

## NUMERICAL METHOD FOR A SYSTEM OF SINGULARLY PERTURBED CONVECTION DIFFUSION EQUATIONS WITH INTEGRAL BOUNDARY CONDITIONS

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**ABSTRACT.** A class of systems of singularly perturbed convection diffusion type equations with integral boundary conditions is considered. A numerical method based on a finite difference scheme on a Shishkin mesh is presented. The suggested method is of almost first order convergence. An error estimate is derived in the discrete maximum norm. Numerical examples are presented to validate the theoretical estimates.

### 1. Introduction

Boundary value problems with integral boundary conditions are an important class of problems which arise in the fields of electro chemistry [10], thermo elasticity [11], heat conduction [7] etc. For a discussion of existence and uniqueness results for applications of second order differential equations with integral boundary conditions see [1, 3, 4, 8, 14]. In [16] the existence of positive solutions of boundary value problems for systems of second order differential equations with integral boundary condition on the half-line was analyzed. In [17], Zhilin Yang considered the existence of positive solutions to a system of second order nonlocal boundary value problems by using fixed point index theory in a cone. Lazhar Bougoffa [5], using the Riesz representation theorem, proved the existence and uniqueness of generalized solutions. The above mentioned papers are mainly concerned with the regular case (without boundary layers). Motivated by above mentioned works, in this paper we consider a class of systems of singularly perturbed convection diffusion equations with integral boundary conditions.

Differential equation with a small parameter  $\varepsilon$  multiplying the leading derivative is called Singularly Perturbed Problem (SPP). Most of the traditional numerical methods are not suitable for SPP because the presence of the parameter  $\varepsilon$  makes the solutions of such equations to have rapid changes in small

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regions (boundary layers) of the domain. Hence it is necessary to develop appropriate numerical methods which converge  $\varepsilon$ -uniformly [2, 9, 12, 15]. In [13] and [6] uniform convergence of the approximate solution on a uniform mesh is proved for second order differential equations with integral boundary condition. In the present paper, a fitted finite difference method is suggested to solve a class of systems of singularly perturbed convection diffusion equations with integral boundary conditions.

This paper is arranged in the following manner. In Section 2 the continuous problem is derived. In Section 3 bounds on the derivatives of the continuous problems are discussed. The numerical method is described in Section 4. In Section 5 the error estimate for approximate solution is presented. Numerical results are given in Section 6. The conclusion is presented in Section 7.

## 2. Statement of the problem

Motivated by the works of [5, 6, 16, 17], we consider the following system of singularly perturbed problem with integral boundary conditions:

$$(2.1) \quad \begin{cases} L_1 \bar{u}(x) = -\varepsilon u_1''(x) + a_1(x)u_1'(x) + b_{11}(x)u_1(x) + b_{12}(x)u_2(x) = f_1(x), \\ L_2 \bar{u}(x) = -\varepsilon u_2''(x) + a_2(x)u_2'(x) + b_{21}(x)u_1(x) + b_{22}(x)u_2(x) = f_2(x), \end{cases}$$

where  $\bar{u}(x) = (u_1(x), u_2(x))$ ,  $x \in \Omega = (0, 1)$ , with the boundary conditions

$$(2.2) \quad \begin{cases} u_1(0) = A_1, & B_1 u_1(1) = u_1(1) - \varepsilon \int_0^1 g_1(x)u_1(x)dx = l_1, \\ u_2(0) = A_2, & B_2 u_2(1) = u_2(1) - \varepsilon \int_0^1 g_2(x)u_2(x)dx = l_2, \end{cases}$$

where  $0 < \varepsilon \ll 1$  is a small positive parameter, the functions  $a_1(x)$ ,  $a_2(x)$ ,  $b_{11}(x)$ ,  $b_{12}(x)$ ,  $b_{21}(x)$ ,  $b_{22}(x)$ ,  $f_1(x)$ ,  $f_2(x)$  are sufficiently smooth on  $\bar{\Omega} = [0, 1]$  and satisfy the following assumptions:

$$a_i(x) \geq \alpha_i > 0, \quad i = 1, 2, \quad x \in \bar{\Omega},$$

$$b_{12}(x) \leq 0, \quad b_{21}(x) \leq 0, \quad b_{11}(x) + b_{12}(x) \geq 0, \quad b_{21}(x) + b_{22}(x) \geq 0, \quad x \in \bar{\Omega},$$

$$g_i \text{ is nonnegative and } 1 - \int_0^1 g_i(x)dx > 0, \quad i = 1, 2 \text{ and } \alpha = \min\{\alpha_1, \alpha_2\}.$$

Throughout the paper, we assume that  $\varepsilon \leq CN^{-1}$ ,  $C$  denotes a positive constant. We are using the supremum norm,  $\|u\|_D = \sup_{x \in D} |u(x)|$ , to establish the convergence of the numerical solution to the exact solution.

