Commun. Korean Math. Soc. **0** (0), No. 0, pp. 1–0
https://doi.org/10.4134/CKMS.c180157
pISSN: 1225-1763 / eISSN: 2234-3024

APPLICATIONS OF JACK'S LEMMA FOR CERTAIN SUBCLASSES OF HOLOMORPHIC FUNCTIONS ON THE UNIT DISC

Batuhan Çatal and Bülent Nafi örnek

Abstract. In this paper, we give some results on $\frac{zf'(z)}{f(z)}$ for the certain classes of holomorphic functions in the unit disc $E = \{z : |z| < 1\}$ and on $\partial E = \{z : |z| = 1\}$. For the function $f(z) = z^2 + c_3z^3 + c_4z^4 + \cdots$ defined in the unit disc E such that $f(z) \in \mathcal{A}_\alpha$, we estimate a modulus of the angular derivative of $\frac{zf'(z)}{f(z)}$ function at the boundary point b with $\frac{bf'(b)}{f(b)} = 1 + \alpha$. Moreover, Schwarz lemma for class $\mathcal{A}_\alpha$ is given. The sharpness of these inequalities is also proved.

1. Introduction

Let $\mathcal{H}$ denote the class of functions $f(z) = z^2 + c_3z^3 + c_4z^4 + \cdots$ which are holomorphic in $E = \{z : |z| < 1\}$. Also, let $\mathcal{A}_\alpha$ be the subclass of $\mathcal{H}$ consisting of all functions $f(z)$ which satisfy

$$\Re\left(z\frac{f''(z)}{f'(z)}\right) > -\frac{2\alpha^2 - \alpha + 1}{2\,(1-\alpha)},\ z \in E, \tag{1.1}$$

where $0 < \alpha < 1$.

One of the simplest results of the complex function theory for holomorphic functions is both the classical Schwarz lemma and Jack's lemma. The Schwarz lemma and Jack's lemma have a very important role in the geometric function theory. A general form for these two lemmas, which are very simple and commonly used, is as follows:

Lemma 1.1 (Schwarz lemma)**.** *Let $f : E \to E$ be a holomorphic function with $f(0) = 0$. Then $|f(z)| \le |z|$ for all $z \in E$, and $|f'(0)| \le 1$. In addition, if the equality $|f(z)| = |z|$ holds for any $z \neq 0$, or $|f'(0)| = 1$ then f is a rotation, that is, $f(z) = ze^{i\sigma}$, σ real* ([7, p. 329]).

Received April 16, 2018; Revised August 10, 2018; Accepted September 7, 2018.
2010 *Mathematics Subject Classification.* Primary 30C80, 32A10.
Key words and phrases. Schwarz lemma, Jack's lemma, angular limit.

Lemma 1.2 (Jack's lemma)**.** *Let $f(z)$ be a non-constant and holomorphic function in the unit disc E with $f(0)=0$. If $|f(z)|$ attains its maximum value on the circle $|z|=r$ at the point z_0, then*

$$\frac{z_0 f'(z_0)}{f(z_0)}=k,$$

where $k\geq 1$ is a real number ([8]).

For historical background about the Schwarz lemma and its applications on the boundary of the unit disc, we refer to [2,6]. Also, a different application of Jack's lemma is shown in [14,19].

In this work, we show an application of Jack's lemma for certain subclasses of holomorphic functions on the unit disc that provide (1.1) inequality. Also, we will give Schwarz lemma for this class. Moreover, we will give at the boundary Schwarz lemma for this class.

Let $f(z)=z^2+c_3z^3+c_4z^4+\cdots$ be a holomorphic function in the unit disc E. Consider the function

$$\Theta(z)=\frac{g(z)-2}{g(z)-2\alpha}, \tag{1.2}$$

where $g(z)=\frac{zf'(z)}{f(z)}$ and $f(z)\in\mathcal{A}_\alpha$. $\Theta(z)$ is holomorphic in the unit disc and $\Theta(0)=0$. We show that $|\Theta(z)|<1$ for $|z|<1$. We suppose that there exists a point $z_0\in E$ such that

$$\max_{|z|\leq|z_0|}|\Theta(z)|=|\Theta(z_0)|=1.$$

From the Jack' lemma, we have

$$\Theta(z_0)=e^{i\theta},\ \frac{z_0\Theta'(z_0)}{\Theta(z_0)}=k.$$

Therefore, from (1.2) we obtain

$$\begin{aligned}\Re\left(z_0\frac{f''(z_0)}{f'(z_0)}\right)&=\Re\left(-\frac{\alpha z_0\Theta'(z_0)}{1-\alpha\Theta(z_0)}+\frac{z_0\Theta'(z_0)}{1-\Theta(z_0)}+\frac{2\left(1-\alpha\Theta(z_0)\right)}{1-\Theta(z_0)}-1\right)\\&=\Re\left(-\frac{\alpha k e^{i\theta}}{1-\alpha e^{i\theta}}+\frac{ke^{i\theta}}{1-e^{i\theta}}+\frac{2\left(1-\alpha e^{i\theta}\right)}{1-e^{i\theta}}-1\right).\end{aligned}$$

