(i) \\
&= \frac{1}{(\bar{2}_{q, \omega})^{n-1}} \sum_{k=0}^{n-1} \binom{n-1}{k}_q (\bar{2}_{q, \omega})^k (\bar{i}_{q, \omega})^{n-k-1} \\
&\quad \times \mathcal{F}_{\text{NWA}, \lambda^2, k, q, \omega} (\bar{i}_{q, \omega} x) s_{\text{NWA}, \lambda^i, n-k-1, q, \omega}(2) \\
&= - \frac{2}{\{n\}_q (\bar{2}_{q, \omega})^{n-1}} \frac{(\bar{i}_{q, \omega})^n}{i} \sum_{m=0}^{i-1} (-1)^m \lambda^{2m} \\
&\quad \times \mathcal{B}_{\text{NWA}, \lambda^i, n, q, \omega} \left(\bar{2}_{q, \omega} x \oplus_{q, \omega} \frac{\bar{2} \bar{m}_{q, \omega}}{\bar{i}_{q, \omega}} \right).
\end{aligned}$$

Proof. Put $j = 2$ in formula (82) and multiply by $-\frac{2}{\{n\}_q (\bar{2}_{q, \omega})^{n-1}}$. \square

Corollary 4.11 (A q, ω -analogue of [15, (32) p. 314]).

$$\begin{aligned}
(90) \quad & \sum_{m=0}^1 (-1)^{m+1} \lambda^m \mathcal{B}_{\text{NWA}, \lambda, n, q, \omega} \left(x \oplus_{q, \omega} \frac{\bar{2} \bar{m}_{q, \omega}}{\bar{2}_{q, \omega}} \right) \\
&= \frac{\{n\}_q (\bar{2}_{q, \omega})^{n-1}}{(\bar{2}_{q, \omega})^n} \sum_{m=0}^1 \lambda^m \mathcal{F}_{\text{NWA}, \lambda, n-1, q, \omega} \left(x \oplus_{q, \omega} \frac{\bar{2} \bar{m}_{q, \omega}}{\bar{2}_{q, \omega}} \right).
\end{aligned}$$

Proof. Put $i = 2$ in formula (89), replace x and λ^2 by $\frac{x}{\bar{2}_{q, \omega}}$ and λ , and multiply by $\frac{\{n\}_q (\bar{2}_{q, \omega})^{n-1}}{(\bar{2}_{q, \omega})^n}$. \square

Corollary 4.12 (A q, ω -analogue of [15, (33) p. 314]).

$$\begin{aligned}
(91) \quad & \sum_{m=0}^1 (-1)^m \lambda^{jm} \mathcal{B}_{\text{NWA}, \lambda^2, n, q, \omega} \left(\bar{j}_{q, \omega} x \oplus_{q, \omega} \frac{\bar{j} \bar{m}_{q, \omega}}{\bar{2}_{q, \omega}} \right) \\
&= - \frac{\{n\}_q}{(\bar{2}_{q, \omega})^n} \sum_{k=0}^{n-1} \binom{n-1}{k}_q \\
&\quad \times (\bar{j}_{q, \omega})^k (\bar{2}_{q, \omega})^{n-k-1} \mathcal{F}_{\text{NWA}, \lambda^j, k, q, \omega} (\bar{2}_{q, \omega} x) s_{\text{NWA}, \lambda^2, n-k-1, q, \omega}(j) \\
&= - \frac{\{n\}_q}{(\bar{2}_{q, \omega})^n} (\bar{j}_{q, \omega})^{n-1} \sum_{m=0}^{j-1} \lambda^{2m} \mathcal{F}_{\text{NWA}, \lambda^j, n-1, q, \omega} \left(\bar{2}_{q, \omega} x \oplus_{q, \omega} \frac{\bar{2} \bar{m}_{q, \omega}}{\bar{j}_{q, \omega}} \right).
\end{aligned}$$

Proof. Put $i = 2$ in formula (82) and multiply by $\frac{2}{(\bar{2}_{q, \omega})^n}$. \square

Corollary 4.13 (A q, ω -analogue of [15, (26) p. 315]). *If i is odd then*

$$\begin{aligned}
& \sum_{k=0}^n \binom{n}{k}_q (\bar{i}_{q,\omega})^k (\bar{j}_{q,\omega})^{n-k} \mathcal{B}_{\text{NWA},\lambda^i,k,q,\omega}(\bar{j}_{q,\omega}x) \sigma_{\text{NWA},\lambda^j,n-k,q,\omega}(i) \\
&= \sum_{k=0}^n \binom{n}{k}_q (\bar{i}_{q,\omega})^k (\bar{j}_{q,\omega})^{n-k} \mathcal{B}_{\text{NWA},\lambda^i,k,q,\omega}(\bar{j}_{q,\omega}x) \sigma_{\text{NWA},\lambda^j,n-k,q,\omega}(i) \\
(92) \quad &= (\bar{i}_{q,\omega})^n \sum_{m=0}^{i-1} (-1)^m \lambda^{jm} \mathcal{B}_{\text{NWA},\lambda^i,n,q,\omega} \left(\bar{j}_{q,\omega}x \oplus_{q,\omega} \frac{\bar{j}m_{q,\omega}}{\bar{i}_{q,\omega}} \right) \\
&= (\bar{i}_{q,\omega})^{n-1} \sum_{m=0}^{i-1} \lambda^{jm} \mathcal{B}_{\text{NWA},\lambda^i,n,q,\omega} \left(\bar{j}_{q,\omega}x \oplus_{q,\omega} \frac{\bar{j}m_{q,\omega}}{\bar{i}_{q,\omega}} \right) (-1)^m.
\end{aligned}$$

Proof. We can rewrite $f_{q,\omega}(t)$ in the following way:

$$\begin{aligned}
(93) \quad f_{q,\omega}(t) &= \frac{(\bar{i}_{q,\omega}t)^1 \mathbb{E}_{q,\omega}(\bar{i}\bar{j}_{q,\omega}xt) (\lambda^{ij} \mathbb{E}_{q,\omega}(\bar{i}\bar{j}_{q,\omega}t) - 1)}{i(\lambda^i \mathbb{E}_{q,\omega}(\bar{i}_{q,\omega}t) - 1) (\lambda^j \mathbb{E}_{q,\omega}(\bar{j}_{q,\omega}t) + 1)} \\
&= \frac{t}{2} \left(\frac{2\mathbb{E}_{q,\omega}(\bar{i}\bar{j}_{q,\omega}xt)}{\lambda^j \mathbb{E}_{q,\omega}(\bar{j}_{q,\omega}t) + 1} \right) \left(\frac{\lambda^{ij} \mathbb{E}_{q,\omega}(\bar{i}\bar{j}_{q,\omega}t) + 1}{\lambda^i \mathbb{E}_{q,\omega}(\bar{i}_{q,\omega}t) - 1} \right).
\end{aligned}$$

By using the formula for a geometric sequence, we can expand (93) in two ways:

$$\begin{aligned}
(94) \quad f_{q,\omega}(t) &= \left(\sum_{\nu=0}^{\infty} \mathcal{B}_{\text{NWA},\lambda^i,\nu,q,\omega}(\bar{j}_{q,\omega}x) \frac{(\bar{i}_{q,\omega}t)^\nu}{\{\nu\}_q!} \right) \left(\sum_{m=0}^{\infty} \sigma_{\text{NWA},\lambda^j,m,q,\omega}(i) \frac{(\bar{j}_{q,\omega}t)^m}{\{m\}_q!} \right) \frac{1}{i} \\
&= \sum_{m=0}^{i-1} \lambda^{jm} \frac{(-1)^m (\bar{i}_{q,\omega}t)^1}{\lambda^i \mathbb{E}_{q,\omega}(\bar{i}_{q,\omega}t) - 1} \mathbb{E}_{q,\omega} \left(\left(\bar{j}_{q,\omega}x \oplus_{q,\omega} \frac{\bar{j}m_{q,\omega}}{\bar{i}_{q,\omega}} \right) \bar{j}_{q,\omega}t \right) \frac{1}{i} \\
&= \sum_{\nu=0}^{\infty} \left((\bar{i}_{q,\omega})^\nu \sum_{m=0}^{i-1} \lambda^{jm} \mathcal{B}_{\text{NWA},\lambda^i,\nu,q,\omega} \left(\bar{j}_{q,\omega}x \oplus_{q,\omega} \frac{\bar{j}m_{q,\omega}}{\bar{i}_{q,\omega}} \right) \right) \frac{t^\nu}{\{\nu\}_q!} \frac{(-1)^m}{i}.
\end{aligned}$$

By equating the coefficients of $\frac{t^\nu}{\{\nu\}_q!}$, we obtain rows 2 and 4 of formula (82). \square