## 3. The continuous problem

**Theorem 3.1** (Maximum Principle). *Let  $\bar{u}(x) \in C^2(\bar{\Omega})$  be any function satisfying  $u_1(0) \geq 0, u_2(0) \geq 0, B_1 u_1(1) \geq 0, B_2 u_2(1) \geq 0, L_1 \bar{u}(x) \geq 0$  and  $L_2 \bar{u}(x) \geq 0, x \in \Omega$ . Then  $\bar{u}(x) \geq 0, x \in \bar{\Omega}$ .*

*Proof.* Define  $\bar{s}(x) = (s_1(x), s_2(x))$  as  $s_1(x) = s_2(x) = 1 + x$ . Note that  $\bar{s}(x) > 0$ ,  $x \in \bar{\Omega}$ ,  $L_1\bar{s}(x) > 0$ ,  $L_2\bar{s}(x) > 0$ ,  $x \in \Omega$ ,  $s_1(0) > 0$ ,  $s_2(0) > 0$ ,  $B_1s_1(1) > 0$  and  $B_2s_2(1) > 0$ . Further we define

$$\gamma = \max \left\{ \max_{x \in \bar{\Omega}} \left( \frac{-u_1(x)}{s_1(x)} \right), \max_{x \in \bar{\Omega}} \left( \frac{-u_2(x)}{s_2(x)} \right) \right\}.$$

Then there exists at least one  $x_0 \in \Omega$  such that  $\left( \frac{-u_1(x_0)}{s_1(x_0)} \right) = \gamma$  or  $\left( \frac{-u_2(x_0)}{s_2(x_0)} \right) = \gamma$  or both. Also  $(\bar{u} + \gamma\bar{s})(x) \geq 0$ ,  $x \in \bar{\Omega}$ . Without loss of generality we assume that  $\left( \frac{-u_1(x_0)}{s_1(x_0)} \right) = \gamma$ . Therefore the function  $(u_1 + \gamma s_1)$  attains its minimum at  $x = x_0$ . It is easy to observe that for each  $x \in \bar{\Omega}$ ,  $\bar{u}(x) \geq 0$  if  $\gamma \leq 0$ . Now we will show that indeed  $\gamma \leq 0$ . Suppose  $\gamma > 0$ .

Case (i): Assume that  $(u_1 + \gamma s_1)(x_0) = 0$  for  $x_0 = 0$ . Then

$$0 = (u_1 + \gamma s_1)(0) = u_1(0) + \gamma s_1(0) > 0.$$

Case (ii): Assume that  $(u_1 + \gamma s_1)(x_0) = 0$  for  $x_0 \in \Omega$ . Then

$$\begin{aligned} 0 < L_1(\bar{u} + \gamma\bar{s})(x_0) &= -\varepsilon(u_1 + \gamma s_1)''(x_0) + a_1(x_0)(u_1 + \gamma s_1)'(x_0) \\ &\quad + b_{11}(x_0)(u_1 + \gamma s_1)(x_0) + b_{12}(x_0)(u_2 + \gamma s_2)(x_0) \leq 0 \end{aligned}$$

Case (iii): Assume that  $(u_1 + \gamma s_1)(x_0) = 0$  for  $x_0 = 1$ . Then

$$0 < B_1(u_1 + \gamma s_1)(1) = (u_1 + \gamma s_1)(1) - \varepsilon \int_0^1 g_1(x)(u_1 + \gamma s_1)(x) dx \leq 0.$$

Observe that in all the three cases we arrived a contradiction. Therefore  $\gamma > 0$  is not possible. This shows that  $u_1(x) \geq 0$ . Similarly we can show that  $u_2(x) \geq 0$ . Hence  $\bar{u}(x) \geq 0$ ,  $x \in \bar{\Omega}$ .  $\square$

**Note.** Since the operators  $L_j$ ,  $j = 1, 2$  satisfy the above maximum principle, the solution  $\bar{u}(x)$  of (2.1)-(2.2) is unique, if it exists.

**Corollary 3.2** (Stability Result). *The solution  $\bar{u}(x)$  of problem (2.1)-(2.2) satisfies the bound*

$$\begin{aligned} |u_i(x)| &\leq C \max\{|u_1(0)|, |u_2(0)|, |B_1u_1(1)|, |B_2u_2(1)|, \\ &\quad \|L_1\bar{u}\|_{\Omega}, \|L_2\bar{u}\|_{\Omega}\}, \quad x \in \bar{\Omega}, \quad i = 1, 2. \end{aligned}$$

*Proof.* Let  $C > 0$  be a constant. Define  $\psi_i^{\pm}(x) = CMs_i(x) \pm u_i(x)$ ,  $x \in \bar{\Omega}$ ,  $i = 1, 2$ , where  $M = \max\{|u_1(0)|, |u_2(0)|, |B_1u_1(1)|, |B_2u_2(1)|, \|L_1\bar{u}\|_{\Omega}, \|L_2\bar{u}\|_{\Omega}\}$ .

Note that  $\psi_1^{\pm}(0) \geq 0$ ,  $B_1\psi_1^{\pm}(1) \geq 0$ ,  $\psi_2^{\pm}(0) \geq 0$ ,  $B_2\psi_2^{\pm}(1) \geq 0$  by proper choice of  $C > 0$ . It is easy to see that  $L_1\psi^{\pm}(x) \geq 0$ ,  $L_2\psi^{\pm}(x) \geq 0$ . Then by maximum principle, we get the required result.  $\square$

**Lemma 3.3.** *Let  $\bar{u}(x)$  be the solution of (2.1)-(2.2). Then, for  $1 \leq k \leq 3$ ,*

$$|u_j^{(k)}(x)| \leq C\varepsilon^{-k}, \quad x \in \bar{\Omega}, \quad j = 1, 2.$$

*Proof.* Using Corollary 3.2 and applying the arguments as given in [12] this lemma can be proved.  $\square$