Since

$$-\frac{\alpha k e^{i\theta}}{1-\alpha e^{i\theta}}=-\alpha k\frac{1}{e^{-i\theta}-\alpha}=\frac{-\alpha k}{\cos\theta-i\sin\theta-\alpha}=-\alpha k\frac{\cos\theta-\alpha+i\sin\theta}{1+\alpha^2-2\alpha\cos\theta},$$

$$\begin{aligned}\frac{ke^{i\theta}}{1-e^{i\theta}}&=\frac{k}{e^{-i\theta}-1}=\frac{k}{\cos\theta-i\sin\theta-1}=k\frac{\cos\theta-1+i\sin\theta}{\left(\cos\theta-1\right)^2+\sin^2\theta}\\&=k\frac{\cos\theta-1+i\sin\theta}{\cos^2\theta-2\cos\theta+1+\sin^2\theta}=k\frac{\cos\theta-1+i\sin\theta}{2\left(1-\cos\theta\right)}\end{aligned}$$

and

$$\begin{aligned}\frac{2\left(1-\alpha e^{i\theta}\right)}{1-e^{i\theta}} &= 2\frac{1-\alpha\cos\theta-\alpha i\sin\theta}{1-\cos\theta-i\sin\theta}\\ &= 2\frac{\left(1-\alpha\cos\theta-\alpha i\sin\theta\right)\left(1-\cos\theta+i\sin\theta\right)}{2\left(1-\cos\theta\right)}\\ &= \frac{1+\alpha-\left(1+\alpha\right)\cos\theta+i\left(1-\alpha\right)\sin\theta}{1-\cos\theta}\end{aligned}$$

we take

$$\begin{aligned}\Re\left(z_0\frac{f''(z_0)}{f'(z_0)}\right) = &-\alpha k\frac{\cos\theta-\alpha}{1+\alpha^2-2\alpha\cos\theta}+k\frac{\cos\theta-1}{2\left(1-\cos\theta\right)}\\ &+\frac{1+\alpha-\left(1+\alpha\right)\cos\theta}{1-\cos\theta}-1\end{aligned}$$

and

$$\Re\left(z_0\frac{f''(z_0)}{f'(z_0)}\right)=-\alpha k\frac{\cos\theta-\alpha}{1+\alpha^2-2\alpha\cos\theta}-\frac{k}{2}+1+\alpha-1. \tag{1.3}$$

Since the right hand side of (1.3) takes its maximum value for $\cos\theta=1$, we have

$$\begin{aligned}\Re\left(z_0\frac{f''(z_0)}{f'(z_0)}\right) &\leq -\alpha k\frac{1-\alpha}{1+\alpha^2-2\alpha}-\frac{k}{2}+\alpha\\ &= -k\frac{\alpha}{1-\alpha}-\frac{k}{2}+\alpha\end{aligned}$$

and

$$\Re\left(z_0\frac{f''(z_0)}{f'(z_0)}\right)\leq -\frac{2\alpha^2-\alpha+1}{2\left(1-\alpha\right)}.$$

This contradicts our condition of the inequality (1.1). This means that there is no point $z_0\in E$ such that $|\Theta(z_0)|=1$ for all $z\in E$. Thus, we obtain $|\Theta(z)|<1$ for $z\in E$. Therefore, the function $\Theta(z)$ defined at the (1.2) provides the conditions of Schwarz lemma. If we apply Schwarz lemma to the function $\Theta(z)$, we obtain

$$|\Theta'(0)|\leq 1$$

and

$$2\left(1-\alpha\right)\frac{|g'(0)|}{|g(0)-2\alpha|^2}\leq 1.$$

Since

$$g(z)=\frac{zf'(z)}{f(z)}=\frac{z\left(2z+3c_3z^2+4c_4z^3+\cdots\right)}{z^2+c_3z^3+c_4z^4+\cdots}=\frac{2+3c_3z+4c_4z^2}{1+c_3z+c_4z^2+\cdots},$$

$$g(0)=2$$

and

$$g'(0)=c_3,$$

we take

$$2\left(1-\alpha\right)\frac{\left|c_{3}\right|}{\left|2-2\alpha\right|^{2}}\leq 1$$

and

$$\left|c_{3}\right|\leq 2\left(1-\alpha\right). \tag{1.4}$$

The result is sharp and the extremal function is

$$f(z)=\frac{z^{2}}{\left(1-z\right)^{2(1-\alpha)}}.$$

This result yields a "$\mathcal{A}_\alpha$ version" of the classical Schwarz lemma for holomorphic function of one complex variable.