5. More expansion formulas

Theorem 5.1. *A triple sum of NWA q, ω -Apostol-Euler polynomials is equal to a double sum of NWA q, ω -Apostol-Euler polynomials.*

$$\begin{aligned}
(95) \quad & \sum_{|\nu|=n} \binom{n}{\bar{\nu}}_q (\bar{i}_{q,\omega})^{\nu_1} (\bar{j}_{q,\omega})^{\nu_2} \mathcal{F}_{\text{NWA},\lambda^i,\nu_1,q,\omega}^{(k)}(\bar{j}_{q,\omega}x) \mathcal{F}_{\text{NWA},\lambda^j,\nu_2,q,\omega}^{(k-1)}(\bar{i}_{q,\omega}y) \\
& \times \sigma_{\text{NWA},\lambda^j,\nu_3,q,\omega}(i) (\bar{j}_{q,\omega})^{\nu_3}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{\nu=0}^n \binom{n}{\nu}_q (\bar{i}_{q,\omega})^\nu (\bar{j}_{q,\omega})^{n-\nu} \mathcal{F}_{\text{NWA},\lambda^j,n-\nu,q,\omega}^{(k-1)}(\bar{i}_{q,\omega}y) \\
&\quad \times \sum_{m=0}^{i-1} \lambda^{jm} (-1)^m \mathcal{F}_{\text{NWA},\lambda^i,\nu,q,\omega}^{(k)} \left(\bar{j}_{q,\omega}x \oplus_{q,\omega} \frac{\bar{j}m_{q,\omega}}{\bar{i}_{q,\omega}} \right).
\end{aligned}$$

Proof. Define the following function, note that $f_{q,\omega}(t)$ is symmetric when i, j have the same parity.

$$\begin{aligned}
(96) \quad f_{q,\omega}(t) &\equiv \frac{\mathbb{E}_{q,\omega t}(\bar{i}j_{q,\omega}(x \oplus_{q,\omega} y)t) ((-1)^{i+1} \lambda^{ij} \mathbb{E}_{q,\omega}(\bar{i}j_{q,\omega}t) + 1)}{(\lambda^i \mathbb{E}_{q,\omega}(\bar{i}_{q,\omega}t) + 1)^k (\lambda^j \mathbb{E}_{q,\omega}(\bar{j}_{q,\omega}t) + 1)^k} \\
&= 2^{1-2k} \mathbb{E}_{q,\omega t}(\bar{i}j_{q,\omega}(x \oplus_{q,\omega} y)t) \left(\frac{2}{\lambda^i \mathbb{E}_{q,\omega}(\bar{i}_{q,\omega}t) + 1} \right)^k \\
&\quad \times \left(\frac{2}{\lambda^j \mathbb{E}_{q,\omega}(\bar{j}_{q,\omega}t) + 1} \right)^{k-1} \left(\frac{(-1)^{i+1} \lambda^{ij} \mathbb{E}_{q,\omega}(\bar{i}j_{q,\omega}t) + 1}{\lambda^j \mathbb{E}_{q,\omega}(\bar{j}_{q,\omega}t) + 1} \right).
\end{aligned}$$

By using the formula for a geometric sequence, we can expand $f_{q,\omega}(t)$ in two ways:

$$\begin{aligned}
(97) \quad f_{q,\omega}(t) &\stackrel{(65), [8]}{=} 2^{1-2k} \left(\sum_{\nu=0}^{\infty} \mathcal{F}_{\text{NWA},\lambda^i,\nu,q,\omega}^{(k)}(\bar{j}_{q,\omega}x) \frac{(\bar{i}_{q,\omega}t)^\nu}{\{\nu\}_q!} \right) \\
&\quad \left(\sum_{m=0}^{\infty} \sigma_{\text{NWA},\lambda^j,m,q,\omega}(i) \frac{(\bar{j}_{q,\omega}t)^m}{\{m\}_q!} \right) \left(\sum_{l=0}^{\infty} \mathcal{F}_{\text{NWA},\lambda^j,l,q,\omega}^{(k-1)}(\bar{i}_{q,\omega}y) \frac{(\bar{j}_{q,\omega}t)^l}{\{l\}_q!} \right) \\
&= 2^{1-2k} \frac{2^k}{(\lambda^i \mathbb{E}_{q,\omega}(\bar{i}_{q,\omega}t) + 1)^k} \frac{2^{k-1}}{(\lambda^j \mathbb{E}_{q,\omega}(\bar{j}_{q,\omega}t) + 1)^{k-1}} \sum_{m=0}^{i-1} (-1)^m \lambda^{jm} \\
&\quad \times \mathbb{E}_{q,\omega t} \left(\left(\bar{j}_{q,\omega}x \oplus_{q,\omega} \bar{j}_{q,\omega}y \oplus_{q,\omega} \frac{\bar{j}m_{q,\omega}}{\bar{i}_{q,\omega}} \right) \bar{i}_{q,\omega}t \right) \\
&= 2^{1-2k} \sum_{m=0}^{i-1} (-1)^m \lambda^{jm} \sum_{l=0}^{\infty} \frac{(\bar{i}_{q,\omega})^{l^2}}{\{l\}_q!} \mathcal{F}_{\text{NWA},\lambda^i,l,q,\omega}^{(k)} \left(\bar{j}_{q,\omega}x \oplus_{q,\omega} \frac{\bar{j}m_{q,\omega}}{\bar{i}_{q,\omega}} \right) \\
&\quad \sum_{n=0}^{\infty} \frac{(\bar{j}_{q,\omega})^{n^2}}{\{n\}_q!} \mathcal{F}_{\text{NWA},\lambda^j,n,q,\omega}^{(k-1)}(\bar{i}_{q,\omega}y).
\end{aligned}$$

The theorem follows by equating the coefficients of $\frac{t^n}{\{n\}_q!}$. \square

Theorem 5.2 (Almost a q, ω -analogue of [10, p. 3351]). *Assume that i and j are either both odd, or both even. Then we have*

$$(98) \quad \sum_{\nu=0}^n \binom{n}{\nu}_q (\bar{j}_{q,\omega})^\nu (\bar{i}_{q,\omega})^{n-\nu} \mathcal{F}_{\text{NWA},\lambda^i,n-\nu,q,\omega}^{(k-1)}(\bar{i}_{q,\omega}y)$$

$$\begin{aligned}
& \sum_{m=0}^{j-1} \lambda^{im} (-1)^m \mathcal{F}_{\text{NWA}, \lambda^i, \nu, q, \omega}^{(k)} \left(\bar{i}_{q, \omega} x \oplus_{q, \omega} \frac{\bar{i}m_{q, \omega}}{\bar{j}_{q, \omega}} \right) \\
&= \sum_{\nu=0}^n \binom{n}{\nu}_q (\bar{i}_{q, \omega})^\nu (\bar{j}_{q, \omega})^{n-\nu} \mathcal{F}_{\text{NWA}, \lambda^j, n-\nu, q, \omega}^{(k-1)} (\bar{i}_{q, \omega} y) \\
& \quad \times \sum_{m=0}^{i-1} \lambda^{jm} (-1)^m \mathcal{F}_{\text{NWA}, \lambda^i, \nu, q, \omega}^{(k)} \left(\bar{j}_{q, \omega} x \oplus_{q, \omega} \frac{\bar{j}m_{q, \omega}}{\bar{i}_{q, \omega}} \right).
\end{aligned}$$

Proof. This follows from the previous proof, and then using the symmetry for i and j . \square

Theorem 5.3. *A triple sum of NWA q, ω -Apostol-Bernoulli polynomials is equal to a double sum of NWA q, ω -Apostol-Bernoulli polynomials.*

$$\begin{aligned}
& \sum_{|\nu|=n} \binom{n}{\nu}_q (\bar{i}_{q, \omega})^{\nu_1} (\bar{j}_{q, \omega})^{\nu_2} (\bar{j}_{q, \omega})^{\nu_3} \mathcal{B}_{\text{NWA}, \lambda^i, \nu_1, q, \omega}^{(k)} (\bar{j}_{q, \omega} x) \\
& \quad \times \mathcal{B}_{\text{NWA}, \lambda^j, \nu_2, q, \omega}^{(k-1)} (\bar{i}_{q, \omega} y) s_{\text{NWA}, \lambda^j, \nu_3, q, \omega}(i) \\
(99) \quad &= \sum_{\nu=0}^n \binom{n}{\nu}_q (\bar{i}_{q, \omega})^\nu (\bar{j}_{q, \omega})^{n-\nu} \mathcal{B}_{\text{NWA}, \lambda^j, n-\nu, q, \omega}^{(k-1)} (\bar{i}_{q, \omega} y) \\
& \quad \sum_{m=0}^{i-1} \lambda^{jm} \mathcal{B}_{\text{NWA}, \lambda^i, \nu, q, \omega}^{(k)} \left(\bar{j}_{q, \omega} x \oplus_{q, \omega} \frac{\bar{j}m_{q, \omega}}{\bar{i}_{q, \omega}} \right).
\end{aligned}$$

