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CONFORMAL SEMI-SLANT SUBMERSIONS FROM LORENTZIAN PARA SASAKIAN MANIFOLDS

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Abstract. In this paper, we introduce conformal semi-slant submersions from Lorentzian para Sasakian manifolds onto Riemannian manifolds. We investigate integrability of distributions and the geometry of leaves of such submersions from Lorentzian para Sasakian manifolds onto Riemannian manifolds. Moreover, we examine necessary and sufficient conditions for such submersions to be totally geodesic where characteristic vector field ξ is vertical.

1. Introduction

Firstly, the notion of Riemannian submersion between Riemannian manifolds was initiated by O' Neill [22] and Grey [15]. Later, this notion was widely studied in differential geometry. In particular, Riemannian submersions are fundamentally important in several areas of Riemannian geometry.

For two Riemannian or semi-Riemannian manifolds (M_1, g_1) and (M_2, g_2), let f be a submersion from (M_1, g_1) onto (M_2, g_2). Then according to the conditions on the map $f : (M_1, g_1) \to (M_2, g_2)$, we have following submersions:

Semi-Riemannian submersions and Lorentzian submersion [9], locally conformal Kahler submersions [23], almost Hermitian submersion ([3, 25]), almost contact submersion [18], Semi-slant submanifolds of a Sasakian manifold [8], semi-slant submersion [24], para contact submersion [12], semi-slant submersions from almost product Riemannian manifolds ([4, 13]) para contact-para complex submersion [21], anti-invariant Riemnnian submersion from cosymplectic manifolds ([2, 14]). The concept of conformal semi-slant submersions was studied by Akyol [1]. In particular, the Riemannian submersions have several important applications both in mathematics and in physics because of their application in supergravity and superstring theories ([17, 26]), Kaluza-Klein theory ([6, 16]), Yang-Mills theory [7] etc. On the other hand, the study

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of Lorentzian para contact manifolds was initiated by K. Matsumoto [19] and Lorentzian para Sasakian manifold was studied by I. Mihai and R. Rosca [20].

In the present paper, we study conformal semi-slant submersion from Lorentzian para Sasakian manifolds onto Riemannian manifolds. The paper is organized as follows: In the second section, we gather main notions and formulae for other sections. In the third section, we give the definition of slant submersions and some results. We also study the integrability of distributions and the geometry of leaves of vertical distribution. In the fourth section we discuss some examples on it. Finally, we obtain certain conditions for such submersions to be totally geodesic.

2. Preliminaries

In this section, we recall main definitions and properties of Lorentzian para Sasakian manifolds and Lorentzian submersions.

An m-dimensional differentiable manifold M admitting a (1,1) tensor field ϕ, a contravariant vector field ξ, a 1-form η is called a Lorentzian para Sasakian manifold with Lorentzian metric g if they satisfy:

$$\phi^2 = I + \eta \otimes \xi, \ \phi \circ \xi = 0, \ \eta \circ \xi = 0, \tag{2.1}$$

$$\eta(\xi) = -1, \ g(X, \xi) = \eta(X), \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \ g(\phi X, Y) = g(X, \phi Y), \tag{2.3}$$

$$\nabla_X \xi = \phi X, \tag{2.4}$$

$$(\nabla_X \phi)Y = \eta(Y)X + g(X, Y)\xi + 2\eta(X)\eta(Y)\xi, \tag{2.5}$$

where ∇ represents the operator of covariant differentiation with respect to the Lorentzian metric g.

In a Lorentzian para Sasakian manifold, it is clear that

$$\text{rank}\phi = m - 1. \tag{2.6}$$

Now, if we put

$$\Phi(X, Y) = \Phi(Y, X) = g(X, \phi Y) = g(\phi X, Y), \tag{2.7}$$

then the tensor field Φ is symmetric (0,2) tensor field, for any vector fields X and Y.

Example 1. Let $R^{2m+1} = \{(x^1, x^2, \ldots, x^m, y^1, y^2, \ldots, y^m, z : x^i, y^i, z \in R, i = 1, 2, \ldots, m)\}$. Consider R^{2m+1} with the following structure:

$$\phi(\sum_{i=1}^{m}(X_i \frac{\partial}{\partial x_i} + Y_i \frac{\partial}{\partial y_i}) + Z\frac{\partial}{\partial z}) = \sum_{i=1}^{m} Y_i \frac{\partial}{\partial x_i} + \sum_{i=1}^{m} X_i \frac{\partial}{\partial y_i} + \sum_{i=1}^{m} Y_i y^i \frac{\partial}{\partial z},$$

$$g = -\eta \otimes \eta + \frac{1}{4}\sum_{i=1}^{m}(dx^i \otimes dx^i + dy^i \otimes dy^i),$$

$$\eta = -\frac{1}{2}(dz - \sum_{i=1}^{m} y^i dx^i),\ \xi = 2\frac{\partial}{\partial z}.$$

Then, $(R^{2m+1}, \phi, \xi, \eta, g)$ is a Lorentzian para-Sasakian manifold. The vector fields $E_i = 2\frac{\partial}{\partial y^i}$, $E_{m+i} = 2(\frac{\partial}{\partial x^i} + y^i \frac{\partial}{\partial z})$ and ξ form a ϕ-basis for the contact metric structure.

