Commun. Korean Math. Soc. **0** (0), No. 0, pp. 1–0
https://doi.org/10.4134/CKMS.c180147
pISSN: 1225-1763 / eISSN: 2234-3024

CERTAIN INTEGRATION FORMULAE FOR THE GENERALIZED k-BESSEL FUNCTIONS AND DELEURE HYPER-BESSEL FUNCTION

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Abstract. Integrals involving a finite product of the generalized Bessel functions have recently been studied by Choi *et al.* [2, 3]. Motivated by these results, we establish certain unified integral formulas involving a finite product of the generalized k-Bessel functions. Also, we consider some integral formulas of the (p, q)-extended Bessel functions $J_{\nu,p,q}(z)$ and the Delerue hyper-Bessel function which are proved in terms of (p, q)-extended generalized hypergeometric functions, and the generalized Wright hypergeometric functions, respectively.

1. Introduction and preliminaries

We begin by recalling the generalized Bessel function $w_\nu(z)$ of the first kind (see, *e.g.*, [1, p. 10, Eq. (1.15)])

$$w_\nu(z) = \sum_{k=0}^{\infty} \frac{(-c)^k \left(\frac{z}{2}\right)^{\nu+2k}}{k!\,\Gamma(\nu+k+\frac{b+1}{2})}, \tag{1}$$

where $z \in \mathbb{C} \setminus \{0\}$ and $b, c, \nu \in \mathbb{C}$ with $\Re(\nu) > -1$. Here and in the following, let $\mathbb{C}$, $\mathbb{R}^+$ and $\mathbb{N}$ be the sets of complex numbers, positive real numbers, and positive integers, respectively, and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The special case of the defining series (1) when $b = 1$ and $c = 1$ reduces to the Bessel function $J_\nu(z)$ (see, *e.g.*, [14, p. 100])

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{\nu+2k}}{k!\,\Gamma(\nu+k+1)}, \tag{2}$$

where $z \in \mathbb{C} \setminus \{0\}$ and $\nu \in \mathbb{C}$ with $\Re(\nu) > -1$.

Received April 11, 2018; Revised August 17, 2018; Accepted September 4, 2018.
2010 *Mathematics Subject Classification.* Primary 33B20, 33C20; Secondary 33B15, 33C05.
Key words and phrases. Gamma function, generalized hypergeometric function ${}_pF_q$, generalized (Wright) hypergeometric functions ${}_p\Psi_q$, generalized Lauricella series in several variables, generalized k-Bessel function of the first kind, Oberhettinger's integral formula.

In 2012, Romero et al. [12] introduced the k-Bessel function of the first kind defined by

$$J_{k,\nu}^{\gamma,\lambda}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{n,k}}{\Gamma_k(\lambda n+\nu+1)}\frac{(-1)^n\ \left(\frac{z}{2}\right)^{\nu+2n}}{(n!)^2} \tag{3}$$

($k\in\mathbb{R}$; $\lambda,\gamma,\nu\in\mathbb{C}$; $\Re(\lambda)>0$ and $\Re(\nu)>0$), whose coefficients are given by [6]

$$(\gamma)_{n,k}:=\begin{cases}\dfrac{\Gamma_k(\gamma+nk)}{\Gamma_k(\gamma)} & (k\in\mathbb{R};\gamma\in\mathbb{C}\setminus\{0,-k,-2k,\dots\})\\ \gamma(\gamma+k)\dots(\gamma+(n-1)k) & (n\in\mathbb{N};\gamma\in\mathbb{C}),\end{cases} \tag{4}$$

and

$$\Gamma_k(z)=k^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right). \tag{5}$$

In addition to the above, we consider the following generalization of k-Bessel function of the first kind [10]

$$w_{k,\nu,b,c}^{\gamma,\alpha}(z)=\sum_{n=0}^{\infty}\frac{(-c)^n(\gamma)_{n,k}}{\Gamma_k(\alpha n+\nu+\frac{b+1}{2})}\frac{\left(\frac{z}{2}\right)^{\nu+2n}}{(n!)^2} \tag{6}$$

($k\in\mathbb{R}$; $\alpha,\gamma,\nu,\ b,\ c\in\mathbb{C}$; $\Re(\nu)>0$). It is noted that $w_{1,\nu,b,c}^{1,1}(z)$ is the generalized Bessel function of the first kind (1). Also, $w_{k,\nu,1,1}^{\gamma,\lambda}(z)$ is the k-Bessel function of the first kind (3) .