Proof. Define the following symmetric function

$$\begin{aligned}
(100) \quad \phi_{q, \omega}(t) &\equiv \frac{\mathbb{E}_{q, \omega t}(\bar{i}\bar{j}_{q, \omega}(x \oplus_{q, \omega} y)t)(\lambda^{ij} \mathbb{E}_{q, \omega}(\bar{i}\bar{j}_{q, \omega}t) - 1)}{(\lambda^i \mathbb{E}_{q, \omega}(\bar{i}_{q, \omega}t) - 1)^k (\lambda^j \mathbb{E}_{q, \omega}(\bar{j}_{q, \omega}t) - 1)^k} t^k \\
&= \mathbb{E}_{q, \omega t}(\bar{i}\bar{j}_{q, \omega}(x \oplus_{q, \omega} y)t) \left(\frac{\bar{i}_{q, \omega}t}{\lambda^i \mathbb{E}_{q, \omega}(\bar{i}_{q, \omega}t) - 1} \right)^k \left(\frac{\bar{j}_{q, \omega}t}{\lambda^j \mathbb{E}_{q, \omega}(\bar{j}_{q, \omega}t) - 1} \right)^{k-1} \\
& \quad \times \left(\frac{\lambda^{ij} \mathbb{E}_{q, \omega}(\bar{i}\bar{j}_{q, \omega}t) - 1}{\lambda^j \mathbb{E}_{q, \omega}(\bar{j}_{q, \omega}t) - 1} \right) \frac{t^{1-2k}}{(\bar{i}_{q, \omega})^k (\bar{j}_{q, \omega})^{k-1}}.
\end{aligned}$$

By using the formula for a geometric sequence, we can expand $\phi_{q, \omega}(t)$ in two ways:

$$\begin{aligned}
(101) \quad \phi_{q, \omega}(t) &\stackrel{\text{by(64)}}{=} \left(\sum_{\nu=0}^{\infty} \mathcal{B}_{\text{NWA}, \lambda^i, \nu, q, \omega}^{(k)} (\bar{j}_{q, \omega} x) \frac{(\bar{i}_{q, \omega}t)^\nu}{\{\nu\}_q!} \right) \\
& \quad \times \left(\sum_{m=0}^{\infty} s_{\text{NWA}, \lambda^j, m, q, \omega}(i) \frac{(\bar{j}_{q, \omega}t)^m}{\{m\}_q!} \right)
\end{aligned}$$

$$\begin{aligned}
& \times \left(\sum_{l=0}^{\infty} \mathcal{B}_{\text{NWA}, \lambda^j, l, q, \omega}^{(k-1)}(\bar{i}_{q, \omega} y) \frac{(\bar{j}_{q, \omega} t)^l}{\{l\}_q!} \right) \frac{t^{1-2k}}{(\bar{i}_{q, \omega})^k (\bar{j}_{q, \omega})^{k-1}} \\
&= \frac{(\bar{i}_{q, \omega} t)^k}{(\lambda^i \mathbb{E}_{q, \omega}(\bar{i}_{q, \omega} t) - 1)^k} \frac{(\bar{j}_{q, \omega} t)^{k-1}}{(\lambda^j \mathbb{E}_{q, \omega}(\bar{j}_{q, \omega} t) - 1)^{k-1}} \\
& \times \sum_{m=0}^{i-1} \lambda^{jm} \mathbb{E}_{q, \omega} t \left(\left(\bar{j}_{q, \omega} x \oplus_{q, \omega} \bar{j}_{q, \omega} y \oplus_{q, \omega} \frac{\bar{j} \bar{m}_{q, \omega}}{\bar{i}_{q, \omega}} \right) \bar{i}_{q, \omega} t \right) \\
& \times \frac{t^{1-2k}}{(\bar{i}_{q, \omega})^k (\bar{j}_{q, \omega})^{k-1}} \\
&= \frac{t^{1-2k}}{(\bar{i}_{q, \omega})^k (\bar{j}_{q, \omega})^{k-1}} \sum_{m=0}^{i-1} \lambda^{jm} \sum_{l=0}^{\infty} \frac{(\bar{i}_{q, \omega})^l t^l}{\{l\}_q!} \mathcal{B}_{\text{NWA}, \lambda^i, l, q, \omega}^{(k)} \\
& \times \left(\bar{j}_{q, \omega} x \oplus_{q, \omega} \frac{\bar{j} \bar{m}_{q, \omega}}{\bar{i}_{q, \omega}} \right) \sum_{n=0}^{\infty} \frac{(\bar{j}_{q, \omega})^n t^n}{\{n\}_q!} \mathcal{B}_{\text{NWA}, \lambda^j, n, q, \omega}^{(k-1)}(\bar{i}_{q, \omega} y).
\end{aligned}$$

The theorem follows by equating the coefficients of $\frac{t^n}{\{n\}_q!}$. \square

Theorem 5.4 (A q, ω -analogue of [17, p. 2994], [16, p. 551]).

$$\begin{aligned}
& \sum_{|\nu|=n} \binom{n}{\bar{\nu}}_q (\bar{i}_{q, \omega})^{\nu_1} (\bar{j}_{q, \omega})^{\nu_2} (\bar{j}_{q, \omega})^{\nu_3} \mathcal{B}_{\text{NWA}, \lambda^i, \nu_1, q, \omega}^{(k)}(\bar{j}_{q, \omega} x) \\
& \times \mathcal{B}_{\text{NWA}, \lambda^j, \nu_2, q, \omega}^{(k-1)}(\bar{i}_{q, \omega} y) s_{\text{NWA}, \lambda^j, \nu_3, q, \omega}(i) \\
(102) \quad &= \sum_{|\nu|=n} \binom{n}{\bar{\nu}}_q (\bar{j}_{q, \omega})^{\nu_1} (\bar{i}_{q, \omega})^{\nu_2} (\bar{i}_{q, \omega})^{\nu_3} \mathcal{B}_{\text{NWA}, \lambda^j, \nu_1, q, \omega}^{(k)}(\bar{i}_{q, \omega} x) \\
& \times \mathcal{B}_{\text{NWA}, \lambda^i, \nu_2, q, \omega}^{(k-1)}(\bar{j}_{q, \omega} y) s_{\text{NWA}, \lambda^i, \nu_3, q, \omega}(j).
\end{aligned}$$

Proof. Use the symmetry in $\phi_{q, \omega}(t)$. \square

Theorem 5.5 (A q, ω -analogue of [17, p. 2996]). *We have*

$$\begin{aligned}
(103) \quad & \sum_{\nu=0}^n \binom{n}{\nu}_q \sum_{l=0}^{i-1} \sum_{m=0}^{j-1} \lambda^{l+m} (\bar{i}_{q, \omega})^\nu (\bar{j}_{q, \omega})^{n-\nu} \\
& \times \mathcal{B}_{\text{NWA}, \lambda, \nu, q, \omega}^{(k)} \left(\bar{j}_{q, \omega} x \oplus_{q, \omega} \frac{\bar{j} \bar{l}_{q, \omega}}{\bar{i}_{q, \omega}} \right) \mathcal{B}_{\text{NWA}, \lambda, n-\nu, q, \omega}^{(k)} \left(\bar{i}_{q, \omega} y \oplus_{q, \omega} \frac{\bar{i} \bar{m}_{q, \omega}}{\bar{j}_{q, \omega}} \right) \\
&= \sum_{\nu=0}^n \binom{n}{\nu}_q \sum_{l=0}^{j-1} \sum_{m=0}^{i-1} \lambda^{l+m} (\bar{j}_{q, \omega})^\nu (\bar{i}_{q, \omega})^{n-\nu} \\
& \times \mathcal{B}_{\text{NWA}, \lambda, \nu, q, \omega}^{(k)} \left(\bar{i}_{q, \omega} x \oplus_{q, \omega} \frac{\bar{i} \bar{l}_{q, \omega}}{\bar{j}_{q, \omega}} \right) \mathcal{B}_{\text{NWA}, \lambda, n-\nu, q, \omega}^{(k)} \left(\bar{j}_{q, \omega} y \oplus_{q, \omega} \frac{\bar{j} \bar{m}_{q, \omega}}{\bar{i}_{q, \omega}} \right).
\end{aligned}$$

