

SOME RESULTS ON PROJECTIVE CURVATURE TENSOR IN SASAKIAN MANIFOLDS

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ABSTRACT. In the present paper, we study certain curvature conditions satisfying by the projective curvature tensor in Sasakian manifolds with respect to the generalized-Tanaka-Webster connection. Finally, we give an example of a 3-dimensional Sasakian manifold with respect to the generalized-Tanaka-Webster connection.

1. Introduction

In 1969, S. Tanno classified connected almost contact metric manifolds whose automorphism groups possess the maximum dimension [15]. For such a manifold, the sectional curvature of plane sections containing ξ is a constant, say c . He showed that they can be divided into three classes: (1) homogeneous normal contact Riemannian manifolds with $c > 0$, (2) global Riemannian products of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvature if $c = 0$ and (3) a warped product space $R \times_f C$ if $c > 0$. It is known that the manifolds of class (1) are characterized by admitting a Sasakian structure.

The Tanaka-Webster connection [14, 17] is defined as the canonical affine connection on a non-degenerate, pseudo-Hermitian CR -submanifold. S. Tanno [16] defined the generalized Tanaka-Webster connection for contact metric manifolds by the canonical connection which coincides with the Tanaka-Webster connection if the associated CR -structure is integrable.

A Riemannian manifold is said to be Ricci semisymmetric if the curvature tensor R satisfies

$$(1) \quad R \cdot S = 0,$$

where $R(X, Y)$ is considered as a field of linear operators acting on S and S is the Ricci tensor of type $(0, 2)$. The class of Ricci semisymmetric manifolds includes the set of Ricci symmetric manifolds ($\nabla S = 0$) as a proper subset.

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In 1977, T. Takahashi [13] introduced the notion of locally ϕ -symmetric Sasakian manifolds as a weaker version of local symmetry of such manifolds. U. C. De [3] and D. G. Prakasha [8] have studied ϕ -symmetric Kenmotsu manifolds and have obtained some interesting results. In 2010, S. S. Shukla and M. K. Shukla [11] have studied ϕ -symmetric para-Sasakian manifolds.

The paper is organized as follows: In Section 2, we give a brief introduction of Sasakian manifolds. In Section 3, we deduce the relation between the curvature tensors with respect to the g -Tanaka-Webster connection and the Levi-Civita connection. In Section 4, we show that the Ricci symmetries in Sasakian manifolds with respect to the g -Tanaka-Webster connection and the Levi-Civita connection are equivalent if and only if the manifold is an Einstein manifold. Section 5 deals with the study of partially Ricci-pseudosymmetric Sasakian manifolds with respect to the g -Tanaka-Webster connection. Section 6 is devoted to study projectively flat, pseudoprojectively flat and ϕ -projectively semisymmetric Sasakian manifolds with respect to the g -Tanaka-Webster connection. In Section 7, we show that a Sasakian manifold is locally projectively ϕ -symmetric with respect to the g -Tanaka-Webster connection if and only if it is so with respect to the Levi-Civita connection. Finally, we construct an example of 3-dimensional Sasakian manifold with the generalized-Tanaka-Webster connection.

2. Preliminaries

Let M be an almost contact metric manifold of dimension $(2n+1)$ equipped with an almost contact metric structure (ϕ, ξ, η, g) admitting a $(1, 1)$ tensor field ϕ , a vector field ξ , a 1-form η and a Riemannian metric g . Then [1]

$$(2) \quad \phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0,$$

$$(3) \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X),$$

$$(4) \quad g(X, \phi Y) = -g(\phi X, Y)$$

for all vector fields X, Y on M .

A normal contact metric manifold is known as a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if ([9, 10])

$$(5) \quad (\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X$$

for all vector fields X, Y on M , where ∇ is the Levi-Civita connection of the Riemannian metric. From the above equations it follows that

$$(6) \quad \nabla_X \xi = -\phi X,$$

$$(7) \quad (\nabla_X \eta)Y = g(X, \phi Y).$$

Moreover, the curvature tensor R , the Ricci tensor S and the Ricci operator Q in a Sasakian manifold M with respect to the Levi-Civita connection satisfy

([2, 12])

$$(8) \quad R(X, Y)\xi = \eta(Y)X - \eta(X)Y,$$

$$(9) \quad S(X, \xi) = 2n\eta(X), \quad Q\xi = 2n\xi,$$

$$(10) \quad S(\phi X, \phi Y) = S(X, Y) - 2n\eta(X)\eta(Y),$$

$$(11) \quad \eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y)$$

for any vector fields X, Y and Z on M .

Sasakian manifolds have been studied by many authors in several ways to a different extent such as U. C. De and G. Ghosh [4], M. K. Dwivedi and J. S. Kim [5], I. Hasegawa and I. Mihai [6], V. A. Khan and M. A. Khan [7] and many others.

Definition. A Sasakian manifold M is said to be an η -Einstein manifold if its Ricci tensor S is of the form

$$(12) \quad S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y),$$

where a and b are scalar functions on M .

A relation between the g -Tanaka-Webster connection $\bar{\nabla}$ and the Levi-Civita connection ∇ in a Sasakian manifold M is given by [4]

$$(13) \quad \bar{\nabla}_X Y = \nabla_X Y + g(X, \phi Y)\xi + \eta(Y)\phi X - \eta(X)\phi Y$$

for all vector fields X, Y on M .