The uniform error estimates can be derived using the sharper bounds on the derivatives of the solution  $\bar{u}(x)$ . To get sharper bounds we write the analytical solution in the form  $\bar{u}(x) = \bar{v}(x) + \bar{w}(x)$ , where  $\bar{v}(x) = (v_1(x), v_2(x))$  and  $\bar{w}(x) = (w_1(x), w_2(x))$ . The regular component  $\bar{v}(x)$  can be written as  $\bar{v}(x) = \bar{v}_0(x) + \varepsilon \bar{v}_1(x) + \varepsilon^2 \bar{v}_2(x)$ , where  $\bar{v}_0(x) = (v_{01}(x), v_{02}(x))$ ,  $\bar{v}_1(x) = (v_{11}(x), v_{12}(x))$ ,  $\bar{v}_2(x) = (v_{21}(x), v_{22}(x))$  satisfy the following equations respectively:

$$(3.1) \quad \begin{cases} a_1(x)v_{01}(x) + b_{11}(x)v_{01}(x) + b_{12}(x)v_{02}(x) = f_1(x), \\ a_2(x)v_{02}(x) + b_{21}(x)v_{01}(x) + b_{22}(x)v_{02}(x) = f_2(x), \\ v_{01}(0) = u_1(0), \quad v_{02}(0) = u_2(0), \end{cases}$$

$$(3.2) \quad \begin{cases} a_1(x)v'_{11}(x) + b_{11}(x)v_{11}(x) + b_{12}(x)v_{12}(x) = v''_{01}(x), \\ a_2(x)v'_{12}(x) + b_{21}(x)v_{11}(x) + b_{22}(x)v_{12}(x) = v''_{02}(x), \\ v_{11}(0) = 0, \quad v_{12}(0) = 0, \end{cases}$$

$$(3.3) \quad \begin{cases} L_1 \bar{v}_2(x) = v''_{11}(x), \quad v_{21}(0) = 0, \quad B_1 v_{21}(1) = 0, \\ L_2 \bar{v}_2(x) = v''_{12}(x), \quad v_{22}(0) = 0, \quad B_2 v_{22}(1) = 0. \end{cases}$$

Thus the regular component  $\bar{v}(x)$  is the solution of

$$(3.4) \quad \begin{cases} L_1 \bar{v}(x) = f_1(x), \quad v_1(0) = u_1(0), \quad B_1 v_1(1) = B_1 v_{01}(1) + \varepsilon B_1 v_{11}(1) + \varepsilon^2 B_1 v_{21}(1) \\ L_2 \bar{v}(x) = f_2(x), \quad v_2(0) = u_2(0), \quad B_2 v_2(1) = B_2 v_{02}(1) + \varepsilon B_2 v_{12}(1) + \varepsilon^2 B_2 v_{22}(1) \end{cases}$$

and layer component  $\bar{w}(x)$  is the solution of

$$(3.5) \quad \begin{cases} L_1 \bar{w}(x) = 0, \quad w_1(0) = 0, \quad B_1 w_1(1) = B_1 u_1(1) - B_1 v_1(1), \\ L_2 \bar{w}(x) = 0, \quad w_2(0) = 0, \quad B_2 w_2(1) = B_2 u_2(1) - B_2 v_2(1). \end{cases}$$

**Theorem 3.4.** *Let  $\bar{u}(x)$  be the solution of the problem (2.1)-(2.2) and  $\bar{v}_0(x)$  be its reduced problem solution defined in (3.1). Then*

$$|u_j(x) - v_{0j}(x)| \leq C(\varepsilon + e^{-\alpha(1-x)/\varepsilon}), \quad x \in \bar{\Omega}, \quad j = 1, 2.$$

*Proof.* Consider the barrier functions  $\bar{\psi}^\pm(x) = (\psi_1^\pm(x), \psi_2^\pm(x))$ , where

$$\psi_j^\pm(x) = C(\varepsilon s_j(x) + e^{-\alpha(1-x)/\varepsilon}) \pm (u_j(x) - v_{0j}(x)), \quad x \in \bar{\Omega}, \quad j = 1, 2.$$

Note that  $\psi_j^\pm(x) \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ . It is easy to see that,  $\psi_1^\pm(0) \geq 0$  for a suitable choice of  $C > 0$ . Further

$$\begin{aligned} B_1 \psi_1^\pm(1) &= \psi_1^\pm(1) - \varepsilon \int_0^1 g_1(x) \psi_1^\pm(x) dx \\ &\geq C(2\varepsilon + 1) - 2C\varepsilon \int_0^1 g_1(x) dx - C\varepsilon \int_0^1 g_1(x) dx \pm B_1(u_1 - v_{01})(1) \end{aligned}$$

$$\geq 0$$

for a suitable choice of  $C > 0$ .

Let  $x \in \Omega$ . Then

$$\begin{aligned} L_1 \bar{\psi}^\pm(x) &= C\varepsilon[a_1(x)s_1'(x) + b_{11}(x)s_1(x) + b_{12}(x)s_2(x)] \\ &\quad + C\left[\frac{\alpha}{\varepsilon}(a_1(x) - \alpha) + b_{11}(x) + b_{12}(x)\right]e^{-\alpha(1-x)/\varepsilon} \\ &\quad \pm L_1(\bar{u} - \bar{v}_{01})(x) \\ &\geq 0, \end{aligned}$$

by a proper choice of  $C > 0$ . Similarly one can prove that  $B_2\psi_2^\pm(1) \geq 0$  and  $L_2\bar{\psi}_2^\pm(x) \geq 0$ . Then by maximum principle we have  $\bar{\psi}^\pm(x) \geq 0$ ,  $x \in \bar{\Omega}$ .  $\square$