The generalized Lauricella function is defined by (see, *e.g.*, [13, p. 36, Eq. (19)])

$$\begin{aligned}&F_{C:D^{(1)};\cdots;D^{(n)}}^{A:B^{(1)};\cdots;B^{(n)}}\begin{pmatrix}z_1\\ \vdots\\ z_n\end{pmatrix}=F_{C:D^{(1)};\cdots;D^{(n)}}^{A:B^{(1)};\cdots;B^{(n)}}\left(\begin{matrix}[(a):\theta^{(1)},\dots,\theta^{(n)}]:\\ [(c):\psi^{(1)},\dots,\psi^{(n)}]:\end{matrix}\right.\\ &\qquad\qquad\left.\begin{matrix}[(b)^{(1)}:\phi^{(1)}];\dots;[(b)^{(n)}:\phi^{(n)}];\\ [(d)^{(1)}:\delta^{(1)}];\dots;[(d)^{(n)}:\delta^{(n)}];\end{matrix}\ z_1,\dots,z_n\right)\\ &\qquad=\sum_{k_1,\dots,k_n=0}^{\infty}\Omega(k_1,\dots,k_n)\,\frac{z_1^{k_1}}{k_1!}\cdots\frac{z_n^{k_n}}{k_n!},\end{aligned} \tag{7}$$

where, for convenience,

$$\Omega(k_1,\dots,k_n)=\frac{\prod_{j=1}^{A}(a_j)_{k_1\theta_j^{(1)}+\cdots+k_n\theta_j^{(n)}}\prod_{j=1}^{B^{(1)}}(b_j^{(1)})_{k_1\phi_j^{(1)}}\cdots\prod_{j=1}^{B^{(n)}}(b_j^{(n)})_{k_n\phi_j^{(n)}}}{\prod_{j=1}^{C}(c_j)_{k_1\psi_j^{(1)}+\cdots+k_n\psi_j^{(n)}}\prod_{j=1}^{D^{(1)}}(d_j^{(1)})_{k_1\delta_j^{(1)}}\cdots\prod_{j=1}^{D^{(n)}}(d_j^{(n)})_{k_n\delta_j^{(n)}}}, \tag{8}$$

the coefficients

$$
(9)\qquad \begin{cases} \theta_j^{(m)}\ (j=1,\ldots,A);\ \phi_j^{(m)}\ (j=1,\ldots,B^{(m)}); \\ \psi_j^{(m)}\ (j=1,\ldots,C);\ \delta_j^{(m)}\ (j=1,\cdots,D^{(m)});\ \forall\, m\in\{1,\ldots,n\} \end{cases}
$$

are real and positive, and (a) abbreviates the array of A parameters $a_1,\ldots,a_A$, $(b^{(m)})$ abbreviates the array of $B^{(m)}$ parameters

$$
b_j^{(m)}\quad (j=1,\ldots,B^{(m)});\quad \forall\, m\in\{1,\ldots,n\},
$$

with similar interpretations for (c) and $(d^{(m)})$ $(m=1,\ldots,n)$; *et cetera.*

The multiple series (7) converges absolutely either

(i) $\Delta_i>0\ \ (i=1,\ldots,n),\ \ \forall\ z_1,\ldots,z_n\in\mathbb{C},$

or

(ii) $\Delta_i=0\ \ (i=1,\ldots,n),\ \ \forall\ z_1,\ldots,z_n\in\mathbb{C},\ \ |z_i|<\varrho_i\ \ (i=1,\ldots,n).$