Proof. We can expand the following symmetric function $\phi'_{q,\omega}(t)$ by using the formula for a geometric sequence:

$$\begin{aligned}
(104) \quad & \phi'_{q,\omega}(t) \\
\equiv & \frac{\mathbb{E}_{q,\omega t}(\bar{i}\bar{j}_{q,\omega}(x \oplus_{q,\omega} y)t)(\lambda^i \mathbb{E}_{q,\omega}(\bar{i}\bar{j}_{q,\omega}t) - 1)(\lambda^j \mathbb{E}_{q,\omega}(\bar{i}\bar{j}_{q,\omega}t) - 1)}{(\lambda \mathbb{E}_{q,\omega}(\bar{i}_{q,\omega}t) - 1)^k (\lambda \mathbb{E}_{q,\omega}(\bar{j}_{q,\omega}t) - 1)^k} t^{2k-2} \\
& \times \mathbb{E}_{q,\omega t}(\bar{i}\bar{j}_{q,\omega}(x \oplus_{q,\omega} y)t) \frac{1}{(\bar{i}_{q,\omega})^{k-1} (\bar{j}_{q,\omega})^{k-1}} \\
& \times \left(\frac{\bar{i}_{q,\omega} t}{\lambda \mathbb{E}_{q,\omega}(\bar{i}_{q,\omega}t) - 1} \right)^{k-1} \left(\frac{\bar{j}_{q,\omega} t}{\lambda \mathbb{E}_{q,\omega}(\bar{j}_{q,\omega}t) - 1} \right)^{k-1} \\
& \times \left(\frac{\lambda^i \mathbb{E}_{q,\omega}(\bar{i}\bar{j}_{q,\omega}t) - 1}{\lambda \mathbb{E}_{q,\omega}(\bar{j}_{q,\omega}t) - 1} \right) \left(\frac{\lambda^j \mathbb{E}_{q,\omega}(\bar{i}\bar{j}_{q,\omega}t) - 1}{\lambda \mathbb{E}_{q,\omega}(\bar{i}_{q,\omega}t) - 1} \right) \\
= & \frac{1}{(\bar{i}_{q,\omega})^{k-1} (\bar{j}_{q,\omega})^{k-1}} \sum_{l=0}^{i-1} \sum_{m=0}^{j-1} \lambda^{l+m} \left(\frac{\bar{i}_{q,\omega} t}{\lambda \mathbb{E}_{q,\omega}(\bar{i}_{q,\omega}t) - 1} \right)^{k-1} \left(\frac{\bar{j}_{q,\omega} t}{\lambda \mathbb{E}_{q,\omega}(\bar{j}_{q,\omega}t) - 1} \right)^{k-1} \\
& \times \mathbb{E}_{q,\omega t} \left(\left(\bar{j}_{q,\omega} x \oplus_{q,\omega} \frac{\bar{j}l_{q,\omega}}{\bar{i}_{q,\omega}} \right) \bar{i}_{q,\omega} t \right) \mathbb{E}_{q,\omega t} \left(\left(\bar{i}_{q,\omega} y \oplus_{q,\omega} \frac{\bar{i}m_{q,\omega}}{\bar{j}_{q,\omega}} \right) \bar{j}_{q,\omega} t \right) \\
= & \frac{1}{(\bar{i}_{q,\omega})^{k-1} (\bar{j}_{q,\omega})^{k-1}} \left(\sum_{l=0}^{i-1} \lambda^l \sum_{\nu_1=0}^{\infty} \frac{(\bar{i}_{q,\omega})^{\nu_1} t^{\nu_1}}{\{\nu_1\}_q!} \mathcal{B}_{\text{NWA},\lambda,\nu_1,q,\omega}^{(k-1)} \left(\bar{j}_{q,\omega} x \oplus_{q,\omega} \frac{\bar{j}l_{q,\omega}}{\bar{i}_{q,\omega}} \right) \right) \\
& \times \left(\sum_{m=0}^{j-1} \lambda^m \sum_{\nu_2=0}^{\infty} \frac{(\bar{j}_{q,\omega})^{\nu_2} t^{\nu_2}}{\{\nu_2\}_q!} \mathcal{B}_{\text{NWA},\lambda,\nu_2,q,\omega}^{(k-1)} \left(\bar{i}_{q,\omega} y \oplus_{q,\omega} \frac{\bar{i}m_{q,\omega}}{\bar{j}_{q,\omega}} \right) \right).
\end{aligned}$$

The theorem follows by using the symmetry in $\phi'_{q,\omega}(t)$ and changing $k-1$ to k . \square

Theorem 5.6 (A q,ω -analogue of [17, p. 2997]). *We have*

$$\begin{aligned}
(105) \quad & \sum_{\nu=0}^n \binom{n}{\nu}_q \sum_{l=0}^{i-1} (\bar{i}_{q,\omega} \bar{j}_{q,\omega})^{n-\nu} \mathcal{B}_{\text{NWA},\lambda,n-\nu,q,\omega}^{(k)}(\bar{i}_{q,\omega} y) \\
& \times \sum_{m=0}^{j-1} \lambda^{l+m} \mathcal{B}_{\text{NWA},\lambda,\nu,q,\omega}^{(k)} \left(\bar{j}_{q,\omega} x \oplus_{q,\omega} \frac{\bar{j}l_{q,\omega}}{\bar{i}_{q,\omega}} \oplus_{q,\omega} \bar{m}_{q,\omega} \right) \\
= & \sum_{\nu=0}^n \binom{n}{\nu}_q \sum_{l=0}^{j-1} (\bar{j}_{q,\omega})^\nu (\bar{i}_{q,\omega})^{n-\nu} \mathcal{B}_{\text{NWA},\lambda,n-\nu,q,\omega}^{(k)}(\bar{j}_{q,\omega} y) \\
& \times \sum_{m=0}^{i-1} \lambda^{l+m} \mathcal{B}_{\text{NWA},\lambda,\nu,q,\omega}^{(k)} \left(\bar{i}_{q,\omega} x \oplus_{q,\omega} \frac{\bar{i}l_{q,\omega}}{\bar{j}_{q,\omega}} \oplus_{q,\omega} \bar{m}_{q,\omega} \right).
\end{aligned}$$

Proof. Similar to above. \square

Theorem 5.7 (A q, ω -analogue of [16, p. 552]). *We have*

$$\begin{aligned}
& \frac{1}{(\bar{i}_{q,\omega})^k (\bar{j}_{q,\omega})^{k-1}} \sum_{m=0}^n \binom{n}{m}_q (\bar{i}_{q,\omega})^m (\bar{j}_{q,\omega})^{n-m} \\
& \times \mathcal{B}_{\text{NWA}, \lambda^j, n-m, q, \omega}^{(k-1)}(\bar{i}_{q,\omega} y) \sum_{l=0}^{i-1} \lambda^{jl} \mathcal{B}_{\text{NWA}, \lambda^i, m, q, \omega}^{(k)} \left(\bar{j}_{q,\omega} x \oplus_{q,\omega} \frac{\bar{j} \bar{l}_{q,\omega}}{\bar{i}_{q,\omega}} \right) \\
(106) \quad & = \frac{1}{(\bar{j}_{q,\omega})^k (\bar{i}_{q,\omega})^{k-1}} \sum_{m=0}^n \binom{n}{m}_q (\bar{j}_{q,\omega})^m (\bar{i}_{q,\omega})^{n-m} \\
& \times \mathcal{B}_{\text{NWA}, \lambda^i, n-m, q, \omega}^{(k-1)}(\bar{j}_{q,\omega} y) \sum_{l=0}^{j-1} \lambda^{il} \mathcal{B}_{\text{NWA}, \lambda^j, m, q, \omega}^{(k)} \left(\bar{i}_{q,\omega} x \oplus_{q,\omega} \frac{\bar{i} \bar{l}_{q,\omega}}{\bar{j}_{q,\omega}} \right).
\end{aligned}$$

Proof. We can expand the following symmetric function $\psi_{q,\omega}(t)$ by using the formula for a geometric sequence:

$$\begin{aligned}
\psi_{q,\omega}(t) & \equiv \frac{\mathbf{E}_{q,\omega t}(\bar{i} \bar{j}_{q,\omega} (x \oplus_{q,\omega} y) t) (\lambda^{ij} \mathbf{E}_{q,\omega}(\bar{i} \bar{j}_{q,\omega} t) - 1)}{(\lambda^i \mathbf{E}_{q,\omega}(\bar{i}_{q,\omega} t) - 1)^k (\lambda^j \mathbf{E}_{q,\omega}(\bar{j}_{q,\omega} t) - 1)^k} t^{2k-1} \\
& = \mathbf{E}_{q,\omega t}(\bar{i} \bar{j}_{q,\omega} (x \oplus_{q,\omega} y) t) \frac{1}{(\bar{i}_{q,\omega})^k (\bar{j}_{q,\omega})^{k-1}} \left(\frac{\bar{i}_{q,\omega} t}{\lambda^i \mathbf{E}_{q,\omega}(\bar{i}_{q,\omega} t) - 1} \right)^k \\
(107) \quad & \times \left(\frac{\bar{j}_{q,\omega} t}{\lambda^j \mathbf{E}_{q,\omega}(\bar{j}_{q,\omega} t) - 1} \right)^{k-1} \left(\frac{\lambda^{ij} \mathbf{E}_{q,\omega}(\bar{i} \bar{j}_{q,\omega} t) - 1}{\lambda^j \mathbf{E}_{q,\omega}(\bar{j}_{q,\omega} t) - 1} \right) \\
& = \frac{1}{(\bar{i}_{q,\omega})^k (\bar{j}_{q,\omega})^{k-1}} \left(\frac{\bar{i}_{q,\omega} t}{\lambda^i \mathbf{E}_{q,\omega}(\bar{i}_{q,\omega} t) - 1} \right)^k \left(\frac{\bar{j}_{q,\omega} t}{\lambda^j \mathbf{E}_{q,\omega}(\bar{j}_{q,\omega} t) - 1} \right)^{k-1} \\
& \times \sum_{l=0}^{i-1} \lambda^{jl} \mathbf{E}_{q,\omega t} \left(\left(\bar{j}_{q,\omega} x \oplus_{q,\omega} \frac{\bar{j} \bar{l}_{q,\omega}}{\bar{i}_{q,\omega}} \right) \bar{i}_{q,\omega} t \right) \mathbf{E}_{q,\omega t} \left((\bar{i}_{q,\omega} y) \bar{j}_{q,\omega} t \right) \\
& = \frac{1}{(\bar{i}_{q,\omega})^k} \frac{1}{(\bar{j}_{q,\omega})^{k-1}} \\
& \times \left(\sum_{l=0}^{i-1} \lambda^{jl} \sum_{\nu_1=0}^{\infty} \frac{(\bar{i}_{q,\omega})^{\nu_1} t^{\nu_1}}{\{\nu_1\}_q!} \mathcal{B}_{\text{NWA}, \lambda^i, \nu_1, q, \omega}^{(k)} \left(\bar{j}_{q,\omega} x \oplus_{q,\omega} \frac{\bar{j} \bar{l}_{q,\omega}}{\bar{i}_{q,\omega}} \right) \right) \\
& \times \sum_{\nu_2=0}^{\infty} \frac{(\bar{j}_{q,\omega})^{\nu_2} t^{\nu_2}}{\{\nu_2\}_q!} \mathcal{B}_{\text{NWA}, \lambda^j, \nu_2, q, \omega}^{(k-1)}(\bar{i}_{q,\omega} y) \\
& = \frac{1}{(\bar{i}_{q,\omega})^k (\bar{j}_{q,\omega})^{k-1}} \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \binom{n}{m}_q \right) \\
& \times \sum_{l=0}^{i-1} \lambda^{jl} (\bar{i}_{q,\omega})^m (\bar{j}_{q,\omega})^{n-m} \mathcal{B}_{\text{NWA}, \lambda^i, m, q, \omega}^{(k)} \left(\bar{j}_{q,\omega} x \oplus_{q,\omega} \frac{\bar{j} \bar{l}_{q,\omega}}{\bar{i}_{q,\omega}} \right)
\end{aligned}$$

$$\times \mathcal{B}_{\text{NWA}, \lambda^j, n-m, q, \omega}^{(k-1)}(\bar{i}_{q, \omega} y) \frac{t^n}{\{n\}_q!}.$$

The theorem follows by using the symmetry in $\psi_{q, \omega}(t)$. \square

6. Mixed formulas

Theorem 6.1 (A q, ω -analogue of [10, (3.9) p. 3356]). *A triple sum of mixed q, ω -Apostol polynomials is equal to a double sum of mixed q, ω -Apostol polynomials.*

$$\begin{aligned} & \sum_{|\nu|=n} \binom{n}{\bar{\nu}}_q (\bar{i}_{q, \omega})^{\nu_1} (\bar{j}_{q, \omega})^{\nu_2} \mathcal{B}_{\text{NWA}, \lambda^i, \nu_1, q, \omega}^{(k)}(\bar{j}_{q, \omega} x) \\ & \times \mathcal{F}_{\text{NWA}, \lambda^j, \nu_2, q, \omega}^{(k-1)}(\bar{i}_{q, \omega} y) \sigma_{\text{NWA}, \lambda^j, \nu_3, q, \omega}(i) (\bar{j}_{q, \omega})^{\nu_3} \\ (108) \quad & = \sum_{\nu=0}^n \binom{n}{\nu}_q (\bar{i}_{q, \omega})^\nu (\bar{j}_{q, \omega})^{n-\nu} \mathcal{F}_{\text{NWA}, \lambda^j, n-\nu, q, \omega}^{(k-1)}(\bar{i}_{q, \omega} y) \sum_{m=0}^{i-1} \lambda^j m (-1)^m \\ & \times \mathcal{B}_{\text{NWA}, \lambda^i, \nu, q, \omega}^{(k)}\left(\bar{j}_{q, \omega} x \oplus_{q, \omega} \frac{\bar{j} m_{q, \omega}}{\bar{i}_{q, \omega}}\right). \end{aligned}$$

Proof. Define the following function

$$\begin{aligned} g_{q, \omega}(t) & \equiv \frac{\mathbf{E}_{q, \omega t}(\bar{i} \bar{j}_{q, \omega}(x \oplus_{q, \omega} y)t) ((-1)^{i+1} \lambda^{ij} \mathbf{E}_{q, \omega}(\bar{i} \bar{j}_{q, \omega} t) + 1)}{(\lambda^i \mathbf{E}_{q, \omega}(\bar{i}_{q, \omega} t) - 1)^k (\lambda^j \mathbf{E}_{q, \omega}(\bar{j}_{q, \omega} t) + 1)^k} \\ (109) \quad & = \frac{2^{1-k}}{(\bar{i}_{q, \omega} t)^k} \mathbf{E}_{q, \omega t}(\bar{i} \bar{j}_{q, \omega}(x \oplus_{q, \omega} y)t) \left(\frac{\bar{i}_{q, \omega} t}{\lambda^i \mathbf{E}_{q, \omega}(\bar{i}_{q, \omega} t) - 1} \right)^k \\ & \times \left(\frac{2}{\lambda^j \mathbf{E}_{q, \omega}(\bar{j}_{q, \omega} t) + 1} \right)^{k-1} \left(\frac{(-1)^{i+1} \lambda^{ij} \mathbf{E}_{q, \omega}(\bar{i} \bar{j}_{q, \omega} t) + 1}{\lambda^j \mathbf{E}_{q, \omega}(\bar{j}_{q, \omega} t) + 1} \right). \end{aligned}$$

By using the formula for a geometric sequence, we can expand $g_{q, \omega}(t)$ in two ways:

$$\begin{aligned} g_{q, \omega}(t) & \stackrel{\text{by (65)}}{=} \frac{2^{1-k}}{(\bar{i}_{q, \omega} t)^k} \left(\sum_{\nu=0}^{\infty} \mathcal{B}_{\text{NWA}, \lambda^i, \nu, q, \omega}^{(k)}(\bar{j}_{q, \omega} x) \frac{(\bar{i}_{q, \omega} t)^\nu}{\{\nu\}_q!} \right) \\ (110) \quad & \times \left(\sum_{m=0}^{\infty} \sigma_{\text{NWA}, \lambda^j, m, q, \omega}(i) \frac{(\bar{j}_{q, \omega} t)^m}{\{m\}_q!} \right) \\ & \times \left(\sum_{l=0}^{\infty} \mathcal{F}_{\text{NWA}, \lambda^j, l, q, \omega}^{(k-1)}(\bar{i}_{q, \omega} y) \frac{(\bar{j}_{q, \omega} t)^l}{\{l\}_q!} \right) \\ & = \frac{2^{1-k}}{(\bar{i}_{q, \omega} t)^k} \left(\frac{\bar{i}_{q, \omega} t}{\lambda^i \mathbf{E}_{q, \omega}(\bar{i}_{q, \omega} t) - 1} \right)^k \frac{2^{k-1}}{(\lambda^j \mathbf{E}_{q, \omega}(\bar{j}_{q, \omega} t) + 1)^{k-1}} \end{aligned}$$

$$\begin{aligned}
& \times \sum_{m=0}^{i-1} (-1)^m \lambda^{jm} E_{q,\omega t} \left(\left(\bar{j}_{q,\omega} x \oplus_{q,\omega} \bar{j}_{q,\omega} y \oplus_{q,\omega} \frac{\bar{j}m_{q,\omega}}{\bar{i}_{q,\omega}} \right) \bar{i}_{q,\omega} t \right) \\
& = \frac{2^{1-k}}{(\bar{i}_{q,\omega} t)^k} \sum_{m=0}^{i-1} (-1)^m \\
& \quad \times \lambda^{jm} \sum_{l=0}^{\infty} \frac{(\bar{i}_{q,\omega})^{lt}}{\{l\}_q!} \mathcal{B}_{\text{NWA},\lambda^i,l,q,\omega}^{(k)} \left(\bar{j}_{q,\omega} x \oplus_{q,\omega} \frac{\bar{j}m_{q,\omega}}{\bar{i}_{q,\omega}} \right) \\
& \quad \times \sum_{n=0}^{\infty} \frac{(\bar{j}_{q,\omega})^{nt}}{\{n\}_q!} \mathcal{F}_{\text{NWA},\lambda^j,n,q,\omega}^{(k-1)}(\bar{i}_{q,\omega} y).
\end{aligned}$$

The theorem follows by equating the coefficients of $\frac{t^n}{\{n\}_q!}$. \square

Theorem 6.2 (A q,ω -analogue of [10, p. 3353]). *Under the assumption that i is even, a triple sum of mixed q,ω -Apostol polynomials is equal to another triple sum of mixed q,ω -Apostol polynomials.*