It is an elementary consequence of Schwarz lemma that if f extends continuously to some boundary point b with $|b|=1$, and if $|f(b)|=1$ and $f'(b)$ exists, then $|f'(b)|\geq 1$, which is known as the Schwarz lemma on the boundary. In [15], R. Osserman proposed the boundary refinement of the classical Schwarz lemma as follows:

Let $f:E\rightarrow E$ be holomorphic function with $f(0)=0$. Assume that there is a $b\in\partial E$ so that f extends continuously to b, $|f(b)|=1$ and $f'(b)$ exists. Then

$$|f'(b)|\geq\frac{2}{1+|f'(0)|}. \tag{1.5}$$

Thus, by the classical Schwarz lemma, it follows that

$$|f'(b)|\geq 1. \tag{1.6}$$

Inequality (1.5) is sharp, with equality possible for each value of $|f'(0)|$. In addition, for $b=1$ in the inequality (1.5), equality occurs for the function $f\left(z\right)=z\frac{z+\gamma}{1+\gamma z}$, $\gamma\in[0,1]$. Also, $|f'(b)|>1$ unless $f(z)=ze^{i\theta}$, θ real. Inequality (1.6) and its generalizations have important applications in geometric theory of functions and they are still hot topics in the mathematics literature [1, 2, 5, 6, 16–18].

Let us give the definitions needed for our results. A *Stolz angle* Δ at $b\in\partial E$ is the interior of any triangle in E symmetric to $[0,b]$ whose closure lies in E except for the vertex b. Basic for this paper is the notions of the angular limit and the angular derivative. Let $b\in\partial E$. We say that the angular limit $f(b)$ exists if

$$f(b)=\lim_{z\rightarrow b,\ z\in\Delta}f(z)$$

for every *Stolz angle* Δ at b and we say that the angular derivative $f'(b)$ exists if the angular limit $f(b)$ exists and

$$f'(b)=\lim_{z\rightarrow b,\ z\in\Delta}\frac{f(z)-f(b)}{z-b}$$

for every *Stolz angle* Δ at b.

The following lemma, known as the Julia-Wolff lemma, is needed in the sequel (see [20]).

Lemma 1.3 (Julia-Wolff lemma)**.** *Let f be a holomorphic function in E, $f(0) = 0$ and $f(E) \subset E$. If, in addition, the function f has an angular limit $f(b)$ at $b \in \partial E$, $|f(b)| = 1$, then the angular derivative $f'(b)$ exists and $1 \leq |f'(b)| \leq \infty$.*

Corollary 1.4. *The holomorphic function f has a finite angular derivative $f'(b)$ if and only if f' has the finite angular limit $f'(b)$ at $b \in \partial E$.*

D. M. Burns and S. G. Krantz [3] and D. Chelst [4] studied the uniqueness part of the Schwarz lemma. According to M. Mateljevic's studies, some other types of results which are related to the subject can be found in ([12] and [13]). In addition, [11] was posed on ResearchGate where is discussed concerning results in more general aspects. The inequality (1.6) is a particular case of a result due to Vladimir N. Dubinin in [5], who strengthened the inequality $|f'(b)| \geq 1$ by involving zeros of the function f. Also, M. Jeong [9] showed some inequalities at a boundary point for different form of holomorphic functions and found the condition for equality and in [10] a holomorphic self map defined on the closed unit disc with fixed points only on the boundary of the unit disc.

2. Main results

In this section, we give some results on $\frac{zf'(z)}{f(z)}$ for the certain subclasses of holomorphic functions in the unit disc on $\partial E = \{z : |z| = 1\}$. For the function $f(z) = z^2 + c_3 z^3 + c_4 z^4 + \cdots$ defined in the unit disc E such that $f(z) \in \mathcal{A}_\alpha$, we estimate a modulus of the angular derivative of $\frac{zf'(z)}{f(z)}$ function at the boundary point b with $\frac{bf'(b)}{f(b)} = 1 + \alpha$. The sharpness of these inequalities is also proved.

