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CESÀRO-HYPERCYCLIC AND HYPERCYCLIC OPERATORS

Mohammed El Berrag and Abdelaziz Tajmouati

Abstract. In this paper we provide a Cesàro-hypercyclicity criterion and offer two examples of this criterion. At the same time, we also characterize other properties of Cesàro-hypercyclic operators.

1. Introduction

Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space over the scalar field $\mathbb{C}$. As usual, $\mathbb{N}$ is the set of all non-negative integers, $\mathbb{Z}$ is the set of all integers, and $B(\mathcal{H})$ is the space of all bounded linear operators on $\mathcal{H}$. A bounded linear operator $T : \mathcal{H} \to \mathcal{H}$ is called hypercyclic if there is some vector $x \in \mathcal{H}$ such that $Orb(T, x) = \{T^n x : n \in \mathbb{N}\}$ is dense in $\mathcal{H}$, where such a vector x is said hypercyclic for T.

The first example of hypercyclic operator was given by Rolewicz in [11]. He proved that if B is a backward shift on the Banach space l^p, then λB is hypercyclic if and only if $|\lambda| > 1$.

Let $\{e_n\}_{n\geq 0}$ be the canonical basis of $l^2(\mathbb{N})$. If $\{w_n\}_{n\in\geq 1}$ is a bounded sequence in $\mathbb{C}\backslash\{0\}$, then the unilateral backward weighted shift $T : l^2(\mathbb{N}) \longrightarrow l^2(\mathbb{N})$ is defined by $Te_n = w_n e_{n-1}$, $n \geq 1$, $Te_0 = 0$, and let $\{e_n\}_{n\in\mathbb{Z}}$ be the canonical basis of $l^2(\mathbb{Z})$. If $\{w_n\}_{n\in\mathbb{Z}}$ is a bounded sequence in $\mathbb{C}\backslash\{0\}$, then the bilateral weighted shift $T : l^2(\mathbb{Z}) \longrightarrow l^2(\mathbb{Z})$ is defined by $Te_n = w_n e_{n-1}$. The definition and the properties of supercyclicity operators were introduced by Hilden and Wallen [9]. They proved that all unilateral backward weighted shifts on a Hilbert space are supercyclic.

A bounded linear operator $T \in B(\mathcal{H})$ is called supercyclic if there is some vector $x \in \mathcal{H}$ such that the projective orbit $\mathbb{C}.Orb(T, x) = \{\lambda T^n x : \lambda \in \mathbb{C}, n \in \mathbb{N}\}$ is dense in X. Such a vector x is said supercyclic for T. Refer to [1,3,8,14] for more informations about hypercyclicity and supercyclicity.

A nice criterion namely hypercyclicity criterion, was developed independently by Kitai [7] and, Gethner and Shapiro [6]. The hypercyclicity criterion

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has been widely used to show that many different types of operators are hypercyclic. For instance hypercyclic operators arise in the classes of composition operators [2], adjoints of multiplication operators [6], cohyponormal operators [5], and weighted shifts [12].

For the following theorem, see [1, 8].

Theorem 1.1 (Hypercyclicity Criterion)**.** *Suppose that $T \in B(\mathcal{H})$. If there exist two dense subsets X_0 and Y_0 in $\mathcal{H}$ and an increasing sequence n_j of positive integer such that*:

(1) $T^{n_j}x \to 0$ *for each* $x \in X_0$*, and*
(2) *there exist mappings* $S_{n_j} : Y_0 \longrightarrow \mathcal{H}$ *such that* $S_{n_j}y \to 0$*, and* $T^{n_j}S_{n_j}y \to y$ *for each* $y \in Y_0$*,*

then T is hypercyclic.

In [12] and [13], Salas characterized the bilateral weighted shifts that are hypercyclic and those that are supercyclic in terms of their weight sequence. In [4], N. Feldman gave a characterization of the invertible bilateral weighted shifts that are hypercyclic or supercyclic.

For the following theorem, see [4, Theorem 4.1].