The multiple series (7) is divergent when $\Delta_i<0\ \ (i=1,\ldots,n)$ except for the trivial case $z_1=0,\ldots,z_n=0$. Here

$$
(10)\qquad \Delta_i\equiv 1+\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{A}\theta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\phi_j^{(i)}\ \ (i=1,\ldots,n)
$$

and

$$
(11)\qquad \varrho_i=\min_{\mu_1,\ldots,\,\mu_n>0}\{E_i\}\quad (i=1,\ldots,n),
$$

with

(12)

$$
E_i=(\mu_i)^{1+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\phi_j^{(i)}}\cdot\frac{\left\{\prod_{j=1}^{C}\left(\sum_{i=1}^{n}\mu_i\psi_j^{(i)}\right)^{\psi_j^{(i)}}\right\}\left\{\prod_{j=1}^{D^{(i)}}\left(\delta_j^{(i)}\right)^{\delta_j^{(i)}}\right\}}{\left\{\prod_{j=1}^{A}\left(\sum_{i=1}^{n}\mu_i\theta_j^{(i)}\right)^{\theta_j^{(i)}}\right\}\left\{\prod_{j=1}^{B^{(i)}}\left(\phi_j^{(i)}\right)^{\phi_j^{(i)}}\right\}}.
$$

We recall the generalized Wright hypergeometric function ${}_p\Psi_q$ (see, *e.g.*, [13, p. 21, Eqs. (38)-(40)])

$$
(13)\qquad {}_p\Psi_q\left[\begin{matrix}(\alpha_1,A_1),\ldots,(\alpha_p,A_p);\\ (\beta_1,B_1),\ldots,(\beta_q,B_q);\end{matrix}\,z\right]=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(\alpha_j+A_jn)}{\prod_{j=1}^{q}\Gamma(\beta_j+B_jn)}\frac{z^n}{n!},
$$

where the coefficients $A_1,\ldots,A_p$ and $B_1,\ldots,B_q$ are positive real numbers such that

$$
(14)\qquad 1+\sum_{j=0}^{q}B_j-\sum_{j=0}^{p}A_j\geq 0.
$$

In special case $A_j = B_j = 1$ for all j in (13), the generalized (Wright) hypergeometric function ${}_p\Psi_q$ (see, *e.g.*, [4, p. 29, Eq. (1.7)]) reduces to the generalized hypergeometric series ${}_pF_q$ as follows:

$$(15)\qquad {}_p\Psi_q\left[\begin{matrix}(\alpha_1,1),\ldots,(\alpha_p,1);\\(\beta_1,1),\ldots,(\beta_q,1);\end{matrix}\,z\right] = \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(\alpha_j)}{\prod_{j=1}^{q}\Gamma(\beta_j)}\,{}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p;\\\beta_1,\ldots,\beta_q;\end{matrix}\,z\right],$$

where ${}_pF_q$ is the generalized hypergeometric series (see, *e.g.*, [13, p. 19, Eq. (23)])

$$(16)\qquad \begin{aligned}{}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p;\\\beta_1,\ldots,\beta_q;\end{matrix}\,z\right] &= {}_pF_q\left[\alpha_1,\ldots\alpha_p;\beta_1,\ldots,\beta_q;z\right]\\ &= \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!},\end{aligned}$$

where $(\lambda)_\nu$ denotes the Pochhammer symbol defined (for λ, $\nu \in \mathbb{C}$), in terms of the familiar Gamma function Γ, by

$$(17)\quad (\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases}1 & (\nu = 0;\ \lambda \in \mathbb{C}\setminus\{0\}),\\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}).\end{cases}$$

For present investigation, we also need to recall Oberhettinger's integral formula [11]:

$$(18)\quad \int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda}dx = 2\lambda\, a^{-\lambda}\left(\frac{a}{2}\right)^{\mu}\frac{\Gamma(2\mu)\,\Gamma(\lambda-\mu)}{\Gamma(1+\lambda+\mu)},$$

provided $0 < \Re(\mu) < \Re(\lambda)$.

Motivated by the works of Choi et al. [2,3] and Purohit et al. [9], here, we aim at presenting certain integral formulas associated with various Bessel functions, which are expressed in terms of the generalized Lauricella functions, the generalized Wright hypergeometric functions and the (p, q)-extended generalized hypergeometric functions, respectively.