(111)

$$\begin{aligned}
& \sum_{|\nu|=n} \binom{n}{\vec{\nu}}_q (\bar{i}_{q,\omega})^{\nu_1} (\bar{j}_{q,\omega})^{\nu_2} \mathcal{B}_{\text{NWA},\lambda^i,\nu_1,q,\omega}^{(k)}(\bar{j}_{q,\omega} x) \mathcal{F}_{\text{NWA},\lambda^j,\nu_2,q,\omega}^{(k-1)}(\bar{i}_{q,\omega} y) \\
& \quad \times s_{\text{NWA},\lambda^j,\nu_3,q,\omega}(i)(\bar{j}_{q,\omega})^{\nu_3} \\
& = - \frac{\{n\}_q (\bar{i}_{q,\omega})^k}{2(\bar{i}_{q,\omega})^{k-1}} \sum_{|\nu|=n-1} \binom{n-1}{\vec{\nu}}_q (\bar{i}_{q,\omega})^{\nu_1} (\bar{j}_{q,\omega})^{\nu_2} \\
& \quad \times (\bar{j}_{q,\omega})^{\nu_3} \mathcal{B}_{\text{NWA},\lambda^i,\nu_1,q,\omega}^{(k-1)}(\bar{j}_{q,\omega} y) \mathcal{F}_{\text{NWA},\lambda^j,\nu_2,q,\omega}^{(k)}(\bar{i}_{q,\omega} x) s_{\text{NWA},\lambda^i,\nu_3,q,\omega}(j).
\end{aligned}$$

Proof. We can write $g_{q,\omega}(t)$ as follows:

$$\begin{aligned}
(112) \quad g_{q,\omega}(t) & \stackrel{\text{by(64),(109)}}{=} \frac{2^{1-k}}{(\bar{i}_{q,\omega} t)^k} \left(\sum_{\nu=0}^{\infty} \mathcal{B}_{\text{NWA},\lambda^i,\nu,q,\omega}^{(k)}(\bar{j}_{q,\omega} x) \frac{(\bar{i}_{q,\omega} t)^\nu}{\{\nu\}_q!} \right) \\
& \quad \times \left(\sum_{m=0}^{\infty} s_{\text{NWA},\lambda^j,m,q,\omega}(i) \frac{(\bar{j}_{q,\omega} t)^m}{\{m\}_q!} \right) \\
& \quad \times \left(\sum_{l=0}^{\infty} \mathcal{F}_{\text{NWA},\lambda^j,l,q,\omega}^{(k-1)}(\bar{i}_{q,\omega} y) \frac{(\bar{j}_{q,\omega} t)^l}{\{l\}_q!} \right) \\
& = - \frac{2^{-k}}{(\bar{i}_{q,\omega} t)^{k-1}} E_{q,\omega t}(\bar{i} \bar{j}_{q,\omega} (x \oplus_{q,\omega} y) t) \left(\frac{\bar{i}_{q,\omega} t}{\lambda^i E_{q,\omega}(\bar{i}_{q,\omega} t) - 1} \right)^{k-1} \\
& \quad \times \left(\frac{2}{\lambda^j E_{q,\omega}(\bar{j}_{q,\omega} t) + 1} \right)^k \left(\frac{\lambda^{ij} E_{q,\omega}(\bar{i} \bar{j}_{q,\omega} t) - 1}{\lambda^i E_{q,\omega}(\bar{i}_{q,\omega} t) - 1} \right) \\
& \stackrel{\text{by(64)}}{=} - \frac{2^{-k}}{(\bar{i}_{q,\omega} t)^{k-1}}
\end{aligned}$$

$$\begin{aligned}
& \times \left(\sum_{\nu=0}^{\infty} \mathcal{F}_{\text{NWA}, \lambda^j, \nu, q, \omega}^{(k)}(\bar{i}_{q, \omega} x) \frac{(\bar{j}_{q, \omega} t)^\nu}{\{\nu\}_q!} \right) \\
& \times \left(\sum_{m=0}^{\infty} s_{\text{NWA}, \lambda^i, m, q, \omega}(j) \frac{(\bar{j}_{q, \omega} t)^m}{\{m\}_q!} \right) \\
& \times \left(\sum_{l=0}^{\infty} \mathcal{B}_{\text{NWA}, \lambda^i, l, q, \omega}^{(k-1)}(\bar{j}_{q, \omega} y) \frac{(\bar{i}_{q, \omega} t)^l}{\{l\}_q!} \right).
\end{aligned}$$

The theorem follows by equating the coefficients of $\frac{t^n}{\{n\}_q!}$. \square

Theorem 6.3 (A q, ω -analogue of [10, p. 3353]). *Under the assumption that i is even, a double sum of mixed q, ω -Apostol polynomials is equal to another double sum of mixed q, ω -Apostol polynomials.*

(113)

$$\begin{aligned}
& \sum_{\nu=0}^n \binom{n}{\nu}_q (\bar{i}_{q, \omega})^\nu (\bar{j}_{q, \omega})^{n-\nu} \mathcal{F}_{\text{NWA}, \lambda^j, n-\nu, q, \omega}^{(k-1)}(\bar{i}_{q, \omega} y) \sum_{m=0}^{i-1} \lambda^{jm} (-1)^m \\
& \times \mathcal{B}_{\text{NWA}, \lambda^i, \nu, q, \omega}^{(k)} \left(\bar{j}_{q, \omega} x \oplus_{q, \omega} \frac{\bar{j} m_{q, \omega}}{\bar{i}_{q, \omega}} \right) \\
& = - \frac{\{n\}_q (\bar{i}_{q, \omega})^k}{2 (\bar{i}_{q, \omega})^{k-1}} \sum_{k=0}^{n-1} \binom{n-1}{k}_q (\bar{i}_{q, \omega})^{n-k-1} (\bar{j}_{q, \omega})^k \mathcal{B}_{\text{NWA}, \lambda^i, n-k-1, q, \omega}^{(k-1)}(\bar{j}_{q, \omega} y) \\
& \times \sum_{m=0}^{j-1} \lambda^{im} \mathcal{F}_{\text{NWA}, \lambda^j, k, q, \omega}^{(k)} \left(\bar{i}_{q, \omega} x \oplus_{q, \omega} \frac{\bar{i} m_{q, \omega}}{\bar{j}_{q, \omega}} \right).
\end{aligned}$$

Proof. We can expand $g_{q, \omega}(t)$ as follows:

$$\begin{aligned}
(114) \quad g_{q, \omega}(t) & \stackrel{\text{by(109)}}{=} - \frac{2^{-k}}{(\bar{i}_{q, \omega} t)^{k-1}} E_{q, \omega t}(\bar{i} \bar{j}_{q, \omega} (x \oplus_{q, \omega} y) t) \\
& \times \left(\frac{\bar{i}_{q, \omega} t}{\lambda^i E_{q, \omega}(\bar{i}_{q, \omega} t) - 1} \right)^{k-1} \left(\frac{2}{\lambda^j E_{q, \omega}(\bar{j}_{q, \omega} t) + 1} \right)^k \\
& \times \left(\frac{\lambda^{ij} E_{q, \omega}(\bar{i} \bar{j}_{q, \omega} t) - 1}{\lambda^i E_{q, \omega}(\bar{i}_{q, \omega} t) - 1} \right) \\
& = - \frac{2^{-k}}{(\bar{i}_{q, \omega} t)^{k-1}} \left(\frac{\bar{i}_{q, \omega} t}{\lambda^i E_{q, \omega}(\bar{i}_{q, \omega} t) - 1} \right)^{k-1} \frac{2^k}{(\lambda^j E_{q, \omega}(\bar{j}_{q, \omega} t) + 1)^k} \\
& \times \sum_{m=0}^{j-1} \lambda^{im} \times E_{q, \omega t} \left(\left(\bar{i}_{q, \omega} x \oplus_{q, \omega} \frac{\bar{i} m_{q, \omega}}{\bar{j}_{q, \omega}} \right) \bar{j}_{q, \omega} t \right) E_{q, \omega t}(\bar{i} \bar{j}_{q, \omega} y t) \\
& = - \frac{2^{-k}}{(\bar{i}_{q, \omega} t)^{k-1}} \sum_{m=0}^{j-1} \lambda^{im} \sum_{l=0}^{\infty} \frac{(\bar{i}_{q, \omega})^l t^l}{\{l\}_q!} \mathcal{B}_{\text{NWA}, \lambda^i, l, q, \omega}^{(k-1)}(\bar{j}_{q, \omega} y)
\end{aligned}$$

$$\times \sum_{n=0}^{\infty} \frac{(\bar{j}_{q,\omega})^n t^n}{\{n\}_q!} \mathcal{F}_{\text{NWA},\lambda^j,n,q,\omega}^{(k)}(\bar{i}_{q,\omega} x \oplus_{q,\omega} \frac{\bar{i}m_{q,\omega}}{\bar{j}_{q,\omega}}).$$

The theorem follows by equating the coefficients of $\frac{t^n}{\{n\}_q!}$. \square