Theorem 2.1. *Let $f(z) \in \mathcal{A}_\alpha$. Suppose that, for some $b \in \partial E$, f has an angular limit $f(b)$ at b, $\frac{bf'(b)}{f(b)} = 1 + \alpha$. Then we have the inequality*

$$\left| \left(\frac{zf'(z)}{f(z)} \right)'_{z=b} \right| \geq \frac{1-\alpha}{2}. \tag{2.1}$$

The inequality (2.1) *is sharp with extremal function*

$$f(z) = \frac{z^2}{(1-z)^{2(1-\alpha)}}.$$

Proof. Let us consider the following function

$$\Theta(z) = \frac{g(z) - 2}{g(z) - 2\alpha},$$

where $g(z) = \frac{zf'(z)}{f(z)}$. Then $\Theta(z)$ is holomorphic function in the unit disc E and $\Theta(0) = 0$. By the Jack's lemma and since $f(z) \in \mathcal{A}_\alpha$, we take $|\Theta(z)| < 1$ for

$|z|<1$. Also, we have $|\Theta(b)|=1$ for $b\in\partial E$. It is clear that

$$\Theta'(z)=2\left(1-\alpha\right)\frac{g'(z)}{\left(g(z)-2\alpha\right)^{2}}.$$

Therefore, we take from (1.6), we obtain

$$1\leq|\Theta'(b)|=2\left(1-\alpha\right)\frac{|g'(b)|}{\left|g(b)-2\alpha\right|^{2}}=\frac{2\left|g'(b)\right|}{1-\alpha}$$

and

$$|g'(b)|\geq\frac{1-\alpha}{2}.$$

Now, we shall show that the inequality (2.1) is sharp. Let

$$f(z)=\frac{z^{2}}{\left(1-z\right)^{2(1-\alpha)}}. \tag{2.2}$$

Differentiating (2.2) logarithmically, we obtain

$$\ln f(z)=\ln\frac{z^{2}}{\left(1-z\right)^{2(1-\alpha)}}=\ln z^{2}-2\left(1-\alpha\right)\ln\left(1-z\right),$$

$$\frac{f'(z)}{f(z)}=\frac{2}{z}+\frac{2\left(1-\alpha\right)}{1-z}$$

and

$$g(z)=\frac{zf'(z)}{f(z)}=2+\frac{2\left(1-\alpha\right)z}{1-z}.$$

Therefore, we take

$$g'(z)=\frac{2\left(1-\alpha\right)}{\left(1-z\right)^{2}}$$

and

$$g'(-1)=\frac{1-\alpha}{2}.$$

□

The inequality (2.1) can be strengthened as below by taking into account c_3 which is second coefficient in the expansion of the function $f(z)=z^{2}+c_{3}z^{3}+c_{4}z^{4}+\cdots$.

Theorem 2.2. *Let $f(z)\in\mathcal{A}_{\alpha}$. Suppose that, for some $b\in\partial E$, f has an angular limit $f(b)$ at b, $\frac{bf'(b)}{f(b)}=1+\alpha$. Then we have the inequality*

$$\left|\left(\frac{zf'(z)}{f(z)}\right)'_{z=b}\right|\geq\frac{2\left(1-\alpha\right)^{2}}{2\left(1-\alpha\right)+|c_{3}|}. \tag{2.3}$$

The inequality (2.3) is sharp with extremal function

$$f(z)=e^{2\int_{0}^{z}\frac{1+at+\alpha at+\alpha t^{2}}{t\left(1+2at+t^{2}\right)}dt},$$

where $a=\frac{|c_{3}|}{2(1-\alpha)}$ is an arbitrary number from $[0,1]$ (see, (1.4)).

Proof. Let $\Theta(z)$ be the same as in the proof of Theorem2.1. Therefore, we take from (1.5), we obtain

$$\frac{2}{1+|\Theta'(0)|} \leq |\Theta'(b)| = \frac{2}{1-\alpha}\left|g'(b)\right|.$$

Since

$$\Theta(z) = \frac{g(z)-2}{g(z)-2\alpha} = \frac{\frac{zf'(z)}{f(z)}-2}{\frac{zf'(z)}{f(z)}-2\alpha}$$
$$= \frac{2+c_3z+\left(2c_4-c_3^2\right)z^2+\cdots-2}{2+c_3z+\left(2c_4-c_3^2\right)z^2+\cdots-2\alpha}$$
$$= \frac{c_3z+\left(2c_4-c_3^2\right)z^2+\cdots}{2\left(1-\alpha\right)+c_3z+\left(2c_4-c_3^2\right)z^2+\cdots},$$

and

$$|\Theta'(0)| = \frac{|c_3|}{2\left(1-\alpha\right)},$$

we take

$$\frac{2}{1+\frac{|c_3|}{2(1-\alpha)}} \leq \frac{2}{1-\alpha}\left|g'(b)\right|$$

and

$$\left|\left(\frac{zf'(z)}{f(z)}\right)'_{z=b}\right| \geq \frac{2\left(1-\alpha\right)^2}{2\left(1-\alpha\right)+|c_3|}.$$

Now, we shall show that the inequality (2.3) is sharp. Let

$$f(z) = e^{2\int_0^z \frac{1+at+\alpha at+\alpha t^2}{t\left(1+2at+t^2\right)}dt}. \tag{2.4}$$