Theorem 1.2. *Suppose that $T : l^2(\mathbb{Z}) \longrightarrow l^2(\mathbb{Z})$ is a bilateral weighted shift with weight sequence $(w_n)_{n\in\mathbb{Z}}$ and either $w_n \geq m > 0$ for all $n < 0$ or $w_n \leq m$ for all $n > 0$. Then*:

(1) *T is hypercyclic if and only if there exists a sequence of integers $n_k \to \infty$ such that* $\lim_{k\to\infty} \prod_{j=1}^{n_k} w_j = 0$ *and* $\lim_{k\to\infty} \prod_{j=1}^{n_k} \frac{1}{w_{-j}} = 0$.
(2) *T is supercyclic if and only if there exists a sequence of integers $n_k \to \infty$ such that* $\lim_{k\to\infty}(\prod_{j=1}^{n_k} w_j)(\prod_{j=1}^{n_k} \frac{1}{w_{-j}}) = 0$.

Let $\mathcal{M}_n(T)$ denote the arithmetic mean of the powers of $T \in B(\mathcal{H})$, that is

$$\mathcal{M}_n(T) = \frac{1 + T + T^2 + \cdots + T^{n-1}}{n},\ n \in \mathbb{N}^*.$$

If the arithmetic means of the orbit of x are dense in $\mathcal{H}$, then the operator T is said to be Cesàro-hypercyclic. In [10], Fernando León-Saavedra proved that an operator is Cesàro-hypercyclic if and only if there exists a vector $x \in \mathcal{H}$ such that the orbit $\{n^{-1}T^n x\}_{n\geq 1}$ is dense in $\mathcal{H}$ and characterized the bilateral weighted shifts that are Cesàro-hypercyclic.

For the following proposition, see [10, Proposition 3.4].

Proposition 1.1. *Let $T : l^2(\mathbb{Z}) \longrightarrow l^2(\mathbb{Z})$ be a bilateral weighted shift with weight sequence $(w_n)_{n\in\mathbb{Z}}$. Then T is Cesàro-hypercyclic if and only if there exists an increasing sequence n_k of positive integers such that for any integer q,*

$$\lim_{k\to\infty} \prod_{i=1}^{n_k} \frac{w_{i+q}}{n_k} = \infty \text{ and } \lim_{k\to\infty} \prod_{i=0}^{n_k-1} \frac{w_{q-i}}{n_k} = 0.$$

In this paper we will give an example of a hypercyclic and supercyclic operator which is not Cesàro-hypercyclic and vice versa. Furthermore, we provide a Cesàro-Hypercyclicity Criterion and offer two examples of this criterion. At the same time, we also characterize other properties of Cesàro-hypercyclic operators.

2. Main results

Suppose $\{n^{-1}T^n : n \geq 1\}$ is a sequence of bounded linear operators on $\mathcal{H}$.

Definition 2.1. An operator $T \in B(\mathcal{H})$ is Cesàro-hypercyclic if and only if there exists a vector $x \in \mathcal{H}$ such that the orbit $\{n^{-1}T^n x\}_{n\geq 1}$ is dense in $\mathcal{H}$.

The following example gives an operator which is Cesàro-hypercyclic but not hypercyclic.

Example 1 ([10])**.** Let T the bilateral backward shift with the weight sequence

$$w_n = \begin{cases} 1 & \text{if } n \leq 0, \\ 2 & \text{if } n \geq 1. \end{cases}$$

Then T is not hypercyclic, but it is Cesàro-hypercyclic.

Now, we will give an example of a hypercyclic and supercyclic operator which is not Cesàro-hypercyclic.

Example 2. Let T the bilateral backward shift with the weight sequence

$$w_n = \begin{cases} 2 & \text{if } n < 0, \\ \frac{1}{2} & \text{if } n \geq 0. \end{cases}$$

Then T is not Cesàro-hypercyclic, but it is hypercyclic and supercyclic.