Theorem 6.4. *Assume that $\vec{\nu}$ on the left hand side is a vector with three components and with length n .*

$$\begin{aligned} (115) \quad & \sum_{|\nu|=n} \binom{n}{\vec{\nu}}_q (\bar{i}_{q,\omega})^{\nu_1} (\bar{j}_{q,\omega})^{\nu_2} (\bar{j}_{q,\omega})^{\nu_3} \mathcal{F}_{\text{NWA},\lambda^i,\nu_1,q,\omega}^{(k)}(\bar{j}_{q,\omega} x) \\ & \times \mathcal{B}_{\text{NWA},\lambda^j,\nu_2,q,\omega}^{(k-1)}(\bar{i}_{q,\omega} y) s_{\text{NWA},\lambda^j,\nu_3,q,\omega}(i) \\ & = \sum_{\nu=0}^n \binom{n}{\nu}_q (\bar{i}_{q,\omega})^\nu (\bar{j}_{q,\omega})^{n-\nu} \mathcal{B}_{\text{NWA},\lambda^j,n-\nu,q,\omega}^{(k-1)}(\bar{i}_{q,\omega} y) \\ & \times \sum_{m=0}^{i-1} \lambda^{jm} \mathcal{F}_{\text{NWA},\lambda^i,\nu,q,\omega}^{(k)}\left(\bar{j}_{q,\omega} x \oplus_{q,\omega} \frac{\bar{j}m_{q,\omega}}{\bar{i}_{q,\omega}}\right). \end{aligned}$$

Proof. Define the following function

$$\begin{aligned} \Psi_{q,\omega}(t) & \equiv \frac{\mathbb{E}_{q,\omega t}(\bar{i}\bar{j}_{q,\omega}(x \oplus_{q,\omega} y)t)(\lambda^{ij} \mathbb{E}_{q,\omega}(\bar{i}\bar{j}_{q,\omega}t) - 1)}{(\lambda^i \mathbb{E}_{q,\omega}(\bar{i}_{q,\omega}t) + 1)^k (\lambda^j \mathbb{E}_{q,\omega}(\bar{j}_{q,\omega}t) - 1)^k} t^{k-1} \\ (116) \quad & = \mathbb{E}_{q,\omega t}(\bar{i}\bar{j}_{q,\omega}(x \oplus_{q,\omega} y)t) \left(\frac{2}{\lambda^i \mathbb{E}_{q,\omega}(\bar{i}_{q,\omega}t) + 1} \right)^k \left(\frac{\bar{j}_{q,\omega}t}{\lambda^j \mathbb{E}_{q,\omega}(\bar{j}_{q,\omega}t) - 1} \right)^{k-1} \\ & \times \left(\frac{\lambda^{ij} \mathbb{E}_{q,\omega}(\bar{i}\bar{j}_{q,\omega}t) - 1}{\lambda^j \mathbb{E}_{q,\omega}(\bar{j}_{q,\omega}t) - 1} \right) \frac{2^{-k}}{(\bar{j}_{q,\omega})^{k-1}}. \end{aligned}$$

By using the formula for a geometric sequence, we can expand $\Psi_{q,\omega}(t)$ in two ways:

$$\begin{aligned} \Psi_{q,\omega}(t) & \stackrel{\text{by(64)}}{=} \left(\sum_{\nu=0}^{\infty} \mathcal{F}_{\text{NWA},\lambda^i,\nu,q,\omega}^{(k)}(\bar{j}_{q,\omega} x) \frac{(\bar{i}_{q,\omega}t)^\nu}{\{\nu\}_q!} \right) \\ (117) \quad & \times \left(\sum_{m=0}^{\infty} s_{\text{NWA},\lambda^j,m,q,\omega}(i) \frac{(\bar{j}_{q,\omega}t)^m}{\{m\}_q!} \right) \\ & \times \left(\sum_{l=0}^{\infty} \mathcal{B}_{\text{NWA},\lambda^j,l,q,\omega}^{(k-1)}(\bar{i}_{q,\omega} y) \frac{(\bar{j}_{q,\omega}t)^l}{\{l\}_q!} \right) \frac{2^{-k}}{(\bar{j}_{q,\omega})^{k-1}} \\ & = \frac{2^k}{(\lambda^i \mathbb{E}_{q,\omega}(\bar{i}_{q,\omega}t) - 1)^k} \frac{(\bar{j}_{q,\omega}t)^{k-1}}{(\lambda^j \mathbb{E}_{q,\omega}(\bar{j}_{q,\omega}t) - 1)^{k-1}} \sum_{m=0}^{i-1} \lambda^{jm} \\ & \times \mathbb{E}_{q,\omega t} \left(\left(\bar{j}_{q,\omega} x \oplus_{q,\omega} \bar{j}_{q,\omega} y \oplus_{q,\omega} \frac{\bar{j}m_{q,\omega}}{\bar{i}_{q,\omega}} \right) \bar{i}_{q,\omega} t \right) \frac{2^{-k}}{(\bar{j}_{q,\omega})^{k-1}} \end{aligned}$$

$$\begin{aligned}
&= \sum_{m=0}^{i-1} \frac{2^{-k} \lambda^{jm}}{(\bar{j}_{q,\omega})^{k-1}} \sum_{l=0}^{\infty} \frac{(\bar{i}_{q,\omega})^l t^l}{\{l\}_q!} \mathcal{F}_{\text{NWA},\lambda^i,l,q,\omega}^{(k)} \left(\bar{j}_{q,\omega} x \oplus_{q,\omega} \frac{\bar{j}m_{q,\omega}}{\bar{i}_{q,\omega}} \right) \\
&\quad \times \sum_{n=0}^{\infty} \frac{(\bar{j}_{q,\omega})^n t^n}{\{n\}_q!} \mathcal{B}_{\text{NWA},\lambda^j,n,q,\omega}^{(k-1)}(\bar{i}_{q,\omega} y).
\end{aligned}$$

The theorem follows by equating the coefficients of $\frac{t^n}{\{n\}_q!}$. \square

Similar formulas with \mathcal{H} polynomials can easily be constructed.

Theorem 6.5. *Assume that $\vec{\nu}$ on the left hand side is a vector with three components and length n .*

$$\begin{aligned}
&\sum_{|\nu|=n} \binom{n}{\nu}_q (\bar{i}_{q,\omega})^{\nu_1} (\bar{j}_{q,\omega})^{\nu_2} (\bar{j}_{q,\omega})^{\nu_3} \mathcal{H}_{\text{NWA},\lambda^i,\nu_1,q,\omega}^{(k)}(\bar{j}_{q,\omega} x) \\
&\quad \times \mathcal{B}_{\text{NWA},\lambda^j,\nu_2,q,\omega}^{(k-1)}(\bar{i}_{q,\omega} y) s_{\text{NWA},\lambda^j,\nu_3,q,\omega}(i) \\
(118) \quad &= \sum_{\nu=0}^n \binom{n}{\nu}_q (\bar{i}_{q,\omega})^{\nu} (\bar{j}_{q,\omega})^{n-\nu} \mathcal{B}_{\text{NWA},\lambda^j,n-\nu,q,\omega}^{(k-1)}(\bar{i}_{q,\omega} y) \\
&\quad \times \sum_{m=0}^{i-1} \lambda^{jm} \mathcal{H}_{\text{NWA},\lambda^i,\nu,q,\omega}^{(k)} \left(\bar{j}_{q,\omega} x \oplus_{q,\omega} \frac{\bar{j}m_{q,\omega}}{\bar{i}_{q,\omega}} \right).
\end{aligned}$$

Proof. Use $\Psi_{q,\omega}(t)$ again. \square

Theorem 6.6 (A q, ω -analogue of [10, (3.11) p. 3356]). *A triple sum of mixed q, ω -Apostol polynomials is equal to a double sum of mixed q, ω -Apostol polynomials.*

$$\begin{aligned}
&\sum_{|\nu|=n} \binom{n}{\nu}_q (\bar{i}_{q,\omega})^{\nu_1} (\bar{j}_{q,\omega})^{\nu_2} (\bar{j}_{q,\omega})^{\nu_3} \mathcal{F}_{\text{NWA},\lambda^i,\nu_1,q,\omega}^{(k)}(\bar{j}_{q,\omega} x) \\
&\quad \times \mathcal{B}_{\text{NWA},\lambda^j,\nu_2,q,\omega}^{(k-1)}(\bar{i}_{q,\omega} y) n_{\text{NWA},\lambda^j,\nu_3,q,\omega}(i) \\
(119) \quad &= \sum_{\nu=0}^n \binom{n}{\nu}_q (\bar{i}_{q,\omega})^{\nu} (\bar{j}_{q,\omega})^{n-\nu} \mathcal{B}_{\text{NWA},\lambda^j,n-\nu,q,\omega}^{(k-1)}(\bar{i}_{q,\omega} y) \\
&\quad \times \sum_{m=0}^{i-1} \lambda^{jm} \mathcal{F}_{\text{NWA},\lambda^i,\nu,q,\omega}^{(k)} \left(\bar{j}_{q,\omega} x \oplus_{q,\omega} \frac{\bar{j}m_{q,\omega}}{\bar{i}_{q,\omega}} \right).
\end{aligned}$$