2. Integral formulas associated with generalized *k*-Bessel function of the first kind

We establish two (presumably) new integral formulae involving a finite product of the generalized k-Bessel function of the first kind and the integrand in the integral formula (18), which are expressed in terms of the generalized Lauricella functions asserted by the following theorems.

Theorem 1. *The following integral formula holds true*: *For* $k \in \mathbb{R}^+$, λ, μ, α, b, c_j, γ_j, $\nu_j \in \mathbb{C}$ *with* $\Re(\nu_j) > -1$, $0 < \Re(\mu) < \Re(\lambda+\nu_j)$ $(j = 1, \ldots, n)$, $\alpha = k$

and $x > 0$,

(19)
$$\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda}\prod_{j=1}^{n} w_{k,\nu_j,b,c_j}^{\gamma_j,\alpha}\left(\frac{y_j}{x+a+\sqrt{x^2+2ax}}\right)dx$$
$$= 2^{1-\mu}\,a^{\mu-\lambda}\left(\prod_{j=1}^{n}\frac{\left(\frac{y_j}{2ak^{1/k}}\right)^{\nu_j}}{\Gamma(\frac{\nu_j}{k}+\frac{b+1}{2k})}\right)\frac{\Gamma(2\mu)\Gamma(\lambda-\mu+\sum_{j=1}^{n}\nu_j)}{k^{\frac{b+1}{2k}-1}\Gamma(1+\lambda+\mu+\sum_{j=1}^{n}\nu_j)}$$
$$\times F^{2:1;\dots;1}_{2:2;\dots;2}\left[\begin{matrix}\left[1+\lambda+\sum_{j=1}^{n}\nu_j:2,\dots,2\right],\left[\lambda-\mu+\sum_{j=1}^{n}\nu_j:2,\dots,2\right]:\\ \left[1+\lambda+\mu+\sum_{j=1}^{n}\nu_j:2,\dots,2\right],\left[\lambda+\sum_{j=1}^{n}\nu_j:2,\dots,2\right]:\end{matrix}\right.$$
$$\left.\begin{matrix}[\frac{\gamma_1}{k}:1];\dots;[\frac{\gamma_n}{k}:1];\\ [1:1],[\frac{\nu_1}{k}+\frac{b+1}{2k}:1];\dots;[1:1],[\frac{\nu_n}{k}+\frac{b+1}{2k}:1];\end{matrix}\; -\frac{c_1y_1^2}{4a^2},\dots,-\frac{c_ny_n^2}{4a^2}\right].$$

Proof. Let $\mathcal{L}$ denote the left hand side of (19). By making use of a finite product of (6) in the integrand of (19) and then interchanging the order of integral sign and summation, which is verified by uniform convergence of the involved series under the given conditions, we get

$$\begin{aligned}\mathcal{L} = &\sum_{n_1,\dots,n_n=0}^{\infty}\prod_{j=1}^{n}\frac{(-c_j)^{n_j}(\gamma_j)_{n_j,k}}{\Gamma_k(\alpha n_j+\nu_j+\frac{b+1}{2})}\frac{(z/2)^{\nu_j+2n_j}}{(n_j!)^2}\\ &\cdot\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda-\nu_1-\cdots-\nu_n-2n_1-\cdots-2n_n}dx.\end{aligned} \tag{20}$$

In view of the conditions given in Theorem 1, since

$$\Re(\nu_j) > -1,\ 0 < \Re(\mu) < \Re(\lambda+\nu_j) \le \Re(\lambda+\nu_j+2n_j)$$
$$(n\in\mathbb{N}_0 \quad \text{and} \quad j=1,\dots,n),$$

we can apply k-Gamma function (5) and the integral formula (18) to the integral in (20) and obtain the following expression:

(21)
$$\mathcal{L} = \sum_{n_1,\dots,n_n=0}^{\infty}\prod_{j=1}^{n}\left(\frac{y_j}{2}\right)^{\nu_j}\frac{(\frac{\gamma_j}{k})_{n_j}}{k^{\frac{\nu_j}{k}+\frac{b+1}{2}-1}\Gamma(\frac{\nu_j}{k}+\frac{b+1}{2k})(\frac{\nu_j}{k}+\frac{b+1}{2k})_{n_j}(n_j!)^2}\left(-\frac{c_jy_j^2}{4}\right)^{n_j}$$
$$\times\frac{\Gamma(2\mu)\Gamma(\lambda-\mu+\nu_1+\cdots+\nu_n+2n_1+\cdots+2n_n)}{\Gamma(1+\lambda+\mu+\nu_1+\cdots+\nu_n+2n_1+\cdots+2n_n)}$$
$$\times\ 2^{1-\mu}(\lambda+\nu_1+\cdots+\nu_n+2n_1+\cdots+2n_n)a^{\mu-(\lambda+\nu_1+\cdots+\nu_n+2n_1+\cdots+2n_n)}.$$

Therefore we find
(22)

$$\mathcal{L} = 2^{1-\mu}\, a^{\mu-\lambda} \prod_{j=1}^{n} \left(\frac{y_j}{2ak^{\frac{1}{k}}} \right)^{\nu_j} \frac{\Gamma(2\mu)\Gamma(\lambda-\mu+\sum_{j=1}^{n}\nu_j)}{k^{\frac{b+1}{2k}-1}\Gamma(\frac{\nu_j}{k}+\frac{b+1}{2k})\Gamma(1+\lambda+\mu+\sum_{j=1}^{n}\nu_j)}$$

$$\cdot \sum_{n_1,\ldots,n_n=0}^{\infty} \frac{(\lambda-\mu+\sum_{j=1}^{n}\nu_j)_{2n_1+\cdots+2n_n}(1+\lambda+\sum_{j=1}^{n}\nu_j)_{2n_1+\cdots+2n_n}}{(\lambda+\sum_{j=1}^{n}\nu_j)_{2n_1+\cdots+2n_n}(1+\lambda+\mu+\sum_{j=1}^{n}\nu_j)_{2n_1+\cdots+2n_n}}$$

$$\cdot \frac{(\frac{\gamma_1}{k})_{n_1}\cdots(\frac{\gamma_n}{k})_{n_n}}{n_1!\cdots n_n!(\frac{\nu_1}{k}+\frac{b+1}{2k})_{n_1}\cdots(\frac{\nu_n}{k}+\frac{b+1}{2k})_{n_n}} \frac{(-c_1y_1^2/4a^2)^{n_1}}{n_1!}\cdots\frac{(-c_ny_n^2/4a^2)^{n_n}}{n_n!}.$$

Finally, we interpret the multiple series in (22) as a special case of the general hypergeometric series in several variables defined by (7). This completes the proof of (19). □

Theorem 2. *The following integral formula holds true: For* $k \in \mathbb{R}^+$, λ, μ, α, b, c_j, γ_j, $\nu_j \in \mathbb{C}$ *with* $\Re(\nu_j) > -1$, $0 < \Re(\mu) < \Re(\lambda+\nu_j)$ $(j = 1, \ldots, n)$, $\alpha = k$ *and* $x > 0$,
(23)

$$\int_0^{\infty} x^{\mu-1} \left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} \prod_{j=1}^{n} w_{k,\nu_j,b,c_j}^{\gamma_j,\alpha}\left(\frac{xy_j}{x+a+\sqrt{x^2+2ax}}\right) dx$$

$$= 2^{1-\mu}\, a^{\mu-\lambda} \left(\prod_{j=1}^{n} \frac{\left(\frac{y_j}{4k^{1/k}}\right)^{\nu_j}}{\Gamma(\frac{\nu_j}{k}+\frac{b+1}{2k})} \right) \frac{\Gamma(\lambda-\mu)\Gamma(2\mu+2\sum_{j=1}^{n}\nu_j)}{k^{\frac{b+1}{2k}-1}\Gamma(1+\lambda+2\sum_{j=1}^{n}\nu_j)}$$