**Lemma 3.5.** *The regular component  $\bar{v}(x)$  and the layer component  $\bar{w}(x)$  of the solution  $\bar{u}(x)$  of the problem (2.1)-(2.2) satisfy the following bounds:*

$$(3.6) \quad \|v_j^{(k)}\|_{\bar{\Omega}} \leq C(1 + \varepsilon^{2-k}),$$

$$(3.7) \quad |w_j^{(k)}(x)| \leq C\varepsilon^{-k}e^{-\alpha(1-x)/\varepsilon}, \quad 0 \leq k \leq 3, \quad x \in \bar{\Omega}, \quad j = 1, 2.$$

*Proof.* Integrating (3.4) and using the stability result one can prove the inequalities (3.6). To prove the inequalities (3.7), consider the functions  $\bar{\psi}^\pm(x) = (\psi_1^\pm(x), \psi_2^\pm(x))$ , where  $\psi_j^\pm(x) = Ce^{-\alpha(1-x)/\varepsilon} \pm w_j(x)$ ,  $j = 1, 2$ . It is easy to see that  $\psi_1^\pm(0) \geq 0$  and  $\psi_2^\pm(0) \geq 0$  for a suitable choice of  $C > 0$ . Further

$$B_1\psi_1^\pm(1) = \psi_1^\pm(1) - \varepsilon \int_0^1 g_1(x)\psi_1^\pm(x)dx \geq C(1 - \varepsilon \int_0^1 g_1(x)dx) \pm B_1w_1(1) \geq 0$$

for a suitable choice of  $C > 0$ . Let  $x \in \Omega$ .

$$L_1\bar{\psi}^\pm(x) = C\left[\frac{\alpha}{\varepsilon}(a_1(x) - \alpha) + b_{11}(x) + b_{12}(x)\right]e^{-\frac{\alpha(1-x)}{\varepsilon}} \pm L_1\bar{w} \geq 0.$$

Similarly we can prove  $B_2\psi_2^\pm(1) \geq 0$  and  $L_2\bar{\psi}^\pm \geq 0$ . Hence the maximum principle gives  $\psi_j^\pm(x) \geq 0$  and so  $|w_j(x)| \leq Ce^{-\alpha(1-x)/\varepsilon}$  for all  $x \in \bar{\Omega}$ ,  $j = 1, 2$ .

Integration of (3.5) yields the estimate of  $|w'(x)|$ . From (3.5), one can derive the rest of derivative estimates (3.7).  $\square$

**Note.** From the above theorem, it is easy to see that,

$$(3.8) \quad |u_j(x) - v_j(x)| \leq Ce^{-\alpha(1-x)/\varepsilon}, \quad x \in \bar{\Omega}, \quad j = 1, 2.$$

#### 4. The discrete problem

On  $\bar{\Omega}$  a piecewise uniform Shishkin mesh of  $N$  ( $\geq 4$ ) mesh intervals is constructed. The domain  $\bar{\Omega}$  is partitioned into two subintervals  $[0, 1 - \sigma]$ , and  $[1 - \sigma, 1]$  where  $\sigma$  is the transition parameter defined by  $\sigma = \min\{\frac{1}{2}, \frac{2\varepsilon \ln N}{\alpha}\}$ . On  $[0, 1 - \sigma]$  and  $[1 - \sigma, 1]$  a uniform mesh with  $\frac{N}{2}$  mesh intervals are placed. The interior mesh points are denoted by  $\Omega^N$ . Let  $h_i = x_i - x_{i-1}$  be the mesh step and  $\bar{h}_i = \frac{h_{i+1} + h_i}{2}$ .

The discrete problem corresponding to (2.1)-(2.2) is:  
Find  $\bar{U}(x_i) = (U_1(x_i), U_2(x_i))$  for  $i = 0, 1, 2, \dots, N$  such that

$$(4.1) \quad \begin{cases} L_1^N \bar{U}(x_i) = -\varepsilon \delta^2 U_1(x_i) + a_1(x_i) D^- U_1(x_i) + b_{11}(x_i) U_1(x_i) \\ \quad + b_{12}(x_i) U_2(x_i) = f_1(x_i), \quad \forall x_i \in \Omega^N \\ L_2^N \bar{U}(x_i) = -\varepsilon \delta^2 U_2(x_i) + a_2(x_i) D^- U_2(x_i) + b_{21}(x_i) U_1(x_i) \\ \quad + b_{22}(x_i) U_2(x_i) = f_2(x_i), \quad \forall x_i \in \Omega^N. \end{cases}$$

$$(4.2) \quad \begin{cases} U_1(x_0) = A_1, \\ B_1^N U_1(x_N) = U_1(x_N) - \varepsilon \sum_{i=1}^N \frac{g_1(x_{i-1}) U_1(x_{i-1}) + g_1(x_i) U_1(x_i)}{2} h_i = l_1, \quad \forall x_i \in \bar{\Omega}^N, \\ U_2(x_0) = A_2, \\ B_2^N U_2(x_N) = U_2(x_N) - \varepsilon \sum_{i=1}^N \frac{g_2(x_{i-1}) U_2(x_{i-1}) + g_2(x_i) U_2(x_i)}{2} h_i = l_2, \quad \forall x_i \in \bar{\Omega}^N, \end{cases}$$

where

$$\delta^2 U_j(x_i) = \frac{1}{h_i} \left( \frac{U_j(x_{i+1}) - U_j(x_i)}{h_{i+1}} - \frac{U_j(x_i) - U_j(x_{i-1})}{h_i} \right),$$

$$D^- U_j(x_i) = \frac{U_j(x_i) - U_j(x_{i-1})}{h_i}, \quad j = 1, 2.$$

**Theorem 4.1** (Discrete Maximum Principle). *Let  $\bar{\Psi}(x_i) = (\Psi_1(x_i), \Psi_2(x_i))$  be the mesh function satisfying  $\Psi_1(x_0) \geq 0$ ,  $\Psi_2(x_0) \geq 0$ ,  $B_1^N \Psi_1(x_N) \geq 0$ ,  $B_2^N \Psi_2(x_N) \geq 0$ ,  $L_1^N \bar{\Psi}(x_i) \geq 0$ , and  $L_2^N \bar{\Psi}(x_i) \geq 0$ . Then  $\bar{\Psi}(x_i) \geq 0$ ,  $x_i \in \bar{\Omega}^N$ .*