Differentiating (2.4) logarithmically, we obtain

$$\ln f(z) = \ln e^{2\int_0^z \frac{1+at+\alpha at+\alpha t^2}{t\left(1+2at+t^2\right)}dt} = 2\int_0^z \frac{1+at+\alpha at+\alpha t^2}{t\left(1+2at+t^2\right)}dt,$$

$$\frac{f'(z)}{f(z)} = 2\frac{1+az+\alpha az+\alpha z^2}{z\left(1+2az+z^2\right)}$$

and

$$g(z) = \frac{zf'(z)}{f(z)} = 2\frac{1+az+\alpha az+\alpha z^2}{\left(1+2az+z^2\right)}.$$

Thus, we get

$$g'(z) = 2\frac{\left(a+\alpha a+2\alpha z\right)\left(1+2az+z^2\right)}{\left(1+2az+z^2\right)^2}$$
$$-2\frac{\left(2a+2z\right)\left(1+az+\alpha az+\alpha z^2\right)}{\left(1+2az+z^2\right)^2}$$

and

$$|g'(1)| = \frac{2(1-\alpha)^2}{2(1-\alpha)+|c_3|}.$$

The last inequality shows that the equality intended is obtained. □

The inequality (2.3) can be strengthened as below by taking into account c_4 which is third coefficient in the expansion of the function $f(z) = z^2 + c_3 z^3 + c_4 z^4 + \cdots$.

Theorem 2.3. *Let $f(z) \in \mathcal{A}_\alpha$. Suppose that, for some $b \in \partial E$, f has an angular limit $f(b)$ at b, $\frac{bf'(b)}{f(b)} = 1+\alpha$. Then we have the inequality*

$$\left|\left(\frac{zf'(z)}{f(z)}\right)'_{z=b}\right| \geq \frac{1-\alpha}{2}\left(1+\frac{2(2(1-\alpha)-|c_3|)^2}{4(1-\alpha)^2-|c_3|^2+\left|4(1-\alpha)c_4+(3-2\alpha)c_3^2\right|}\right). \tag{2.5}$$

The equality in (2.5) *occurs for the function*

$$f(z) = z^2\left(z^2-1\right)^{\alpha-1}.$$

Proof. Let $\Theta(z)$ be the same as in the proof of Theorem2.1. Let us consider the function

$$\vartheta(z) = \frac{\Theta(z)}{B(z)},$$

where $B(z) = z$. The function $\vartheta(z)$ is holomorphic in E. According to the maximum principle, we have $|\vartheta(z)| < 1$ for each $z \in E$. In particular, we have

$$|\vartheta(0)| = \frac{|c_3|}{2(1-\alpha)} \leq 1 \tag{2.5}$$

and

$$|\vartheta'(0)| = \frac{\left|4(1-\alpha)c_4+(3-2\alpha)c_3^2\right|}{4(1-\alpha)^2}.$$

Furthermore, it can be seen that

$$\frac{b\Theta'(b)}{\Theta(b)} = |\Theta'(b)| \geq |B'(b)| = \frac{bB'(b)}{B(b)}.$$

Consider the function

$$d(z) = \frac{\vartheta(z)-\vartheta(0)}{1-\overline{\vartheta(0)}\vartheta(z)}.$$

This function is holomorphic in E, $|d(z)| \leq 1$ for $|z| < 1$, $d(0) = 0$, and $|d(b)| = 1$ for $b \in \partial E$. From (1.5), we obtain

$$\frac{2}{1+|d'(0)|} \leq |d'(b)| = \frac{1-|\vartheta(0)|^2}{\left|1-\overline{\vartheta(0)}\vartheta(b)\right|^2}|\vartheta'(b)|$$

$$\leq \frac{1+|\vartheta(0)|}{1-|\vartheta(0)|}\left\{|\Theta'(b)|-|B'(b)|\right\}.$$

Since

$$d'(z)=\frac{1-|\vartheta(0)|^2}{\left(1-\overline{\vartheta(0)}\vartheta(z)\right)^2}\vartheta'(z)$$

and

$$|d'(0)|=\frac{|\vartheta'(0)|}{1-|\vartheta(0)|^2}=\frac{\frac{\left|4(1-\alpha)c_4+(3-2\alpha)c_3^2\right|}{4(1-\alpha)^2}}{1-\left(\frac{|c_3|}{2(1-\alpha)}\right)^2}=\frac{\left|4\left(1-\alpha\right)c_4+\left(3-2\alpha\right)c_3^2\right|}{4\left(1-\alpha\right)^2-|c_3|^2},$$