Proof. By applying Theorem 1.2 and taking $n_k = n$, we have

$$\lim_{n\to\infty} \prod_{j=1}^{n} w_j = \lim_{n\to\infty} \frac{1}{2^n} = 0;$$

and

$$\lim_{n\to\infty} \prod_{j=1}^{n} \frac{1}{w_{-j}} = \lim_{n\to\infty} \frac{1}{2^n} = 0.$$

Furthermore, we have

$$\lim_{n\to\infty} (\prod_{j=1}^{n} w_j)(\prod_{j=1}^{n} \frac{1}{w_{-j}}) = \lim_{n\to\infty} (\frac{1}{2^n})(\frac{1}{2^n}) = 0.$$

Therefore by Theorem 1.2 the operator T is hypercyclic and supercyclic. However, for all increasing sequence $n_k = n$ of positive integers and taking $q = 0$, we have

$$\lim_{n\to\infty} \prod_{i=1}^{n} \frac{w_{i+q}}{n} = \lim_{n\to\infty} \frac{1}{n2^n} = 0,$$

from Proposition 1.1, T is not Cesàro-hypercyclic. □

The following example gives us an operator which is Cesàro-hypercyclic but not hypercyclic and supercyclic.

Example 3. Let T the bilateral backward shift with the weight sequence

$$w_n = \begin{cases} \frac{1}{2} & \text{if } n < 0, \\ n+1 & \text{if } n \geq 0. \end{cases}$$

Then T is Cesàro-hypercyclic, but it is not hypercyclic and supercyclic.

Proof. By applying Proposition 1.1 and taking $n_k = n$ and $q = 0$, we have

$$\lim_{n\to\infty} \prod_{i=1}^{n} \frac{w_{i+q}}{n} = \lim_{n\to\infty} \frac{(n+1)!}{n} = \infty,$$

and

$$\lim_{n\to\infty} \prod_{i=0}^{n} \frac{w_{q-i}}{n} = \lim_{n\to\infty} \frac{1}{n2^n} = 0.$$

Therefore by Proposition 1.1 the operator T is Cesàro-hypercyclic. On the other hand, we have

$$\lim_{n\to\infty} \prod_{j=1}^{n} w_j = \lim_{n\to\infty} ((n+1)!) = \infty;$$

and

$$\lim_{n\to\infty} (\prod_{j=1}^{n} w_j)(\prod_{j=1}^{n} \frac{1}{w_{-j}}) = \lim_{n\to\infty} ((n+1)!)(2^n) = \infty.$$

Therefore by Theorem 1.2 the operator T is not hypercyclic and supercyclic. □

Definition 2.2. We say that $T \in B(\mathcal{H})$ is Cesàro-topologically transitive if for every nonempty open subsets U and V of $\mathcal{H}$ there exists $n \geq 1$ such that $\frac{T^n}{n}(U) \cap V \neq \emptyset$.

Definition 2.3. We say that $T \in B(\mathcal{H})$ is Cesàro-mixing if for every nonempty open subsets U and V of $\mathcal{H}$ there exists $m \geq 1$ such that $\frac{T^n}{n}(U) \cap V \neq \emptyset$, $\forall n \geq m$.

In the proof of the following lemma, we use a method of the proof of [6, theorem 1.2]. The set of Cesàro-hypercyclic vectors for T is denoted by $CH(T)$.

Lemma 2.1. *An operator $T \in B(\mathcal{H})$ is Cesàro-topologically transitive if and only if $CH(T)$ is dense in $\mathcal{H}$.*

Proof. Fix an enumeration $\{B_n, n \geq 1\}$ of the open balls in $\mathcal{H}$ with rational radii, and centers in a countable dense subset of $\mathcal{H}$. By the continuity of the sequence $\frac{T^n}{n}$ the sets

$$G_k = \bigcup\{(\frac{T^n}{n})^{-1}(B_k) : n \in \mathbb{N}^*\}$$

ar open. Clearly $CH(T)$ equal to

$$\bigcap\{G_k : k \in \mathbb{N}^*\}.$$

Now let T be Cesàro-topologically transitive and let U by an arbitrary nonempty open set in $\mathcal{H}$. Then for all $k \in \mathbb{N}^*$, there exist $n(k)$ in $\mathbb{N}^*$ such that

$$(\frac{T^{n(k)}}{n(k)})^{-1}(U) \cap B_k \neq \emptyset$$

which implies that $U \cap G_k \neq \emptyset$ for all k. Thus each G_k is dense in $\mathcal{H}$ and so by the Bair Category Theorem $CH(T)$ is also dense in $\mathcal{H}$.