*Proof.* Define  $\bar{S}(x_i) = (S_1(x_i), S_2(x_i))$ , where  $S_1(x_i) = S_2(x_i) = 1 + x_i$ . Note that  $\bar{S}(x_i) > 0$ ,  $\forall x_i \in \bar{\Omega}^N$ ,  $B_1^N S_1(x_N) > 0$ ,  $B_2^N S_2(x_N) > 0$ ,  $L_1^N \bar{S}_1(x_i) > 0$  and  $L_2^N \bar{S}_2(x_i) > 0$ ,  $\forall x_i \in \bar{\Omega}^N$ . Let

$$\mu = \max \left\{ \max_{x_i \in \bar{\Omega}^N} \left( \frac{-\Psi_1(x_i)}{S_1(x_i)} \right), \max_{x_i \in \bar{\Omega}^N} \left( \frac{-\Psi_2(x_i)}{S_2(x_i)} \right) \right\}.$$

Then there exists one  $x_k \in \bar{\Omega}^N$  such that  $\Psi_1(x_k) + \mu S_1(x_k) = 0$  or  $\Psi_2(x_k) + \mu S_2(x_k) = 0$  or both. Also  $\Psi_j(x_i) + \mu S_j(x_i) \geq 0$ ,  $x_i \in \bar{\Omega}^N$ ,  $j = 1, 2$ . Without loss of generality we assume that  $\left( \frac{-\Psi_1(x_i)}{S_1(x_i)} \right) = \mu$ . Therefore the function  $(\Psi_1 + \mu S_1)$  attains minimum at  $x_i = x_k$ . It is easy to observe that for each  $x_i \in \bar{\Omega}^N$ ,  $\bar{\Psi}(x_i) \geq 0$  if  $\mu \leq 0$ . Now we will show that indeed  $\mu \leq 0$ . Suppose  $\mu > 0$ .

Case (i): Assume that  $(\Psi_1 + \mu S_1)(x_k) = 0$  for  $x_k = 0$ . Then

$$0 = (\Psi_1 + \mu S_1)(x_k) = \Psi_1(x_k) + \mu S_1(x_k) > 0.$$

Case(ii) Assume that  $(\Psi_1 + \mu S_1)(x_k) = 0$  for  $x_k \in \Omega^N$ . Then

$$\begin{aligned} 0 &< L_1^N (\bar{\Psi} + \mu \bar{S})(x_k) \\ &= -\varepsilon \delta^2 (\Psi_1 + \mu S_1)(x_k) + a_1(x_k) D^- (\Psi_1 + \mu S_1)(x_k) \end{aligned}$$

$$+ b_{11}(x_k)(\Psi_1 + \mu S_1)(x_k) + b_{12}(x_k)(\Psi_2 + \mu S_2)(x_k) \leq 0.$$

Case(iii) Assume that  $(\Psi_1 + \mu S_1)(x_k) = 1$  for  $x_k = 1$ . Then

$$\begin{aligned} 0 &< B_1^N(\Psi_1 + \mu S_1)(x_k) \\ &= (\Psi_1 + \mu S_1)(x_k) \\ &\quad - \varepsilon \sum_{i=1}^N \frac{g_1(x_{i-1})(\Psi_1 + \mu S_1)(x_{i-1}) + g_1(x_i)(\Psi_1 + \mu S_1)(x_i)}{2} h_i \leq 0. \end{aligned}$$

Observe that in all the three cases we arrived a contradiction. Therefore  $\mu > 0$  is not possible. This shows that  $\Psi_1(x_i) \geq 0$ . Similarly we can show that  $\Psi_2(x_i) \geq 0$ . Hence  $\bar{\Psi}(x_i) \geq 0$ ,  $x_i \in \bar{\Omega}^N$ .  $\square$

**Lemma 4.2** (Discrete Stability Result). *Let  $\bar{U}(x_i) = (U_1(x_i), U_2(x_i))$  be any mesh function. Then*

$$\begin{aligned} |U_j(x_i)| &\leq C \max \left\{ |U_1(x_0)|, |U_2(x_0)|, |B_1 U_1(x_N)|, |B_2 U_2(x_N)|, \right. \\ &\quad \left. \max_{x_i \in \bar{\Omega}^N} |L_1^N \bar{U}(x_i)|, \max_{x_i \in \bar{\Omega}^N} |L_2^N \bar{U}(x_i)| \right\}, \quad x_i \in \bar{\Omega}^N, \quad j = 1, 2. \end{aligned}$$

*Proof.* By choosing suitable barrier functions and using Theorem 4.1, one can establish the above inequality.  $\square$