we take

$$\frac{2}{1+\frac{\left|4(1-\alpha)c_4+(3-2\alpha)c_3^2\right|}{4(1-\alpha)^2-|c_3|^2}}\leq\frac{2(1-\alpha)+|c_3|}{2(1-\alpha)-|c_3|}\left\{\frac{2\left|g'(b)\right|}{1-\alpha}-1\right\},$$

$$\frac{2\left(4(1-\alpha)^2-|c_3|^2\right)}{4(1-\alpha)^2-|c_3|^2+\left|4(1-\alpha)c_4+(3-2\alpha)c_3^2\right|}\leq\frac{2(1-\alpha)+|c_3|}{2(1-\alpha)-|c_3|}\left\{\frac{2\left|g'(b)\right|}{1-\alpha}-1\right\},$$

$$\frac{2(2(1-\alpha)-|c_3|)^2}{4(1-\alpha)^2-|c_3|^2+\left|4(1-\alpha)c_4+(3-2\alpha)c_3^2\right|}\leq\frac{2\left|g'(b)\right|}{1-\alpha}-1,$$

$$1+\frac{2(2(1-\alpha)-|c_3|)^2}{4(1-\alpha)^2-|c_3|^2+\left|4(1-\alpha)c_4+(3-2\alpha)c_3^2\right|}\leq\frac{2\left|g'(b)\right|}{1-\alpha}$$

and

$$|g'(b)|\geq\frac{1-\alpha}{2}\left(1+\frac{2\left(2\left(1-\alpha\right)-|c_3|\right)^2}{4\left(1-\alpha\right)^2-|c_3|^2+\left|4\left(1-\alpha\right)c_4+\left(3-2\alpha\right)c_3^2\right|}\right).$$

Now, we shall show that the inequality (2.3) is sharp. Let

$$f(z)=z^2\left(z^2-1\right)^{\alpha-1}. \tag{2.6}$$

Differentiating (2.6) logarithmically, we obtain

$$\ln f(z)=\ln z^2+\ln\left(z^2-1\right)^{\alpha-1},$$

$$\frac{f'(z)}{f(z)}=\frac{2}{z}+(\alpha-1)\frac{2z}{z^2-1}$$

and

$$g(z)=\frac{zf'(z)}{f(z)}=2+(\alpha-1)\frac{2z^2}{z^2-1}.$$

Therefore, we take

$$g'(z)=(\alpha-1)\frac{4z\left(z^2-1\right)-4z^3}{\left(z^2-1\right)^2}=(1-\alpha)\frac{4z}{\left(z^2-1\right)^2}$$

and

$$|g'(i)|=1-\alpha.$$

Since $c_3 = 0$ and $c_4 = 1-\alpha$, we take

$$\frac{1-\alpha}{2}\left(1+\frac{2\left(2\left(1-\alpha\right)-\left|c_3\right|\right)^2}{4\left(1-\alpha\right)^2-\left|c_3\right|^2+\left|4\left(1-\alpha\right)c_4+\left(3-2\alpha\right)c_3^2\right|}\right)=1-\alpha. \qquad \square$$

In the following theorem, the relation between the Taylor coefficients c_3 and c_4 are given for $f(z) = z^2 + c_3z^3 + c_4z^4 + \cdots$.

Theorem 2.4. *Let $f(z) \in \mathcal{A}_\alpha$, $\frac{zf'(z)}{f(z)} - 2$ has no zeros in E except $z = 0$ and $c_3 > 0$. Suppose that, for some $b \in \partial E$, f has an angular limit $f(b)$ at b, $\frac{bf'(b)}{f(b)} = 1+\alpha$. Then we have the inequality*

$$\left|4\left(1-\alpha\right)c_4+\left(3-2\alpha\right)c_3^2\right| \leq 4\left(1-\alpha\right)\left|c_3\ln\frac{c_3}{2\left(1-\alpha\right)}\right|. \tag{2.7}$$

The results (2.7) is sharp for the function given by

$$f(z)=e^{\int_0^z 2\frac{1-\alpha t e^{\frac{1+t}{1-t}\ln\left(\frac{c_3}{2(1-\alpha)}\right)}}{\left(1-te^{\frac{1+t}{1-t}\ln\left(\frac{c_3}{2(1-\alpha)}\right)}\right)t}dt},$$

where $0 < c_3 < 1$ and $\ln\left(\frac{c_3}{2(1-\alpha)}\right) < 0$.