$$\times F^{2:1;\ldots;1}_{2:2;\ldots;2}\left[\begin{matrix} \left[1+\lambda+\sum_{j=1}^{n}\nu_j : 2,\ldots,2\right],\ \left[2\mu+2\sum_{j=1}^{n}\nu_j : 4,\ldots,4\right] : \\ \left[\lambda+2\sum_{j=1}^{n}\nu_j : 2,\ldots,2\right],\ \left[1+\lambda+2\sum_{j=1}^{n}\nu_j : 4,\ldots,4\right] : \end{matrix}\right.$$

$$\left.\begin{matrix} [\frac{\gamma_1}{k} : 1];\ \ldots;\ [\frac{\gamma_n}{k} : 1]; \\ [1:1],[\frac{\nu_1}{k}+\frac{b+1}{2k} : 1];\ \ldots;\ [1:1],[\frac{\nu_n}{k}+\frac{b+1}{2k} : 1]; \end{matrix}\ -\frac{c_1y_1^2}{16},\ldots,-\frac{c_ny_n^2}{16}\right].$$

Proof. We omit the details. A similar argument as in the proof of Theorem 1 will establish the integral formula (23). □

Remark. It is easily seen that if we set $c_i = \gamma_i = b = k = 1$ in (19) and (23) for all i we can arrive at the Equations (2.1) and (2.2) in Choi and Agarwal [2].

3. Integral formulas associated with the (p, q)-extended Bessel function

Considering the transformation formula $J_\nu(z)$, which occurs the question about transforming the (p, q)-extended Bessel function into the proper (p, q)-extended generalized hypergeometric function, which defined by [8, Eq. (2.8)]

$$\begin{aligned} &{}_{r+k}F_{s+k}\left[\begin{matrix} a_1, \ldots, a_k, \alpha_1, \ldots, \alpha_k; \\ c_1, \ldots, c_s \gamma_1, \ldots, \gamma_k; \end{matrix} z,\, p,\, q\right] \\ &= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{r}(a_j)_n}{\prod_{j=1}^{s}(c_j)_n} \prod_{j=1}^{k} \frac{B(\alpha_j + n, \gamma_j - \alpha_j; p, q)}{B(\alpha_j, \gamma_j - \alpha_j)} \frac{z^n}{n!}, \\ &(\Re(p) \geq 0, \Re(q) \geq 0,\ \ 0 < \Re(\alpha_j) < \Re(\gamma_j),\ |z| < 1, \\ &k, r, s \in \mathbb{N}_0 \quad \text{and} \quad j = 1, \ldots, k). \end{aligned} \tag{24}$$

Choi et al. [5] have introduced and investigated the following extended beta function

$$B\,(x,\, y,\, ;\, p,\, q) = \int_0^1 t^{x-1}\,(1-t)^{y-1} e^{-\frac{p}{t} - \frac{q}{1-t}}\, dt \tag{25}$$

$$(\min\{\Re(p), \Re(q)\} > 0, \min\{\Re(x), \Re(y)\} > 0)\,.$$

In terms of the extended beta function $B\,(x,\, y,\, ;\, p,\, q)$ defined by (25), we introduce the $(p,\, q)$-extended Bessel function $J_{\nu,p,q}(z)$ [8, Eq. (2.9)] in the form

$$J_{\nu,p,q}(z) = \frac{\left(\frac{z}{2}\right)^n}{\Gamma\,(\nu+1)} \sum_{n=0}^{\infty} \frac{B(n+\frac{1}{2}, \nu+\frac{1}{2}; p, q)}{(\frac{1}{2})_n\, B(\frac{1}{2}, \nu+\frac{1}{2})} \left(-\frac{z^2}{4}\right)^n, \tag{26}$$

when $\min\{\Re(p), \Re(q)\} > 0$, and for $\Re(\nu) > -\frac{1}{2}$ if $p = q = 0$. We obtain the following interesting integral formulas involving the $(p,\, q)$-extended Bessel functions, which expressed in terms of the $(p,\, q)$-extended generalized hypergeometric functions.