3. Curvature tensor and Ricci tensor in Sasakian manifolds with respect to the g -Tanaka-Webster connection

Let $\bar{R}, \bar{S}, \bar{r}$ and \bar{Q} be the curvature tensor, the Ricci tensor, the scalar curvature and the Ricci operator with respect to the g -Tanaka-Webster connection $\bar{\nabla}$ in a Sasakian manifold M , respectively. Then we have [4]

$$(14) \quad \begin{aligned} \bar{R}(X, Y)Z &= R(X, Y)Z + [g(X, Z)\eta(Y) - g(Y, Z)\eta(X)]\xi - g(Y, \phi Z)\phi X \\ &\quad + g(X, \phi Z)\phi Y + 2g(Y, \phi X)\phi Z - \eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y, \end{aligned}$$

$$(15) \quad \bar{R}(X, Y)\xi = \bar{R}(\xi, Y)Z = \eta(\bar{R}(X, Y)Z) = 0,$$

$$(16) \quad \bar{S}(Y, Z) = S(Y, Z) - 2g(Y, Z) - 2(n-1)\eta(Y)\eta(Z),$$

$$(17) \quad \bar{S}(X, \xi) = 0,$$

$$(18) \quad \bar{Q}Y = QY - 2Y - 2(n-1)\eta(Y)\xi, \quad \bar{Q}\xi = 0,$$

$$(19) \quad \bar{r} = r - 6n$$

for all vector fields $X, Y, Z \in \chi(M)$.

Now, let R and \bar{R} be the curvature tensors of type $(0, 4)$ of the connections ∇ and $\bar{\nabla}$, respectively on M given by

$$R(X, Y, Z, W) = g(R(X, Y)Z, W) \text{ and } \bar{R}(X, Y, Z, W) = g(\bar{R}(X, Y)Z, W).$$

Therefore from (14), it follows that

$$\begin{aligned} \bar{R}(X, Y, Z, U) &= R(X, Y, Z, U) + [g(X, Z)\eta(Y) - g(Y, Z)\eta(X)]\eta(U) \\ &\quad - g(Y, \phi Z)g(\phi X, U) + g(X, \phi Z)g(\phi Y, U) \\ (20) \quad &\quad + 2g(Y, \phi X)g(\phi Z, U) - \eta(Y)\eta(Z)g(X, U) \\ &\quad + \eta(X)\eta(Z)g(Y, U). \end{aligned}$$

Interchanging X and Y in (20), we have

$$\begin{aligned} \bar{R}(Y, X, Z, U) &= R(Y, X, Z, U) + [g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]\eta(U) \\ &\quad - g(X, \phi Z)g(\phi Y, U) + g(Y, \phi Z)g(\phi X, U) \\ (21) \quad &\quad + 2g(X, \phi Y)g(\phi Z, U) - \eta(X)\eta(Z)g(Y, U) \\ &\quad + \eta(Y)\eta(Z)g(X, U). \end{aligned}$$

On adding (20) and (21) and using the fact that $R(X, Y, Z, U) + R(Y, X, Z, U) = 0$, we get

$$(22) \quad \bar{R}(X, Y, Z, U) + \bar{R}(Y, X, Z, U) = 0.$$

Now interchanging U and Z in (20), we have

$$\begin{aligned} \bar{R}(X, Y, U, Z) &= R(X, Y, U, Z) + [g(X, U)\eta(Y) - g(Y, U)\eta(X)]\eta(Z) \\ &\quad - g(Y, \phi U)g(\phi X, Z) + g(X, \phi U)g(\phi Y, Z) \\ (23) \quad &\quad + 2g(Y, \phi X)g(\phi U, Z) - \eta(Y)\eta(U)g(X, Z) \\ &\quad + \eta(X)\eta(U)g(Y, Z). \end{aligned}$$

By adding (20) and (23) and using the fact that $R(X, Y, Z, U) + R(X, Y, U, Z) = 0$, we get

$$(24) \quad \bar{R}(X, Y, Z, U) + \bar{R}(X, Y, U, Z) = 0.$$

Interchanging pair of slots in (20), we have

$$\begin{aligned} \bar{R}(Z, U, X, Y) &= R(Z, U, X, Y) + [g(Z, X)\eta(U) - g(U, X)\eta(Z)]\eta(Y) \\ &\quad - g(U, \phi X)g(\phi Z, Y) + g(Z, \phi X)g(\phi U, Y) \\ (25) \quad &\quad + 2g(U, \phi Z)g(\phi X, Y) - \eta(U)\eta(X)g(Z, Y) \\ &\quad + \eta(Z)\eta(X)g(U, Y). \end{aligned}$$

By subtracting (25) from (20) and using fact that $R(X, Y, Z, U) - R(Z, U, X, Y) = 0$, we get

$$(26) \quad \bar{R}(X, Y, Z, U) - \bar{R}(Z, U, X, Y) = 0.$$

Interchanging X and Y in (23), we have

$$(27) \quad \bar{R}(Y, X, U, Z) = R(Y, X, U, Z) + [g(Y, U)\eta(X) - g(X, U)\eta(Y)]\eta(Z)$$

$$\begin{aligned}
& -g(X, \phi U)g(\phi Y, Z) + g(Y, \phi U)g(\phi X, Z) \\
& + 2g(X, \phi Y)g(\phi U, Z) - \eta(X)\eta(U)g(Y, Z) \\
& + \eta(Y)\eta(U)g(X, Z).
\end{aligned}$$

By subtracting (27) from (23) and using the fact that

$$R(X, Y, Z, U) - R(Y, X, U, Z) = 0,$$

we get

$$(28) \quad \bar{R}(X, Y, Z, U) - \bar{R}(Y, X, U, Z) = 0.$$

Thus in view of (22), (24), (26) and (28), we can state the following theorem:

Theorem 3.1. *In a Sasakian manifold with respect to the g -Tanaka-Webster connection, we have*

- (i) $\bar{R}(X, Y, Z, U) + \bar{R}(Y, X, Z, U) = 0,$
- (ii) $\bar{R}(X, Y, Z, U) + \bar{R}(X, Y, U, Z) = 0,$
- (iii) $\bar{R}(X, Y, Z, U) - \bar{R}(Z, U, X, Y) = 0,$
- (iv) $\bar{R}(X, Y, Z, U) - \bar{R}(Y, X, U, Z) = 0$

for any vector fields $X, Y, Z, U \in \chi(M)$.