Analogous to the continuous case, the discrete solution  $\bar{U}(x_i)$  can be decomposed as

$$\bar{U}(x_i) = \bar{V}(x_i) + \bar{W}(x_i),$$

where  $V(x_i)$  and  $W(x_i)$  are respectively the solutions of the problems:

$$(4.3) \quad \begin{cases} L_1^N \bar{V}(x_i) = f_1(x_i), & x_i \in \Omega^N, \quad V_1(x_0) = v_1(0), \quad B_1^N V_1(x_N) = B_1 v_1(1) \\ L_2^N \bar{V}(x_i) = f_2(x_i), & x_i \in \Omega^N, \quad V_2(x_0) = v_2(0), \quad B_2^N V_2(x_N) = B_2 v_2(1). \end{cases}$$

$$(4.4) \quad \begin{cases} L_1^N \bar{W} = 0, & x_i \in \Omega^N, \quad W_1(x_0) = w_1(0), \quad B_1^N W_1(x_N) = B_1 w_1(1) \\ L_2^N \bar{W} = 0, & x_i \in \Omega^N, \quad W_2(x_0) = w_2(0), \quad B_2^N W_2(x_N) = B_1 w_1(1). \end{cases}$$

The following theorem gives an estimate for the difference of the solutions of (4.1)-(4.2) and (4.3).

**Theorem 4.3.** *Let  $\bar{U}(x_i)$  be a numerical solution of (2.1)-(2.2) defined by (4.1)-(4.2) and  $\bar{V}(x_i)$  be a numerical solution of (3.4) defined by (4.3). Then*

$$|U_j(x_i) - V_j(x_i)| \leq C \begin{cases} N^{-1}, & i = 0, 1, \dots, \frac{N}{2} \\ N^{-1} + |l_j - B_j^N V_j(x_N)| & i = \frac{N}{2} + 1, \dots, N, \end{cases} \quad j = 1, 2.$$

*Proof.* Consider mesh functions  $\bar{\Psi}^\pm(x_i) = (\Psi_1^\pm(x_i), \Psi_2^\pm(x_i))$ , where

$$\Psi_j^\pm(x_i) = CN^{-1} S_j(x_i) + C x_i \varphi(x_i) \pm (U_j(x_i) - V_j(x_i)), \quad x_i \in \bar{\Omega}^N,$$

$$\varphi(x_i) = \begin{cases} 0, & i = 0, 1, \dots, \frac{N}{2} \\ |l_j - B_j^N V_j(x_N)| & i = \frac{N}{2} + 1, \dots, N, \end{cases} \quad j = 1, 2.$$

Now

$$\begin{aligned} L_1^N \bar{\Psi}^\pm(x_i) &= CN^{-1}[a_1(x_i) + b_{11}(x_i)S_1(x_i) + b_{12}(x_i)s_2(x_i)] \\ &\quad + CN^{-1}\varphi(x_i)[a_1(x_i) + x_i(b_{11}(x_i) + b_{12}(x_i))] \geq 0, \quad x_i \in \Omega^N. \end{aligned}$$

Similarly one can prove that  $L_2^N \bar{\Psi}^\pm(x_i) \geq 0$ ,  $x_i \in \Omega^N$ . Then by Theorem 4.1 we get the result.  $\square$

### 5. Error estimates for the solution

We obtain separate error estimates for each component of the numerical solution.

**Lemma 5.1.** *Let  $\bar{V}(x_i)$  be a numerical solution of (3.4) defined by (4.3). Then*

$$|(v_j - V_j)(x_i)| \leq CN^{-1}, \quad x_i \in \bar{\Omega}^N, \quad j = 1, 2.$$

*Proof.* By [15] we have

$$|L_j^N(\bar{v} - \bar{V})(x_i)| \leq CN^{-1}, \quad j = 1, 2.$$

Further

$$\begin{aligned} B_j^N(v_j - V_j)(x_N) &= L_1^N v_j(x_N) - L_1^N V_j(x_N) \\ &= L_1^N v_j(x_N) - l_j \\ |B_j^N(v_j - V_j)(x_N)| &\leq C\varepsilon(h_1^3 v''(\chi_1) + \dots + h_N^3 v''(\chi_N)) \\ &\leq CN^{-2}, \end{aligned}$$

where  $x_{i-1} \leq \chi_i \leq x_i$ ,  $1 \leq i \leq N$ ,  $j = 1, 2$ . Then by discrete stability result, we have  $|(v_j - V_j)(x_i)| \leq CN^{-1}$ ,  $x_i \in \bar{\Omega}^N$ ,  $j = 1, 2$ .  $\square$

**Lemma 5.2.** *Let  $\bar{W}(x_i)$  be a numerical solution of (3.5) defined in (4.4). Then*

$$|(w_j - W_j)(x_i)| \leq CN^{-1}(\ln N)^2, \quad x_i \in \bar{\Omega}^N, \quad j = 1, 2.$$

*Proof.* Note that

$$|w_j(x_i) - W_j(x_i)| \leq |u_j(x_i) - U_j(x_i)| + |v_j(x_i) - V_j(x_i)|, \quad j = 1, 2.$$

Then by (3.8), Theorem 3.4 and Lemma 5.1, we have

$$|u_j(x_i) - U_j(x_i)| \leq |U_j(x_i) - V_j(x_i)| + |v_j(x_i) - V_j(x_i)| + |u_j(x_i) - v_j(x_i)|, \quad j = 1, 2.$$

Therefore

$$\begin{aligned} |w_j(x_i) - W_j(x_i)| &\leq |u_j(x_i) - U_j(x_i)| + |v_j(x_i) - V_j(x_i)| \\ &\leq Ce^{-\alpha(1-x_i)/\varepsilon} + CN^{-1} \\ &\leq Ce^{-\alpha\sigma/\varepsilon} + CN^{-1} \leq CN^{-1}, \quad 0 \leq i \leq \frac{N}{2}. \end{aligned}$$