Theorem 3. *The following integral formula holds true: For $x > 0$, $0 < \Re(\mu) < \Re(\lambda + \nu)$, $\min\{\Re(p), \Re(q)\} \geq 0$ and $\Re(\nu) > -\frac{1}{2}$ if $p = q = 0$,*

$$\begin{aligned} &\int_0^{\infty} x^{\mu-1} \left(x + a + \sqrt{x^2 + 2ax}\right)^{-\lambda} J_{\nu,\,p,\,q}\left(\frac{y}{x + a + \sqrt{x^2 + 2ax}}\right) dx \\ &= 2^{1-\nu-\mu}\, a^{\mu-\lambda-\nu}\, y^{\nu}\, \frac{\Gamma(2\mu)\Gamma(\lambda+\nu+1)\Gamma(\lambda+\nu-\mu)}{\Gamma(1+\nu)\Gamma(\lambda+\nu)\Gamma(\lambda+\nu+\mu+1)} \\ &\times {}_3F_4\left[\begin{matrix} \frac{\lambda+\nu}{2}+1, \frac{\lambda+\nu-\mu}{2}, \frac{\lambda+\nu-\mu+1}{2}; \\ \frac{\lambda+\nu}{2}, \frac{\lambda+\nu+\mu+1}{2}, \frac{\lambda+\nu+\mu}{2}+1,\, \nu+1; \end{matrix} -\frac{y^2}{4a^2}; p,\, q\right]. \end{aligned} \tag{27}$$

Proof. By making use of the $(p,\, q)$-extended Bessel functions (26) in the integrand of (27) and then interchanging the order of integral sign and summation, which is verified by uniform convergence of the involved series under the given conditions, we get

$$\tag{28} \int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} J_{\nu,\,p,\,q}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right) dx$$
$$= \frac{y^\nu 2^{-\nu}}{\Gamma(\nu+1)} \sum_{n=0}^\infty \frac{B(n+\frac{1}{2},\nu+\frac{1}{2};p,q)}{(\frac{1}{2})_n\, B(\frac{1}{2},\nu+\frac{1}{2})}\left(-\frac{y^2}{4}\right)^n$$
$$\times \int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-(\lambda+\nu+2n)} dx,$$

which, using integral formula (18), $(\alpha)_{2n} = 2^{2n}\left(\frac{\alpha}{2}\right)_n \left(\frac{\alpha+1}{2}\right)_n$, and on using (24), we obtain the desired result. □

Theorem 4. *The following integral formula holds true: For* $x > 0$, $0 < \Re(\mu) < \Re(\lambda+\nu)$, $\min\{\Re(p),\Re(q)\} \geq 0$ *and* $\Re(\nu) > -\frac{1}{2}$ *if* $p = q = 0$,

$$\tag{29} \int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} J_{\nu,\,p,\,q}\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right) dx$$
$$= 2^{2\lambda-\mu}\, a^{\mu-\lambda}\, y^\nu \frac{\Gamma(\lambda+\nu+\frac{1}{2})\Gamma(\lambda+\nu+1)\Gamma(\lambda-\mu)}{\Gamma(\frac{1}{2})\Gamma(1+\nu)\Gamma(\lambda+2\nu+\mu+1)}$$
$$\times\, {}_4F_5\left[\begin{array}{c} \frac{\lambda+\nu+1}{2}, \frac{\lambda+\nu}{2}+1, \frac{\lambda+\nu}{2}+\frac{1}{4}, \frac{\lambda+\nu}{2}+\frac{3}{4}; \\ \frac{\lambda+\mu+2\nu+1}{4}, \frac{\lambda+\mu+2\nu+2}{4}, \frac{\lambda+\mu+2\nu+3}{4}, \frac{\lambda+\mu+2\nu+4}{4}, \\ \nu+1; -\frac{y^2}{64}; p, q \end{array}\right].$$

Proof. A similar argument as in the proof of Theorem 3 is seen to establish the integral formula (29). □

4. Integral formulas associated with the Deleure hyper-Bessel function

It is worth to mention the Delerue hyper-Bessel function [6,7]

$$\tag{30} J^{(m)}_{\nu_1,\ldots,\nu_m}(z) = \left(\frac{z}{m+1}\right)^{\sum_{j=1}^m \nu_j} \sum_{n=0}^\infty \frac{(-1)^n \left(\frac{z}{m+1}\right)^{n(m+1)}}{n!\,\prod_{j=1}^m \Gamma(n+\nu_j+1)},$$

which is multi-index analogue of the Bessel function J_ν. Here $z, \nu_j \in \mathbb{C}$ and $\Re(\nu_j) > -1$ for $j = 1, \ldots, n$.