Proof. Define the following function

$$\begin{aligned}
f_{q,\omega}(t) &\equiv \frac{E_{q,\omega t}(\bar{i}\bar{j}_{q,\omega}(x \oplus_{q,\omega} y)t)(\lambda^{ij}E_{q,\omega}(\bar{i}\bar{j}_{q,\omega}t) - 1)}{(\lambda^i E_{q,\omega}(\bar{i}_{q,\omega}t) + 1)^k (\lambda^j E_{q,\omega}(\bar{j}_{q,\omega}t) - 1)^k} t^k \\
(120) \quad &= E_{q,\omega t}(\bar{i}\bar{j}_{q,\omega}(x \oplus_{q,\omega} y)t) \left(\frac{2}{\lambda^i E_{q,\omega}(\bar{i}_{q,\omega}t) + 1} \right)^k
\end{aligned}$$

$$\times \left(\frac{\bar{j}_{q,\omega} t}{\lambda^j \mathbb{E}_{q,\omega}(\bar{j}_{q,\omega} t) - 1} \right)^{k-1} \left(\frac{\lambda^{ij} \mathbb{E}_{q,\omega}(\bar{i} \bar{j}_{q,\omega} t) - 1}{\lambda^j \mathbb{E}_{q,\omega}(\bar{j}_{q,\omega} t) - 1} \right) \frac{1}{2^k (\bar{j}_{q,\omega})^{k-1}}.$$

By using the formula for a geometric sequence, we can expand $f_{q,\omega}(t)$ in two ways:

$$\begin{aligned} (121) \quad f_{q,\omega}(t) &\stackrel{\text{by(64)}}{=} \left(\sum_{\nu=0}^{\infty} \mathcal{B}_{\text{NWA},\lambda^i,\nu,q,\omega}^{(k)}(\bar{j}_{q,\omega} x) \frac{(\bar{i}_{q,\omega} t)^\nu}{\{\nu\}_q!} \right) \\ &\times \left(\sum_{m=0}^{\infty} s_{\text{NWA},\lambda^j,m,q,\omega}(i) \frac{(\bar{j}_{q,\omega} t)^m}{\{m\}_q!} \right) \\ &\times \left(\sum_{l=0}^{\infty} \mathcal{B}_{\text{NWA},\lambda^j,l,q,\omega}^{(k-1)}(\bar{i}_{q,\omega} y) \frac{(\bar{j}_{q,\omega} t)^l}{\{l\}_q!} \right) \frac{1}{2^k (\bar{j}_{q,\omega})^{k-1}} \\ &= \frac{2^k}{(\lambda^i \mathbb{E}_{q,\omega}(\bar{i}_{q,\omega} t) + 1)^k} \frac{(\bar{j}_{q,\omega} t)^{k-1}}{(\lambda^j \mathbb{E}_{q,\omega}(\bar{j}_{q,\omega} t) - 1)^{k-1}} \sum_{m=0}^{i-1} \lambda^{jm} \\ &\times \mathbb{E}_{q,\omega t} \left(\left(\bar{j}_{q,\omega} x \oplus_{q,\omega} \bar{j}_{q,\omega} y \oplus_{q,\omega} \frac{\bar{j} m_{q,\omega}}{\bar{i}_{q,\omega}} \right) \bar{i}_{q,\omega} t \right) \frac{1}{2^k (\bar{j}_{q,\omega})^{k-1}} \\ &= \sum_{m=0}^{i-1} \frac{\lambda^{jm}}{2^k (\bar{j}_{q,\omega})^{k-1}} \sum_{l=0}^{\infty} \frac{(\bar{i}_{q,\omega})^l t^l}{\{l\}_q!} \mathcal{F}_{\text{NWA},\lambda^i,l,q,\omega}^{(k)} \left(\bar{j}_{q,\omega} x \oplus_{q,\omega} \frac{\bar{j} m_{q,\omega}}{\bar{i}_{q,\omega}} \right) \\ &\times \sum_{n=0}^{\infty} \frac{(\bar{j}_{q,\omega})^n t^n}{\{n\}_q!} \mathcal{B}_{\text{NWA},\lambda^j,n,q,\omega}^{(k-1)}(\bar{i}_{q,\omega} y). \end{aligned}$$

The theorem follows by equating the coefficients of $\frac{t^n}{\{n\}_q!}$. \square

7. Discussion

Many of the proofs use the formula for a geometric sequence in q, ω -form and the generating function for the q, ω -Appell polynomials and the power sums. The integers i and j are crucial for the formulas; by the generating function, if λ^i, ν appears as index in a polynomial, certainly the factor $(\bar{i}_{q,\omega})^\nu$ will also appear. If the orders of two polynomials in a formula are k and $k-1$, the last one with index λ^j , and argument $\bar{i}_{q,\omega} y$, a function $\sigma_{\text{NWA},\lambda^j,m,q,\omega}(i)$ or $s_{\text{NWA},\lambda^j,m,q,\omega}(i)$, together with $(\bar{j}_{q,\omega})^m$ will appear. If a polynomial has λ^i, ν as index, it will have $(\bar{j}_{q,\omega})$ in the function argument, and vice versa. Most of the q -transformations can be generalized to the q, ω case, a general exception is when the q, ω -addition is expanded and coefficients of the dummy variable t are equated.

References

- [1] M. H. Annaby, A. E. Hamza, and K. A. Aldwoah, *Hahn difference operator and associated Jackson-Nörlund integrals*, J. Optim. Theory Appl. **154** (2012), no. 1, 133–153. <https://doi.org/10.1007/s10957-012-9987-7>
- [2] T. Ernst, *A comprehensive treatment of q-calculus*, Birkhäuser/Springer Basel AG, Basel, 2012. <https://doi.org/10.1007/978-3-0348-0431-8>
- [3] ———, *Multiplication formulas for q-Appell polynomials and the multiple q-power sums*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **70** (2016), no. 1, 1–18.
- [4] ———, *Expansion formulas for Apostol type q-Appell polynomials, and their special cases*, Matematiche (Catania) **73** (2018), no. 1, 3–24. <https://doi.org/10.4418/2018.73.1.1>
- [5] ———, *A new semantics for special functions*. In preparation.
- [6] ———, *On the exponential and trigonometric q, ω -special functions*, to appear in Springer: SPAS2017 Volume 2, Algebraic structure and applications.
- [7] ———, *On the ring of q, ω -Appell polynomials and the related q, ω -calculus*, In preparation.
- [8] ———, *On three general q, ω -Apostol polynomials and their connection to q, ω -power sums*, Adv. Dyn. Syst. Appl. **14** (2019), no. 2, 119–148.
- [9] W. P. Johnson, *q-extensions of identities of Abel-Rothe type*, Discrete Math. **159** (1996), no. 1-3, 161–177. [https://doi.org/10.1016/0012-365X\(95\)00108-9](https://doi.org/10.1016/0012-365X(95)00108-9)
- [10] H. Liu and W. Wang, *Some identities on the Bernoulli, Euler and Genocchi polynomials via power sums and alternate power sums*, Discrete Math. **309** (2009), no. 10, 3346–3363. <https://doi.org/10.1016/j.disc.2008.09.048>
- [11] Q.-M. Luo, *The multiplication formulas for the Apostol-Bernoulli and Apostol-Euler polynomials of higher order*, Integral Transforms Spec. Funct. **20** (2009), no. 5-6, 377–391. <https://doi.org/10.1080/10652460802564324>
- [12] L. M. Milne-Thomson, *The Calculus of Finite Differences*, Macmillan and Co., Ltd., London, 1951.
- [13] T. Nagell, *Textbook in Algebra* (Dutch), Almqvist & Wiksells Akademiska Handböcker. Hugo Gebers Förlag. Uppsala,], 1949.
- [14] S. Varma, B. Yasar, and M. Özarslan, *Hahn-Appell polynomials and their d-orthogonality*, Revista de la Real Academia de Ciencias Exb, 2018.
- [15] W. Wang and W. Wang, *Some results on power sums and Apostol-type polynomials*, Integral Transforms Spec. Funct. **21** (2010), no. 3-4, 307–318. <https://doi.org/10.1080/10652460903169288>
- [16] S. Yang, *An identity of symmetry for the Bernoulli polynomials*, Discrete Math. **308** (2008), no. 4, 550–554. <https://doi.org/10.1016/j.disc.2007.03.030>
- [17] Z. Zhang and H. Yang, *Several identities for the generalized Apostol-Bernoulli polynomials*, Comput. Math. Appl. **56** (2008), no. 12, 2993–2999. <https://doi.org/10.1016/j.camwa.2008.07.038>

THOMAS ERNST
 DEPARTMENT OF MATHEMATICS
 UPPSALA UNIVERSITY
 P.O. BOX 480, SE-751 06 UPPSALA, SWEDEN
 Email address: thomas@math.uu.se