Now consider mesh functions

$$\begin{aligned} \psi_j^\pm(x_i) &= CN^{-1}S_j(x_i) + CN^{-1}\frac{\sigma}{\varepsilon^2}(x_i - (1 - \sigma)) \pm (w_j - W_j)(x_i), \\ x_i &\in [1 - \sigma, 1], \quad j = 1, 2. \end{aligned}$$

It is easy to see that  $\Psi_j^\pm(x_{N/2}) \geq 0$  and  $B_j\Psi^\pm(x_N) \geq 0$ ,  $j = 1, 2$ .

$$\begin{aligned} L_1^N\bar{\Psi}^\pm(x_i) &= CN^{-1}a_1(x_i)(1 + \frac{\sigma}{\varepsilon^2}) \pm (L_1^N - L_1)\bar{w}(x_i) \\ &\geq CN^{-1}\alpha(1 + \frac{\sigma}{\varepsilon^2}) \pm CN^{-1}\varepsilon^{-2} \geq 0. \end{aligned}$$

Similarly one can prove that  $L_2\bar{\Psi}^\pm(x_i) \geq 0$ . Then by discrete maximum principle, we have  $\psi_j^\pm(x_i) \geq 0$ ,  $x_i \in [1 - \sigma, 1]$ ,  $j = 1, 2$ .

Therefore  $|w_j(x_i) - W_j(x_i)| \leq CN^{-1}(\ln N)^2$ ,  $x_i \in [1 - \sigma, 1]$ ,  $j = 1, 2$ .  $\square$

**Theorem 5.3.** *Let  $\bar{U}(x_i)$  be the solution of (2.1)-(2.2) defined in (4.1)-(4.2). Then*

$$|u_j(x_i) - U_j(x_i)| \leq CN^{-1}(\ln N)^2, \quad x_i \in \bar{\Omega}^N, \quad j = 1, 2.$$

*Proof.* Combining Lemma 5.1 and Lemma 5.2, the proof gets completed.  $\square$

## 6. Numerical results

**Example 6.1.**

$$\begin{aligned} -\varepsilon u_1''(x) + (1+x)u'(x) + 6u_1(x) - 2u_2(x) &= 2, \quad x \in \Omega, \\ -\varepsilon u_1''(x) + (1+x)u'(x) - 2u_1(x) + 5u_2(x) &= 3, \quad x \in \Omega, \end{aligned}$$

with the boundary conditions

$$\begin{aligned} u_1(0) = 0, \quad u_1(1) - \varepsilon \int_0^1 \frac{x}{2} u_1(x) dx &= 0, \\ u_2(0) = 0, \quad u_2(1) - \varepsilon \int_0^1 \frac{x}{2} u_2(x) dx &= 0. \end{aligned}$$

**Example 6.2.**

$$\begin{aligned} -\varepsilon u_1''(x) + u'(x) + 6u_1(x) - 2u_2(x) &= \exp(x), \quad x \in \Omega, \\ -\varepsilon u_1''(x) + u'(x) - 2u_1(x) + 5u_2(x) &= \cos(x), \quad x \in \Omega, \end{aligned}$$

with the boundary conditions

$$\begin{aligned} u_1(0) = 0, \quad u_1(1) - \varepsilon \int_0^1 \frac{x}{2} u_1(x) dx &= 1, \\ u_2(0) = 0, \quad u_2(1) - \varepsilon \int_0^1 \frac{x}{2} u_2(x) dx &= 1. \end{aligned}$$

The analytical solution of the above examples are not available. Therefore, we estimate the error by using double mesh principle which is defined as  $D_\varepsilon^N = \max_{x_i \in \bar{\Omega}^N} |U^N(x_i) - U^{2N}(x_i)|$  and  $D^N = \max_\varepsilon D_\varepsilon^N$  where  $U^N(x_i)$  and  $U^{2N}(x_i)$  denote the numerical solution computed using  $N$  and  $2N$  mesh points. From these quantities the order of convergence is defined as  $P^N = \log_2(\frac{D^N}{D^{2N}})$ . In Tables 1 and 2,  $D_1^N$  and  $D_2^N$  denote the maximum pointwise errors of  $U_1$  and  $U_2$  respectively,  $P_1^N$  and  $P_2^N$  denote the order of convergence with respect to  $U_1$  and  $U_2$  respectively. The assumption  $\varepsilon \leq CN^{-1}$  is made for only theoretical purpose. The numerical method works for all  $\varepsilon$  for our examples.

The numerical solutions are plotted in Figure 1 and Figure 3. Loglog plot of the maximum pointwise errors of Example 6.1 is given in Figure 2. The maximum pointwise errors for Example 6.2 through loglog plot is presented in Figure 4.