4. Ricci symmetry in a Sasakian manifold with respect to the connections $\bar{\nabla}$ and ∇

Assuming that the manifold is Ricci symmetric with respect to the g -Tanaka-Webster connection $\bar{\nabla}$, we have

$$(29) \quad (\bar{R}(X, Y) \cdot \bar{S})(U, V) = -\bar{S}(\bar{R}(X, Y)U, V) - \bar{S}(U, \bar{R}(X, Y)V)$$

for all $X, Y, U, V \in \chi(M)$. In view of (14) and (16), (29) takes the form

$$\begin{aligned}
(30) \quad (\bar{R}(X, Y) \cdot \bar{S})(U, V) = & -\bar{S}(R(X, Y)U, V) + g(Y, \phi U)\bar{S}(\phi X, V) \\
& - g(X, \phi U)\bar{S}(\phi Y, V) - 2g(Y, \phi X)\bar{S}(\phi U, V) \\
& + \eta(Y)\eta(U)\bar{S}(X, V) - \eta(X)\eta(U)\bar{S}(Y, V) \\
& - \bar{S}(U, R(X, Y)V) + g(Y, \phi V)\bar{S}(\phi X, U) \\
& - g(X, \phi V)\bar{S}(\phi Y, U) - 2g(Y, \phi X)\bar{S}(\phi V, U) \\
& + \eta(Y)\eta(V)\bar{S}(X, U) - \eta(X)\eta(V)\bar{S}(Y, U).
\end{aligned}$$

By using (4) and (16) in (30), we have

$$\begin{aligned}
(31) \quad (\bar{R}(X, Y) \cdot \bar{S})(U, V) = & (R(X, Y) \cdot S)(U, V) + 2[R(X, Y, U, V) + R(X, Y, V, U)] \\
& + 2(n-1)[\eta(R(X, Y)U)\eta(V) + \eta(R(X, Y)V)\eta(U)] \\
& + S(\phi X, V)g(Y, \phi U) - g(X, \phi U)S(\phi Y, V) \\
& - 2g(Y, \phi X)S(\phi U, V) + \eta(Y)\eta(U)S(X, V) \\
& - 2\eta(Y)\eta(U)g(X, V) - \eta(X)\eta(U)S(Y, V) \\
& + 2\eta(X)\eta(U)g(Y, V) + g(Y, \phi V)S(\phi X, U)
\end{aligned}$$

$$\begin{aligned}
& -g(X, \phi V)S(\phi Y, U) - 2g(Y, \phi X)S(\phi V, U) \\
& + \eta(Y)\eta(V)S(X, U) - 2\eta(Y)\eta(V)g(X, U) \\
& - \eta(X)\eta(V)S(Y, U) + 2\eta(X)\eta(V)g(Y, U)
\end{aligned}$$

which by using (11) and the fact that $R(X, Y, U, V) + R(X, Y, V, U) = 0$ turns to

$$\begin{aligned}
(32) \quad (\bar{R}(X, Y) \cdot \bar{S})(U, V) &= (R(X, Y) \cdot S)(U, V) + S(\phi X, V)g(Y, \phi U) \\
& - g(X, \phi U)S(\phi Y, V) + g(Y, \phi V)S(\phi X, U) \\
& - g(X, \phi V)S(\phi Y, U) + \eta(Y)\eta(U)S(X, V) \\
& - \eta(X)\eta(U)S(Y, V) + \eta(Y)\eta(V)S(X, U) \\
& - \eta(X)\eta(V)S(Y, U) + 2n[g(Y, U)\eta(X)\eta(V) \\
& - g(X, U)\eta(Y)\eta(V) + g(Y, V)\eta(X)\eta(U) \\
& - g(X, V)\eta(Y)\eta(U)].
\end{aligned}$$

Suppose that $(\bar{R}(X, Y) \cdot \bar{S})(U, V) = (R(X, Y) \cdot S)(U, V)$, then from (32), it follows that

$$\begin{aligned}
(33) \quad & g(Y, \phi U)S(\phi X, V) - g(X, \phi U)S(\phi Y, V) + g(Y, \phi V)S(\phi X, U) \\
& - g(X, \phi V)S(\phi Y, U) + \eta(Y)\eta(U)S(X, V) - \eta(X)\eta(U)S(Y, V) \\
& + \eta(Y)\eta(V)S(X, U) - \eta(X)\eta(V)S(Y, U) + 2n[g(Y, U)\eta(X)\eta(V) \\
& - g(X, U)\eta(Y)\eta(V) + g(Y, V)\eta(X)\eta(U) - g(X, V)\eta(Y)\eta(U)] = 0.
\end{aligned}$$

Taking $X = U = \xi$ in (33) and then using (2), (3) and (9), we obtain

$$S(Y, V) = 2ng(Y, V).$$

If $S(Y, V) = 2ng(Y, V)$, then from (32), it follows that

$$(\bar{R}(X, Y) \cdot \bar{S})(U, V) = (R(X, Y) \cdot S)(U, V).$$

Thus we have the following theorem:

Theorem 4.1. *Ricci symmetries in a Sasakian manifold with respect to the g -Tanaka-Webster connection and the Levi-Civita connection are equivalent if and only if the manifold is an Einstein manifold with respect to the Levi-Civita connection.*