Proof. Let $c_3 > 0$ and let us consider the function $\vartheta(z)$ as in Theorem 2.3. Taking account of the equality (2.5), we denote by $\ln\vartheta(z)$ the holomorphic branch of the logarithm normed by condition

$$\ln\vartheta(0)=\ln\left(\frac{c_3}{2\left(1-\alpha\right)}\right)=\ln\left|\frac{c_3}{2\left(1-\alpha\right)}\right|+i\arg\left(\frac{c_3}{2\left(1-\alpha\right)}\right)<0,\ c_3>0$$

and

$$\ln\left(\frac{c_3}{2\left(1-\alpha\right)}\right)<0.$$

Take the following auxiliary function

$$L(z)=\frac{\ln\vartheta(z)-\ln\vartheta(0)}{\ln\vartheta(z)+\ln\vartheta(0)}.$$

It is obvious that $L(z)$ is a holomorphic function in E, $L(0) = 0$, $|L(z)| < 1$ for $|z| < 1$. Therefore, the function $L(z)$ satisfies the assumptions of the Schwarz lemma.

Since

$$L'(z)=\frac{2\ln\vartheta(0)}{\left(\ln\vartheta(z)+\ln\vartheta(0)\right)^2}\frac{\vartheta'(z)}{\vartheta(z)}$$

and

$$L'(0)=\frac{2\ln\vartheta(0)}{\left(\ln\vartheta(0)+\ln\vartheta(0)\right)^2}\frac{\vartheta'(0)}{\vartheta(0)},$$

we obtain

$$1 \geq \frac{-1}{2\ln \vartheta(0)} \left| \frac{\vartheta'(0)}{\vartheta(0)} \right| = \frac{-1}{2\ln \vartheta(0)} \frac{\frac{\left|4(1-\alpha)c_4+(3-2\alpha)c_3^2\right|}{4(1-\alpha)^2}}{\frac{c_3}{2(1-\alpha)}}$$
$$= \frac{-1}{2\ln \frac{c_3}{2(1-\alpha)}} \frac{\left|4\left(1-\alpha\right)c_4+\left(3-2\alpha\right)c_3^2\right|}{2\left(1-\alpha\right)c_3}$$

and

$$\left|4\left(1-\alpha\right)c_4+\left(3-2\alpha\right)c_3^2\right| \leq 4\left(1-\alpha\right)\left|c_3 \ln \frac{c_3}{2\left(1-\alpha\right)}\right|.$$

Now, we shall show that the inequality (2.7) is sharp. Let

$$f(z) = e^{\int_0^z 2\frac{1-\alpha t e^{\frac{1+t}{1-t}\ln\left(\frac{c_3}{2(1-\alpha)}\right)}}{\left(1-te^{\frac{1+t}{1-t}\ln\left(\frac{c_3}{2(1-\alpha)}\right)}\right)t}dt.} \tag{2.8}$$

Differentiating (2.8) logarithmically, we obtain

$$\ln f(z) = \ln e^{\int_0^z 2\frac{1-\alpha t e^{\frac{1+t}{1-t}\ln\left(\frac{c_3}{2(1-\alpha)}\right)}}{\left(1-te^{\frac{1+t}{1-t}\ln\left(\frac{c_3}{2(1-\alpha)}\right)}\right)t}dt.} = \int_0^z 2\frac{1-\alpha t e^{\frac{1+t}{1-t}\ln\left(\frac{c_3}{2(1-\alpha)}\right)}}{\left(1-te^{\frac{1+t}{1-t}\ln\left(\frac{c_3}{2(1-\alpha)}\right)}\right)t}dt,$$

$$\frac{f'(z)}{f(z)} = 2\frac{1-\alpha z e^{\frac{1+z}{1-z}\ln\left(\frac{c_3}{2(1-\alpha)}\right)}}{\left(1-ze^{\frac{1+z}{1-z}\ln\left(\frac{c_3}{2(1-\alpha)}\right)}\right)z}$$

and

$$g(z) = \frac{zf'(z)}{f(z)} = 2\frac{1-\alpha z e^{\frac{1+z}{1-z}\ln\left(\frac{c_3}{2(1-\alpha)}\right)}}{1-ze^{\frac{1+z}{1-z}\ln\left(\frac{c_3}{2(1-\alpha)}\right)}}.$$

So, we get

$$g(z) = 2 + 2\left(1-\alpha\right)zp(z),$$

where

$$p(z) = \frac{e^{\frac{1+z}{1-z}\ln\left(\frac{c_3}{2(1-\alpha)}\right)}}{1-ze^{\frac{1+z}{1-z}\ln\left(\frac{c_3}{2(1-\alpha)}\right)}}. \tag{2.9}$$

Therefore, we take

$$\frac{g(z)-2}{z} = 2\left(1-\alpha\right)p(z),$$
$$\frac{c_3 z + \left(2c_4 - c_3^2\right)z^2 + \cdots}{z} = 2\left(1-\alpha\right)p(z),$$
$$c_3 + \left(2c_4 - c_3^2\right)z + \cdots = 2\left(1-\alpha\right)p(z)$$

and

$$p'(0) = \frac{2c_4 - c_3^2}{2\left(1-\alpha\right)}.$$

From (2.9), after simple calculations, we get

$$p'(0) = 2\left\{\ln\left(\frac{c_3}{2(1-\alpha)}\right)\right\}\left(\frac{c_3}{2(1-\alpha)}\right) + \left(\frac{c_3}{2(1-\alpha)}\right)^2.$$

Thus, we obtain

$$\left|4(1-\alpha)c_4 + (3-2\alpha)c_3^2\right| = 4(1-\alpha)\left|c_3 \ln\frac{c_3}{2(1-\alpha)}\right|.$$

□

Acknowledgement. The authors would like to thank the referee for his/her constructive comments and suggestions on the earlier version of this paper.