Conversely, if $CH(T)$ is dense in $\mathcal{H}$, then each set G_k. This implies that T is Cesàro-topologically transitive. □

Theorem 2.1 (Cesàro-Hypercyclicity Criterion)**.** *Suppose that $T \in B(\mathcal{H})$. If there exist two dense subsets X_0 and Y_0 in $\mathcal{H}$ and an increasing sequence n_j of positive integer such that*:

(1) $\frac{T^{n_j}}{n_j}x \to 0$ *for each* $x \in X_0$, *and*

(2) *there exist mappings* $S_{n_j} : Y_0 \longrightarrow \mathcal{H}$ *such that* $S_{n_j}y \to 0$, *and* $\frac{T^{n_j}}{n_j}S_{n_j}y \to y$ *for each* $y \in Y_0$,

then T is Cesàro-hypercyclic.

Proof. Let U and V are two nonempty open sets in $\mathcal{H}$, then chose $x \in X_0 \cap U$ and $y \in V \cap Y_0$ and let $z_j = x + S_{n_j}y/$ Then as $j \to \infty$, $z_j \to x$ and $\frac{T^{n_j}}{n_j}z_j = \frac{T^{n_j}}{n_j}x + \frac{T^{n_j}}{n_j}S_{n_j}y \to y$. Thus for large j we have $z_j \in U$ and $\frac{T^{n_j}}{n_j}z_j \in V$. By Lemma 2.1, $CH(T)$ is dense in $\mathcal{H}$ and this implies clearly that T is Cesàro-hypercyclic. □

Suppose $T : l^2(\mathbb{N}) \longrightarrow l^2(\mathbb{N})$ be a unilateral weighted shift given by $Te_n = w_n e_{n-1}$, $n \geq 1, Te_0 = 0$. Let $\{e_n\}_{n\geq 0}$ be the canonical basis of $l^2(\mathbb{N})$. We may define a right inverse S of T as $Se_j = \frac{\sqrt[n]{n}}{w_{j+1}}e_{j+1}$.

Example 4. Taking $n_j = n \geq 1$ and suppose $\lim_{n\to\infty}\prod_{i=1}^{n}\frac{w_{j+i}}{n} = \infty$ and $\lim_{n\to\infty}\prod_{i=0}^{n-1}\frac{w_{j-i}}{n} = 0$. Let $X_0 = Y_0 = \text{span}\{e_j : j \in \mathbb{N}\}$ and $S_n = S^n$, where S is the right inverse of T. So we get

$$\frac{T^n}{n}e_j = \prod_{i=0}^{n-1}\frac{w_{j-i}}{n}e_{j-n} \to 0 \text{ for all } j \in \mathbb{N}.$$

Furthermore, we have

$$S_n e_j = S^n e_j = \frac{n}{\prod_{i=1}^n w_{j+i}} \to 0,$$

and

$$||\frac{T^n}{n} S_n e_j - e_j|| = ||\frac{T^n}{n}.\frac{n}{\prod_{i=1}^n w_{j+i}} e_{j+n} - e_j|| \to 0.$$

Hence $\frac{T^n}{n} S_n e_j \to e_j$ for all $j \in \mathbb{N}$. Thus T satisfies the Cesàro-Hypercyclicity Criterion with respect to $n_j = n$.

Example 5. Let $B : l^2(\mathbb{N}) \longrightarrow l^2(\mathbb{N})$ is a backward shift with weight $w_n = 1$, $n \geq 1$ and $T = \lambda B$, where $|\lambda| > 1$. Then T is Cesàro-hypercyclic.