5. Partially Ricci-pseudosymmetric Sasakian manifolds with respect to the g -Tanaka-Webster connection

Definition. A Sasakian manifold M is said to be partially Ricci-pseudosymmetric if and only if the relation

$$(34) \quad R \cdot S = f(p)Q(g, S)$$

holds on the set $A = [x \in M : Q(g, S) \neq 0 \text{ at } x]$, where $f \in C^\infty(A)$ for $p \in (A)$, $R \cdot S$ and $Q(g, S)$ are respectively defined by

$$(35) \quad (R(X, Y) \cdot S)(U, V) = -S(R(X, Y)U, V) - S(U, R(X, Y)V)$$

and

$$(36) \quad Q(g, S) = ((X \wedge_g Y) \cdot S)(U, V), \text{ where } (X \wedge_g Y)Z = g(Y, Z)X - g(X, Z)Y$$

for all X, Y, U and $V \in \chi(M)$.

Let a $(2n + 1)$ -dimensional Sasakian manifold with respect to the g -Tanaka-Webster connection be partially Ricci-pseudosymmetric. Then we have

$$(37) \quad (\bar{R}(X, Y) \cdot \bar{S})(U, V) = f(p)\bar{Q}(g, \bar{S})(X, Y; U, V).$$

It is equivalent to

$$(\bar{R}(X, Y) \cdot \bar{S})(U, V) = f(p)[((X \wedge_g Y) \cdot \bar{S})(U, V)]$$

for all $X, Y, U, V \in \chi(M)$. From the above relation it follows that

$$(38) \quad \begin{aligned} & \bar{S}[\bar{R}(X, Y)U, V] + \bar{S}[U, \bar{R}(X, Y)V] \\ &= f(p)[\bar{S}((X \wedge_g Y)U, V) + \bar{S}(U, (X \wedge_g Y)V)] \end{aligned}$$

which by taking $X = U = \xi$ and then using (15) and (17) reduces to

$$(39) \quad f(p)\bar{S}(Y, V) = 0.$$

In view of (16), we have from (39)

$$(40) \quad f(p)[S(Y, V) - 2g(Y, V) - 2(n - 1)\eta(Y)\eta(V)] = 0.$$

Then we have, either $f(p) = 0$ or, the manifold is an η -Einstein manifold of the form

$$(41) \quad S(Y, V) = 2g(Y, V) + 2(n - 1)\eta(Y)\eta(V).$$

Thus we can state the following theorem:

Theorem 5.1. *A partially Ricci-pseudosymmetric Sasakian manifold with respect to the g -Tanaka-Webster connection is an η -Einstein manifold with respect to the Levi-Civita connection, provided $f(p) \neq 0$.*

6. Projectively flat, pseudo projectively flat and ϕ -projectively semisymmetric Sasakian manifolds with respect to the g -Tanaka-Webster connection

Let M be a $(2n + 1)$ -dimensional Riemannian manifold. If there exists a one to one correspondence between each coordinate neighbourhood of M and a domain in Euclidean space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n \geq 1$, M is locally projectively flat if and only if

the projective curvature tensor vanishes. Here the projective curvature tensor P with respect to the Levi-Civita connection ∇ is defined by [18]

$$(42) \quad P(X, Y)Z = R(X, Y)Z - \frac{1}{2n}[S(Y, Z)X - S(X, Z)Y]$$

for all $X, Y, Z \in \chi(M)$, where R is the Riemannian curvature tensor and S is the Ricci tensor with respect to the Levi-Civita connection.

Now the projective curvature tensor with respect to the g -Tanaka-Webster connection $\bar{\nabla}$ is given by

$$(43) \quad \bar{P}(X, Y)Z = \bar{R}(X, Y)Z - \frac{1}{2n}[\bar{S}(Y, Z)X - \bar{S}(X, Z)Y]$$

for all $X, Y, Z \in \chi(M)$, where \bar{R} is the Riemannian curvature tensor and \bar{S} is the Ricci tensor with respect to the g -Tanaka-Webster connection.

Definition. A Sasakian manifold is said to be

(i) projectively flat with respect to the g -Tanaka-Webster connection, if

$$(44) \quad \bar{P}(X, Y)Z = 0, \quad X, Y, Z \in \chi(M),$$

(ii) pseudoprojectively flat with respect to the g -Tanaka-Webster connection, if

$$(45) \quad g(\bar{P}(\phi X, Y)Z, \phi W) = 0, \quad X, Y, Z, W \in \chi(M),$$

(iii) ϕ -projectively semisymmetric with respect to the g -Tanaka-Webster connection, if

$$(46) \quad (\bar{P}(X, Y) \cdot \phi)Z = 0, \quad X, Y \in \chi(M).$$

First, we consider that the manifold M with respect to the g -Tanaka-Webster connection is projectively flat. Then from (43) and (44) it follows that

$$(47) \quad \bar{R}(X, Y)Z = \frac{1}{2n}[\bar{S}(Y, Z)X - \bar{S}(X, Z)Y].$$

Taking inner product of (47) with ξ and using (3), we have

$$(48) \quad g(\bar{R}(X, Y)Z, \xi) = \frac{1}{2n}[\bar{S}(Y, Z)\eta(X) - \bar{S}(X, Z)\eta(Y)]$$

which in view of (15) turns to

$$(49) \quad \bar{S}(Y, Z)\eta(X) - \bar{S}(X, Z)\eta(Y) = 0.$$

By taking $Y = \xi$ in (49) and then using (2) and (15), we get

$$(50) \quad \bar{S}(X, Z) = 0.$$

In view of (16), (50) yields

$$(51) \quad S(X, Z) = 2g(X, Z) + 2(n-1)\eta(X)\eta(Z).$$

Thus we can state the following theorem:

Theorem 6.1. *A projectively flat Sasakian manifold with respect to the g -Tanaka-Webster connection is an η -Einstein manifold with respect to the Levi-Civita connection.*

Secondly, we consider that the manifold M with respect to the g -Tanaka-Webster connection is pseudoprojectively flat. Then from (43) and (45) it follows that

$$(52) \quad g[\bar{R}(\phi X, Y)Z, \phi W] = \frac{1}{2n}[\bar{S}(Y, Z)g(\phi X, \phi W) - \bar{S}(\phi X, Z)g(Y, \phi W)].$$

By using (14) and (16) in the above equation, we have

$$(53) \quad \begin{aligned} g[R(\phi X, Y)Z, \phi W] &= g(Y, \phi Z)g(\phi^2 X, \phi W) - g(\phi X, \phi Z)g(\phi Y, \phi W) \\ &\quad - 2g(Y, \phi^2 X)g(\phi Z, \phi W) + \eta(Y)\eta(Z)g(\phi X, \phi W) \\ &\quad + \frac{1}{2n}[(S(Y, Z) - 2g(Y, Z) \\ &\quad - 2(n-1)\eta(Y)\eta(Z))g(\phi X, \phi W) \\ &\quad - (S(\phi X, Z) - 2g(\phi X, Z))g(Y, \phi W)]. \end{aligned}$$

Using the relation $g(\phi^2 X, \phi W) = -g(X, \phi W)$, (53) becomes

$$(54) \quad \begin{aligned} g[R(\phi X, Y)Z, \phi W] &= -g(Y, \phi Z)g(X, \phi W) - g(\phi X, \phi Z)g(\phi Y, \phi W) \\ &\quad + 2g(\phi X, \phi Y)g(\phi Z, \phi W) + \eta(Y)\eta(Z)g(\phi X, \phi W) \\ &\quad + \frac{1}{2n}[(S(Y, Z) - 2g(Y, Z) \\ &\quad - 2(n-1)\eta(Y)\eta(Z))g(\phi X, \phi W) \\ &\quad - (S(\phi X, Z) - 2g(\phi X, Z))g(Y, \phi W)]. \end{aligned}$$

Let $\{e_1, e_2, \dots, e_{2n}, \xi\}$ be a local orthonormal basis of the vector fields in M . Then $\{\phi e_1, \phi e_2, \dots, \phi e_{2n}, \xi\}$ is also a local orthonormal basis. If we put $X = W = e_i$ in (54) and sum up with respect to i , then we have

$$(55) \quad \begin{aligned} &\sum_{i=1}^{2n} g[R(\phi e_i, Y)Z, \phi e_i] \\ &= -g(Y, \phi Z) \sum_{i=1}^{2n} g(e_i, \phi e_i) - \sum_{i=1}^{2n} g(\phi e_i, \phi Z)g(\phi Y, \phi e_i) \\ &\quad + 2 \sum_{i=1}^{2n} g(\phi e_i, \phi Y)g(\phi Z, \phi e_i) + \eta(Y)\eta(Z) \sum_{i=1}^{2n} g(\phi e_i, \phi e_i) \\ &\quad + \frac{1}{2n}[(S(Y, Z) - 2g(Y, Z) - 2(n-1)\eta(Y)\eta(Z)) \sum_{i=1}^{2n} g(\phi e_i, \phi e_i) \\ &\quad - \sum_{i=1}^{2n} (S(\phi e_i, Z) - 2g(\phi e_i, Z))g(Y, \phi e_i)]. \end{aligned}$$

It can be easily verified that

$$(56) \quad \sum_{i=1}^{2n} g[R(\phi e_i, Y)Z, \phi e_i] = S(Y, Z) - g(Y, Z) + \eta(Y)\eta(Z),$$

$$(57) \quad \sum_{i=1}^{2n} g(e_i, \phi e_i) = 0,$$

$$(58) \quad \sum_{i=1}^{2n} g(\phi e_i, \phi Z)g(\phi Y, \phi e_i) = g(\phi Y, \phi Z),$$

$$(59) \quad \sum_{i=1}^{2n} g(\phi e_i, \phi e_i) = 2n,$$

$$(60) \quad \sum_{i=1}^{2n} g(\phi e_i, Z)g(Y, \phi e_i) = g(Y, Z) - \eta(Y)\eta(Z),$$

$$(61) \quad \sum_{i=1}^{2n} S(\phi e_i, Z)g(Y, \phi e_i) = S(Y, Z) - 2n\eta(Y)\eta(Z).$$

By virtue of (56)-(61), the equation (55) takes the form

$$(62) \quad \begin{aligned} S(Y, Z) &= 2g(\phi Y, \phi Z) + 2n\eta(Y)\eta(Z) \\ &+ \frac{2n-1}{2n}[S(Y, Z) - 2g(Y, Z) - 2(n-1)\eta(Y)\eta(Z)] \end{aligned}$$

which after simplification gives

$$(63) \quad S(Y, Z) = 2g(Y, Z) + 2(n-1)\eta(Y)\eta(Z).$$

Thus we can state the following theorem:

Theorem 6.2. *A pseudoprojectively flat Sasakian manifold with respect to the g -Tanaka-Webster connection is an η -Einstein manifold with respect to the Levi-Civita connection.*

Now, we consider that the manifold M with respect to the g -Tanaka-Webster connection is ϕ -projectively semisymmetric. Then from (46), we have

$$(64) \quad (\bar{P}(X, Y) \cdot \phi)Z = \bar{P}(X, Y)\phi Z - \phi\bar{P}(X, Y)Z = 0.$$