References

[1] T. A. Azeroğlu and B. N. Örnek, *A refined Schwarz inequality on the boundary*, Complex Var. Elliptic Equ. **58** (2013), no. 4, 571–577.

[2] H. P. Boas, *Julius and Julia: mastering the art of the Schwarz lemma*, Amer. Math. Monthly **117** (2010), no. 9, 770–785.

[3] D. M. Burns and S. G. Krantz, *Rigidity of holomorphic mappings and a new Schwarz lemma at the boundary*, J. Amer. Math. Soc. **7** (1994), no. 3, 661–676.

[4] D. Chelst, *A generalized Schwarz lemma at the boundary*, Proc. Amer. Math. Soc. **129** (2001), no. 11, 3275–3278.

[5] V. N. Dubinin, *The Schwarz inequality on the boundary for functions regular in the disc*, J. Math. Sci. (N. Y.) **122** (2004), no. 6, 3623–3629; translated from Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **286** (2002), Anal. Teor. Chisel i Teor. Funkts. 18, 74–84, 228–229.

[6] M. Elin, F. Jacobzon, M. Levenshtein, and D. Shoikhet, *The Schwarz lemma: rigidity and dynamics*, in Harmonic and complex analysis and its applications, 135–230, Trends Math, Birkhäuser/Springer, Cham, 2014.

[7] G. M. Goluzin, *Geometrical Theory of Functions of Complex Variable* (Russian), Second edition. Edited by V. I. Smirnov. With a supplement by N. A. Lebedev, G. V. Kuzmina and Ju. E. Alenicyn, Izdat. "Nauka", Moscow, 1966.

[8] I. S. Jack, *Functions starlike and convex of order α*, J. London Math. Soc. (2) **3** (1971), 469–474.

[9] M. Jeong, *The Schwarz lemma and boundary fixed points*, J. Korean Soc. Math. Educ. Ser. B Pure Appl. Math. **18** (2011), no. 3, 275–284.

[10] ——, *The Schwarz lemma and its application at a boundary point*, J. Korean Soc. Math. Educ. Ser. B Pure Appl. Math. **21** (2014), no. 3, 219–227.

[11] M. Mateljević, *Note on Rigidity of Holomorphic Mappings & Schwarz and Jack Lemma* (in preparation), ResearchGate 2015.

[12] ——, *Schwarz lemma, the Carath ´ eodory and Kobayashi Metrics and Applications in Complex Analysis*, XIX GEOMETRICAL SEMINAR, At Zlatibor. (2016), 1–12.

[13] ——, *Hyperbolic geometry and Schwarz lemma*, ResearchGate 2016.

[14] M. Nunokawa and J. Sokól, *On multivalent functions in the unit disc*, Tsukuba J. Math. **41** (2017), no. 2, 251–263.

[15] R. Osserman, *A sharp Schwarz inequality on the boundary*, Proc. Amer. Math. Soc. **128** (2000), no. 12, 3513–3517.

[16] B. N. Örnek, *Sharpened forms of the Schwarz lemma on the boundary*, Bull. Korean Math. Soc. **50** (2013), no. 6, 2053–2059.

[17] ——, *Inequalities for the non-tangential derivative at the boundary for holomorphic function*, Commun. Korean Math. Soc. **29** (2014), no. 3, 439–449.

[18] ______, *Inequalities for the angular derivatives of certain classes of holomorphic functions in the unit disc*, Bull. Korean Math. Soc. **53** (2016), no. 2, 325–334.
[19] ______, *Estimates for holomorphic functions concerned with Jack's lemma*, Publications de l'Institut Mathématique, In press.
[20] Ch. Pommerenke, *Boundary Behaviour of Conformal Maps*, Grundlehren der Mathematischen Wissenschaften, **299**, Springer-Verlag, Berlin, 1992.

Batuhan Çatal
Department of Mathematics
Amasya University
Merkez-Amasya 05100, Turkey
Email address: batuhancatal0591@gmail.com

Bülent Nafi örnek
Department of Computer Engineering
Amasya University
Merkez-Amasya 05100, Turkey
Email address: nafiornek@gmail.com, nafi.ornek@amasya.edu.tr