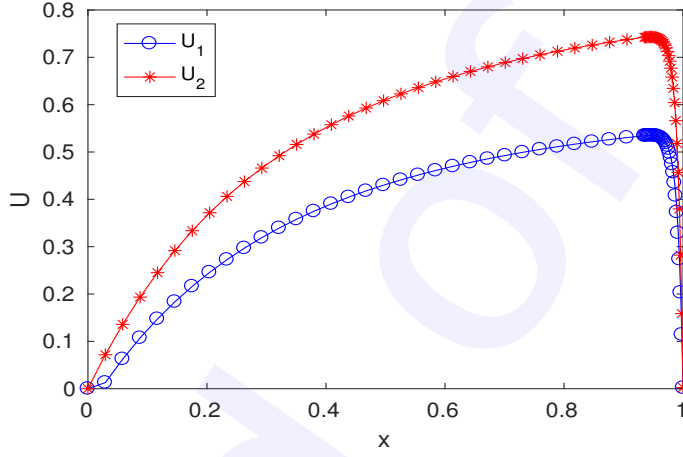


FIGURE 1. Numerical solution graph of the Example 6.1 for  $\varepsilon = 2^{-7}$  and  $N = 64$

TABLE 1. Maximum pointwise errors and order of convergence for Example 6.1

	Number of mesh points $N$						
	16	32	64	128	256	512	1024
$D_1^N$	5.075e-02	3.286e-02	2.128e-02	1.280e-02	7.167e-03	3.993e-03	2.134e-03
$P_1^N$	0.6268	0.6267	0.7329	0.8374	0.8437	0.9035	-
$D_2^N$	3.788e-02	2.574e-02	1.563e-02	9.878e-03	5.850e-03	3.378e-03	1.911e-03
$P_2^N$	0.5573	0.7194	0.6624	0.7558	0.7921	0.8219	-

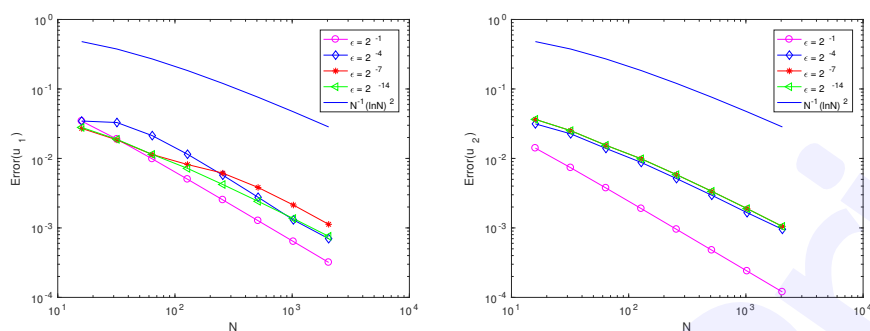


FIGURE 2. Maximum pointwise errors as a function of  $N$  and  $\varepsilon$  for the solution  $U_1$  and  $U_2$  for Example 6.1

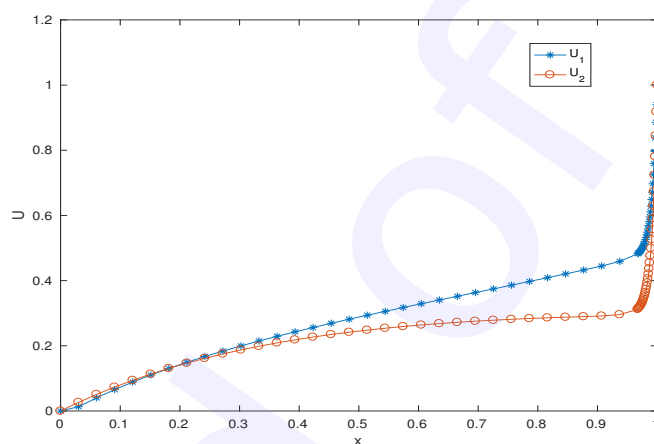


FIGURE 3. Numerical solution graph of the Example 6.2 for  $\varepsilon = 2^{-7}$  and  $N = 64$

TABLE 2. Maximum pointwise errors and order of convergence for Example 6.2

	Number of mesh points $N$						
	16	32	64	128	256	512	1024
$D_1^N$	3.253e-02	1.924e-02	1.170e-02	6.769e-03	3.693e-03	2.040e-03	1.079e-03
$P_1^N$	0.7576	0.7165	0.7904	0.8741	0.8557	0.9186	-
$D_2^N$	1.425e-02	1.380e-02	9.353e-03	5.653e-03	3.220e-03	1.790e-03	9.819e-04
$P_2^N$	0.4595	0.5620	0.7264	0.8120	0.8470	0.8662	-

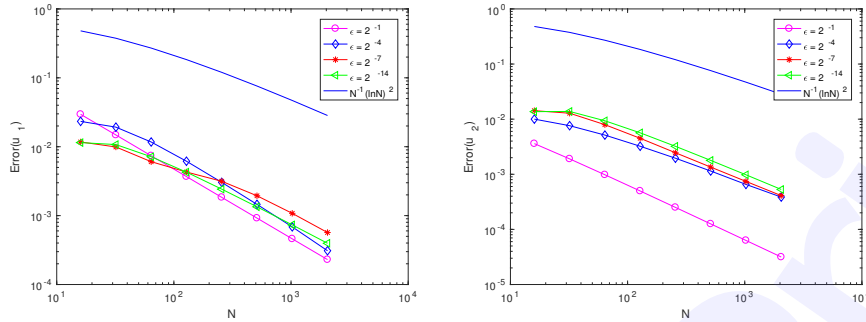


FIGURE 4. Maximum pointwise errors as a function of  $N$  and  $\epsilon$  for the solution  $U_1$  and  $U_2$  for Example 6.2

## 7. Conclusion

We have solved a class of system of singularly perturbed boundary value problem (2.1)-(2.2) with integral boundary conditions, using a finite difference method on a Shishkin mesh. The method is shown to be of order  $O(N^{-1} \ln^2 N)$  (See Tables 1 and 2). Two examples are given to illustrate the numerical method. Our numerical results reflect the theoretical estimates. We are developing a numerical method for the class of systems of singularly perturbed differential equations with integral boundary conditions for two different parameters.

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