Taking $X = \xi$ in (64), we have

$$(65) \quad (\bar{P}(\xi, Y) \cdot \phi)Z = \bar{P}(\xi, Y)\phi Z - \phi\bar{P}(\xi, Y)Z = 0.$$

From (43), we find

$$(66) \quad \bar{P}(\xi, Y)\phi Z = -\frac{1}{2n}\bar{S}(Y, \phi Z)\xi, \quad \phi\bar{P}(\xi, Y)Z = 0.$$

By combining (65) and (66), we obtain $\bar{S}(Y, \phi Z) = 0$, which in view of (16) gives

$$(67) \quad S(Y, \phi Z) = 2g(Y, \phi Z).$$

Now by replacing Y by ϕY then using (3) and (10), (67) takes the form

$$(68) \quad S(Y, Z) = 2g(Y, Z) + 2(n-1)\eta(Y)\eta(Z).$$

Thus we can state the following theorem:

Theorem 6.3. *A ϕ -projectively semisymmetric Sasakian manifold with respect to the g -Tanaka-Webster connection is an η -Einstein manifold with respect to the Levi-Civita connection.*

Remark 6.4. A Sasakian manifold with respect to the g -Tanaka-Webster connection is always ξ -projectively flat.

7. Locally projectively ϕ -symmetric Sasakian manifolds with respect to the g -Tanaka-Webster connection

Definition. A Sasakian manifold with respect to the g -Tanaka-Webster connection is said to be locally projectively ϕ -symmetric if

$$(69) \quad \phi^2((\bar{\nabla}_W \bar{P})(X, Y)Z) = 0$$

for all vector fields X, Y, Z, W orthogonal to ξ .

Using (13), we can write

$$(70) \quad \begin{aligned} & (\bar{\nabla}_W \bar{P})(X, Y)Z + \bar{P}(\bar{\nabla}_W X, Y)Z + \bar{P}(X, \bar{\nabla}_W Y)Z + \bar{P}(X, Y)\bar{\nabla}_W Z \\ &= (\nabla_W \bar{P})(X, Y)Z + \bar{P}(\nabla_W X, Y)Z + \bar{P}(X, \nabla_W Y)Z + \bar{P}(X, Y)\nabla_W Z \\ & \quad + g(W, \phi \bar{P}(X, Y)Z)\xi + \eta(\bar{P}(X, Y)Z)\phi W - \eta(W)\phi \bar{P}(X, Y)Z \end{aligned}$$

which in view of (13) can be written as

$$(71) \quad \begin{aligned} & (\bar{\nabla}_W \bar{P})(X, Y)Z \\ &= (\nabla_W \bar{P})(X, Y)Z + g(W, \phi \bar{P}(X, Y)Z)\xi + \eta(\bar{P}(X, Y)Z)\phi W \\ & \quad - \eta(W)\phi \bar{P}(X, Y)Z - g(W, \phi X)\bar{P}(\xi, Y)Z - \eta(X)\bar{P}(\phi W, Y)Z \\ & \quad + \eta(W)\bar{P}(\phi X, Y)Z - g(W, \phi Y)\bar{P}(X, \xi)Z - \eta(Y)\bar{P}(X, \phi W)Z \\ & \quad + \eta(W)\bar{P}(X, \phi Y)Z - \eta(Z)\bar{P}(X, Y)\phi W + \eta(W)\bar{P}(X, Y)\phi Z. \end{aligned}$$

By Differentiating (43), (14) and (16) with respect to W , we have

$$(72) \quad \begin{aligned} & (\nabla_W \bar{P})(X, Y)Z \\ &= (\nabla_W \bar{R})(X, Y)Z - \frac{1}{2n}[(\nabla_W \bar{S})(Y, Z)X - (\nabla_W \bar{S})(X, Z)Y], \end{aligned}$$

$$(73) \quad \begin{aligned} & (\nabla_W \bar{R})(X, Y)Z \\ &= (\nabla_W R)(X, Y)Z + g(X, Z)g(W, \phi Y)\xi - g(Y, Z)g(W, \phi X)\xi \end{aligned}$$

$$\begin{aligned}
& -g(X, Z)\eta(Y)\phi W + g(Y, Z)\eta(X)\phi W - g(W, Z)\eta(Y)\phi X \\
& + \eta(Z)g(Y, W)\phi X + g(Y, \phi Z)g(X, W)\xi - g(Y, \phi Z)\eta(X)W \\
& + g(W, Z)\eta(X)\phi Y - \eta(Z)g(X, W)\phi Y + g(X, \phi Z)g(Y, W)\xi \\
& - g(X, \phi Z)\eta(Y)W + 2g(X, W)\eta(Y)\phi Z - 2\eta(X)g(Y, W)\phi Z \\
& + 2g(Y, \phi X)g(W, Z)\xi - 2g(Y, \phi X)\eta(Z)W - g(W, \phi Y)\eta(Z)X \\
& - g(W, \phi Z)\eta(Y)X + g(W, \phi X)\eta(Z)Y + g(W, \phi Z)\eta(X)Y,
\end{aligned}$$

$$\begin{aligned}
(74) \quad & (\nabla_W \bar{S})(Y, Z)X \\
& = (\nabla_W S)(Y, Z)X - 2(n-1)[g(W, \phi Y)\eta(Z)X + g(W, \phi Z)\eta(Y)X]
\end{aligned}$$

respectively.