For $m = 1$ we have arrive at the classical Bessel function. We establish two generalized integral formulas in Theorem 5 and Theorem 6 below, which are expressed in terms of the generalized Wright hypergeometric function (15), by inserting the Deleure hyper-Bessel function (30) with suitable arguments into the integrand of the integral (18).

Theorem 5. *The following integral formula holds true: For $\nu_j \in \mathbb{C}$, $\Re(\nu_j) > -1 (j = 1, \ldots, n)$ and $0 < \Re(\mu) < \Re(\lambda + \nu)$,*

(31)
$$\int_0^\infty x^{\mu-1} \left(x + a + \sqrt{x^2 + 2ax}\right)^{-\lambda} J^{(m)}_{\nu_1,\ldots,\nu_m} \left(\frac{y}{x + a + \sqrt{x^2 + 2ax}}\right) dx$$
$$= 2^{1-\nu-\mu} a^{\mu-\lambda-\nu} \left(\frac{y}{m+1}\right)^\nu \frac{\Gamma(2\mu)\Gamma(\lambda+\nu+1)\Gamma(\lambda+\nu-\mu)}{\prod_{j=1}^m \Gamma(\nu_j+1)\Gamma(\lambda+\nu)\Gamma(\lambda+\nu+\mu+1)}$$
$$\times {}_2\Psi_{m+2} \left[\begin{matrix} (\lambda+\nu+1, m+1),\ (\lambda-\mu+\nu, m+1); \\ \prod_{j=1}^m (\nu_j+1, 1), (\lambda+\nu, m+1),\ (\lambda+\nu+\mu+1, m+1); \end{matrix} \left(-\frac{y}{a(m+1)}\right)^{m+1} \right],$$

where $\nu = \sum_{j=1}^m \nu_j$.

Theorem 6. *The following integral formula holds true: For $\nu_j \in \mathbb{C}$, $\Re(\nu_j) > -1 (j = 1, \ldots, n)$ and $0 < \Re(\mu) < \Re(\lambda + \nu)$,*

(32)
$$\int_0^\infty x^{\mu-1} \left(x + a + \sqrt{x^2 + 2ax}\right)^{-\lambda} J^{(m)}_{\nu_1,\ldots,\nu_m} \left(\frac{xy}{x + a + \sqrt{x^2 + 2ax}}\right) dx$$
$$= 2^{1-\nu-\mu} a^{\mu-\lambda-\nu} \left(\frac{y}{m+1}\right)^\nu \frac{\Gamma(2\mu)\Gamma(\lambda+\nu+1)\Gamma(\lambda+\nu-\mu)}{\prod_{j=1}^m \Gamma(\nu_j+1)\Gamma(\lambda+\nu)\Gamma(\lambda+\nu+\mu+1)}$$
$$\times {}_3\Psi_{m+3} \left[\begin{matrix} (\lambda+\nu+1, m+1), (\mu, m+1), (\mu+\frac{1}{2}, m+1); \\ \prod_{j=1}^m (\nu_j+1, 1), (\frac{\lambda+\mu+\nu+1}{2}, m+1), (\frac{\lambda+\mu+\nu}{2}, m+1), \\ (\lambda+\nu, m+1); \end{matrix} \left(-\frac{y}{2a(m+1)}\right)^{m+1} \right],$$

where $\nu = \sum_{j=1}^m \nu_j$.

Proof. Using the Pochhammer symbol:

$$(\lambda)_{mn} = m^{mn} \prod_{j=1}^m \left(\frac{\lambda+j-1}{m}\right)_n,$$

we get the integral formula (32). □

Acknowledgement. This paper was supported by Wonkwang University in 2019.

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