Proof. Let $B(x_1, x_2, \dots, x_n, \dots) = (x_2, x_3, \dots, x_n, \dots)$ for all $(x_i)_{i\in\mathbb{N}} \in l^2(\mathbb{N}$ and $S^n(x_0, x_1, \dots) = \frac{n}{\lambda^n}(0, 0, \dots, x_0, x_1, \dots)$. Let $Y_0 = X_0$ be the set of all vectors in $l^2(\mathbb{N})$, where $Y_0 = \{(y_1, y_2, \dots, y_n, 0, 0, \dots) \in l^2(\mathbb{N}) : n \in \mathbb{N}\}$. Now Y_0 is dense in $l^2(\mathbb{N})$, and $\frac{T^n}{n}x = \frac{(\lambda B)^n}{n} = 0$ for every $x \in Y_0$, and also we have $S^n y = \frac{n}{\lambda^n}(0, 0, \dots, y_0, y_1, \dots) \to 0$ as $n \to \infty$, since $|\lambda| > 1$. Moreover, $\frac{T^n}{n} S^n y = \frac{(\lambda B)^n S^n}{n} y = B^n(0, 0, \dots, y_0, y_1, \dots) = (y_1, y_2, \dots) = y$. Therefore, by Theorem 2.1, $T = \lambda B$ is Cesàro-hypercyclic. □

Proposition 2.1. *Let $T \in B(\mathcal{H})$ satisfy the Hypercyclicity Criterion with respect to a sequence $\{n_j\}$. Then T is Cesàro-mixing.*

Proof. We show that T is Cesàro-mixing. Let X_0 and Y_0 be dense sets in $\mathcal{H}$, that are given in the Cesàro-hypercyclicity Criterion. Let U and V are two nonempty open sets in $\mathcal{H}$, then choose $x \in X_0 \cap U$ and $y \in V \cap Y_0$ and $\varepsilon > 0$ such that $B(x, \varepsilon) \subset U$ and $B(y, \varepsilon) \subset V$. By Theorem 2.1, there exists $j_0 \in \mathbb{N}^*$ so that for all $j \geq j_0$, $||\frac{T^{n_j}}{n_j} x|| \leq \varepsilon, ||S_{n_j}(y)|| \leq \varepsilon$, and $||\frac{T^{n_j}}{n_j} S_{n_j}(y) - y|| \leq \varepsilon$. Then for each $j \geq j_0$ we have $z_j = x + S_{n_j} y \in B(x, \varepsilon) \subset U$ and $\frac{T^{n_j}}{n_j} z_j \in B(y, \varepsilon) \subset V$. That is, $\frac{T^{n_j}}{n_j}(U) \cap V \neq \emptyset, \forall j \geq j_0$. Hence T is Cesàro-mixing. □

Let $\mathbb{J} := \{(x, y) \in \mathcal{H} \times \mathcal{H}; \exists (u_n)_{n\in\mathbb{N}^*} \subset X : u_n \to x \text{ and } \frac{T^n}{n} u_n \to y\}$

Proposition 2.2. *Let $T \in B(\mathcal{H})$ and $\mathbb{J}$ is dense in $\mathcal{H} \times \mathcal{H}$. Then T is Cesàro-mixing.*

Proof. Let U and V are two nonempty open sets in $\mathcal{H}$. Since $\mathbb{J}$ is dense in $\mathcal{H} \times \mathcal{H}$, we can find $x \in U$ and $y \in V$ such that $(x, y) \in \mathbb{J}$. By definition of $\mathbb{J}$, there is a sequence $(u_n)_{n\in\mathbb{N}^*} \subset X$ such that $u_n \to x$ and $\frac{T^n}{n} u_n \to y$. Then, there exists $k_0 \in \mathbb{N}^*$ such that $u_n \in U$ and $\frac{T^n}{n} u_n \in V \ \forall k \geq k_0$. Hence $\frac{T^n}{n}(U) \cap V \neq \emptyset, \ \forall k \geq k_0$. That is T is Cesàro-mixing. □

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Mohammed El Berrag
Sidi Mohamed Ben Abdellah University
Faculty of Sciences Dhar El Mahraz
Fez, Morocco
Email address: `mohammed.elberrag@usmba.ac.ma`

Abdelaziz Tajmouati
Sidi Mohamed Ben Abdellah University
Faculty of Sciences Dhar El Mahraz
Fez, Morocco
Email address: `abdelaziz.tajmouati@usmba.ac.ma`