Now combining the equations (72)-(74), we obtain

$$\begin{aligned}
(75) \quad & (\nabla_W \bar{P})(X, Y)Z \\
& = (\nabla_W R)(X, Y)Z + g(X, Z)g(W, \phi Y)\xi - g(Y, Z)g(W, \phi X)\xi \\
& - g(X, Z)\eta(Y)\phi W + g(Y, Z)\eta(X)\phi W - g(W, Z)\eta(Y)\phi X \\
& + \eta(Z)g(Y, W)\phi X + g(Y, \phi Z)g(X, W)\xi - g(Y, \phi Z)\eta(X)W \\
& + g(W, Z)\eta(X)\phi Y - \eta(Z)g(X, W)\phi Y + g(X, \phi Z)g(Y, W)\xi \\
& - g(X, \phi Z)\eta(Y)W + 2g(X, W)\eta(Y)\phi Z - 2\eta(X)g(Y, W)\phi Z \\
& + 2g(Y, \phi X)g(W, Z)\xi - 2g(Y, \phi X)\eta(Z)W - g(W, \phi Y)\eta(Z)X \\
& - g(W, \phi Z)\eta(Y)X + g(W, \phi X)\eta(Z)Y + g(W, \phi Z)\eta(X)Y \\
& - \frac{1}{2n}[(\nabla_W S)(Y, Z)X - 2(n-1)g(W, \phi Y)\eta(Z)X \\
& - 2(n-1)g(W, \phi Z)\eta(Y)X - (\nabla_W S)(X, Z)Y \\
& + 2(n-1)g(W, \phi X)\eta(Z)Y + 2(n-1)g(W, \phi Z)\eta(X)Y].
\end{aligned}$$

Now differentiating (42) with respect to W , we have

$$\begin{aligned}
(76) \quad & (\nabla_W P)(X, Y)Z \\
& = (\nabla_W R)(X, Y)Z - \frac{1}{2n}[(\nabla_W S)(Y, Z)X - (\nabla_W S)(X, Z)Y].
\end{aligned}$$

By the use of (76), (75) takes the form

$$\begin{aligned}
(77) \quad & (\nabla_W \bar{P})(X, Y)Z \\
& = (\nabla_W P)(X, Y)Z + g(X, Z)g(W, \phi Y)\xi - g(Y, Z)g(W, \phi X)\xi \\
& - g(X, Z)\eta(Y)\phi W + g(Y, Z)\eta(X)\phi W - g(W, Z)\eta(Y)\phi X \\
& + \eta(Z)g(Y, W)\phi X + g(Y, \phi Z)g(X, W)\xi - g(Y, \phi Z)\eta(X)W \\
& + g(W, Z)\eta(X)\phi Y - \eta(Z)g(X, W)\phi Y + g(X, \phi Z)g(Y, W)\xi \\
& - g(X, \phi Z)\eta(Y)W + 2g(X, W)\eta(Y)\phi Z - 2\eta(X)g(Y, W)\phi Z
\end{aligned}$$

$$+ 2g(Y, \phi X)g(W, Z)\xi - 2g(Y, \phi X)\eta(Z)W - \frac{1}{n}[g(W, \phi Y)\eta(Z)X \\ + g(W, \phi Z)\eta(Y)X - g(W, \phi X)\eta(Z)Y - g(W, \phi Z)\eta(X)Y].$$

Operating ϕ^2 to the both sides of (71), we have

$$(78) \quad \phi^2(\bar{\nabla}_W \bar{P})(X, Y)Z \\ = \phi^2(\nabla_W \bar{P})(X, Y)Z + \frac{1}{2n}[\bar{S}(Y, Z)\eta(X) - \bar{S}(X, Z)\eta(Y)]\phi W \\ + \eta(W)\phi \bar{P}(X, Y)Z - \eta(X)\phi^2 \bar{P}(\phi W, Y)Z + \eta(W)\phi^2 \bar{P}(\phi X, Y)Z \\ - \eta(Y)\phi^2 \bar{P}(X, \phi W)Z + \eta(W)\phi^2 \bar{P}(X, \phi Y)Z - \eta(Z)\phi^2 \bar{P}(X, Y)\phi W \\ + \eta(W)\phi^2 \bar{P}(X, Y)\phi Z$$

which by using (77) becomes

$$(79) \quad \phi^2(\bar{\nabla}_W \bar{P})(X, Y)Z \\ = \phi^2(\nabla_W P)(X, Y)Z + g(X, Z)\eta(Y)\phi W \\ - g(Y, Z)\eta(X)\phi W + g(W, Z)\eta(Y)\phi X - \eta(Z)g(Y, W)\phi X \\ - g(Y, \phi Z)\eta(X)\phi^2 W - g(W, Z)\eta(X)\phi Y + \eta(Z)g(X, W)\phi Y \\ - g(X, \phi Z)\eta(Y)\phi^2 W - 2g(X, W)\eta(Y)\phi Z + 2\eta(X)g(Y, W)\phi Z \\ - 2g(Y, \phi X)\eta(Z)\phi^2 W - \frac{1}{n}[g(W, \phi Y)\eta(Z)\phi^2 X \\ + g(W, \phi Z)\eta(Y)\phi^2 X - g(W, \phi X)\eta(Z)\phi^2 Y - g(W, \phi Z)\eta(X)\phi^2 Y] \\ + \frac{1}{2n}[\bar{S}(Y, Z)\eta(X) - \bar{S}(X, Z)\eta(Y)]\phi W + \eta(W)\phi \bar{P}(X, Y)Z \\ - \eta(X)\phi^2 \bar{P}(\phi W, Y)Z + \eta(W)\phi^2 \bar{P}(\phi X, Y)Z - \eta(Y)\phi^2 \bar{P}(X, \phi W)Z \\ + \eta(W)\phi^2 \bar{P}(X, \phi Y)Z - \eta(Z)\phi^2 \bar{P}(X, Y)\phi W + \eta(W)\phi^2 \bar{P}(X, Y)\phi Z.$$

By taking X, Y, Z and W orthogonal to ξ , (79) reduces to

$$(80) \quad \phi^2(\bar{\nabla}_W \bar{P})(X, Y)Z = \phi^2(\nabla_W P)(X, Y)Z.$$

Thus we can state the following theorem:

Theorem 7.1. *A Sasakian manifold is locally projectively ϕ -symmetric with respect to the g -Tanaka-Webster connection if and only if it is so with respect to the Levi-Civita connection.*

Example. We consider the 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3\}$, where (x, y, z) are the standard coordinates in \mathbb{R}^3 . Let e_1, e_2 and e_3 be the vector fields on M given by

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y} - 2x \frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial z} = \xi,$$

which are linearly independent at each point of M and hence form a basis of T_pM . Let g be the Riemannian metric defined by

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1, \quad g(e_1, e_2) = g(e_1, e_3) = g(e_2, e_3) = 0.$$

Let η be the 1-form on M defined by $\eta(X) = g(X, e_3) = g(X, \xi)$ for all $X \in \chi(M)$. Let ϕ be the $(1, 1)$ tensor field on M defined by

$$\phi e_1 = -e_2, \quad \phi e_2 = e_1, \quad \phi e_3 = 0.$$

The linearity property of ϕ and g yields

$$\begin{aligned} \eta(\xi) &= g(\xi, \xi) = 1, \quad \phi^2 X = -X + \eta(X)\xi, \quad \eta(\phi X) = 0, \\ g(X, \xi) &= \eta(X), \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \end{aligned}$$

for all $X, Y \in \chi(M)$.

Now, by direct computations we obtain

$$[e_1, e_2] = -2e_3, \quad [e_1, e_3] = 0, \quad [e_2, e_3] = 0.$$

The Riemannian connection ∇ of the metric tensor g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) \\ &\quad + g(Y, [Z, X]) + g(Z, [X, Y]), \end{aligned}$$

which is known as Koszul's formula. Using Koszul's formula, we can easily calculate

$$(81) \quad \begin{aligned} \nabla_{e_1} e_1 &= 0, \quad \nabla_{e_2} e_1 = e_3, \quad \nabla_{e_3} e_1 = e_2, \quad \nabla_{e_1} e_2 = -e_3, \quad \nabla_{e_2} e_2 = 0, \\ \nabla_{e_3} e_2 &= -e_1, \quad \nabla_{e_1} e_3 = e_2, \quad \nabla_{e_2} e_3 = -e_1, \quad \nabla_{e_3} e_3 = 0. \end{aligned}$$

Let $X = \sum_{i=1}^3 X^i e_i = X^1 e_1 + X^2 e_2 + X^3 e_3 \in \chi(M)$.

Also, one can easily verify that

$$\nabla_X \xi = -\phi X \quad \text{and} \quad (\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X.$$

Thus the manifold M is a Sasakian manifold. Using (13) in (81), we obtain

$$(82) \quad \begin{aligned} \bar{\nabla}_{e_1} e_1 &= 0, \quad \bar{\nabla}_{e_2} e_1 = 0, \quad \bar{\nabla}_{e_3} e_1 = 2e_2, \quad \bar{\nabla}_{e_1} e_2 = 0, \quad \bar{\nabla}_{e_2} e_2 = 0, \\ \bar{\nabla}_{e_3} e_2 &= -2e_1, \quad \bar{\nabla}_{e_1} e_3 = 0, \quad \bar{\nabla}_{e_2} e_3 = 0, \quad \bar{\nabla}_{e_3} e_3 = 0. \end{aligned}$$

It is known that

$$(83) \quad R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

By using the above results, we can easily obtain the components of the curvature tensors as follows:

$$(84) \quad \begin{aligned} R(e_1, e_2)e_1 &= 3e_2, \quad R(e_1, e_2)e_2 = -3e_1, \quad R(e_1, e_2)e_3 = 0, \\ R(e_2, e_3)e_1 &= 0, \quad R(e_2, e_3)e_2 = -e_3, \quad R(e_2, e_3)e_3 = e_2 \\ R(e_1, e_3)e_1 &= -e_3, \quad R(e_1, e_3)e_2 = 0, \quad R(e_1, e_3)e_3 = e_1 \end{aligned}$$

and

$$(85) \quad \begin{aligned} \bar{R}(e_1, e_2)e_1 &= 4e_2, \quad \bar{R}(e_1, e_2)e_2 = -4e_1, \quad \bar{R}(e_1, e_2)e_3 = 0, \\ \bar{R}(e_2, e_3)e_1 &= 0, \quad \bar{R}(e_2, e_3)e_2 = 0, \quad \bar{R}(e_2, e_3)e_3 = 0, \end{aligned}$$

$$\bar{R}(e_1, e_3)e_1 = 0, \bar{R}(e_1, e_3)e_2 = 0, \bar{R}(e_1, e_3)e_3 = 0.$$

From these curvature tensors, we calculate the Ricci tensors and the scalar curvatures as follows:

$$S(e_1, e_1) = -2, S(e_2, e_2) = -2, S(e_3, e_3) = 2, r = -2, \\ \bar{S}(e_1, e_1) = -4, \bar{S}(e_2, e_2) = -4, \bar{S}(e_3, e_3) = 0, \bar{r} = -8.$$

From the expressions of the curvature tensors and the Ricci tensors, we have

$$\bar{P}(e_1, e_2)e_3 = \bar{P}(e_1, e_3)e_2 = \bar{P}(e_2, e_3)e_1 = 0,$$

that is, the manifold M is ξ -projectively flat with respect to the g -Tanaka-Webster connection. Hence Remark 6.4 is verified.

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