Lemma 1. *Let $(M, \phi, \xi, \eta, g_M)$ be an m-dimensional Lorentzian para Sasakian manifold and (N, g_N) be an n-dimensional Riemannian manifold. Let $f : M \to N$ be a differentiable map and $p \in M$. Then f is called horizontally weakly conformal or semi-conformal at p if either $df_p = 0$, or df_p maps the horizontal space $\mathcal{H} = ((\ker f_*)_p)^\perp$ conformally onto $T_{f(p)}$.*

The second condition in the above definition exactly is the same as df_p is symmetric and there exists a number $\chi(p) \neq 0$ such that

$$g_N(f_*X, f_*Y) = \chi(p) g_M(X, Y) \quad \text{for } X, Y \in ((\ker f_*)_p)^\perp. \tag{2.8}$$

Here $\chi(p)$ is called the square dilation of f at p and its square root $\lambda(p) = \sqrt{\chi(p)}$ is called the dilation of f at p. The map f is called horizontally weakly conformal or semi-conformal on M if it is horizontally weakly conformal at every point on M. If f has no critical point, then it is said to be a (horizontally) conformal Lorentzian submersion [4].

We should mention that a horizontally conformally Lorentzian submersion $f : M \to N$ is called horizontally homothetic if the gradient of its dilation λ is vertical, i.e.,

$$\mathcal{H}(grad\lambda) = 0, \tag{2.9}$$

at $p \in M$, where $\mathcal{H}$ is the complement orthogonal distribution to $\mathcal{V} = \ker f_*$ in $\Gamma(T_pM)$.

Again, we recall the following definition from [22].

Let $f : M \to N$ be a conformal Lorentzian submersion. A vector field E on M is called projectiable if there exists a vector field $\widehat{E}$ on N such that $f_*(E_p) = \widehat{E}_{f(p)}$ for any $p \in M$. In this case E and $\widehat{E}$ are called f-related. A horizontal vector field Y on M is called basic, if it is projectiable. It is a well known fact that if $\widehat{Z}$ is a vector field on N, then there exists a unique basic vector field Z which is called the horizontal lift of $\widehat{Z}$.

The fundamental tensors $\mathcal{T}$ and $\mathcal{A}$ defined by O'Nill's [22], for vector field E and F on M such that

$$\mathcal{A}_E F = \mathcal{H}\nabla^M_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla^M_{\mathcal{H}E}\mathcal{H}F, \tag{2.10}$$

$$\mathcal{T}_E F = \mathcal{H}\nabla^M_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla^M_{\mathcal{V}E}\mathcal{H}F, \tag{2.11}$$

where $\mathcal{V}$ and $\mathcal{H}$ are the vertical and horizontal projections. On the other hand, from equations (2.10) and (2.11), we have

$$\nabla_U V = \mathcal{T}_U V + \widehat{\nabla}_U V, \tag{2.12}$$

$$\nabla_U X = \mathcal{H}\nabla_U X + \mathcal{T}_U X, \tag{2.13}$$

$$\nabla_X U = \mathcal{A}_X U + \mathcal{V}\nabla_X U, \tag{2.14}$$

$$\nabla_X Y = \mathcal{H}\nabla_X Y + \mathcal{A}_X Y, \tag{2.15}$$

for all $U, V \in \Gamma(\ker \pi_*)$ and $X, Y \in \Gamma(\ker \pi_*)^\perp$, where $\mathcal{V}\nabla_U V = \widehat{\nabla}_U V$. If X is basic, then $\mathcal{A}_X V = \mathcal{H}\nabla_X V$.

It is easily seen that for $p \in M$, $V \in \mathcal{V}_p$ and $X \in \mathcal{H}_p$ the linear operators

$$\mathcal{T}_V, \mathcal{A}_X : T_pM \to T_pM$$

are skew-symmetric, that is

$$g(\mathcal{A}_X E, F) = -g(E, \mathcal{A}_X F) \text{ and } g(\mathcal{T}_V E, F) = -g(E, \mathcal{T}_V F) \tag{2.16}$$

for all $E, F \in T_pM$. We also see that the restriction of $\mathcal{T}$ to the vertical distribution $\mathcal{T}$ is the second fundamental form of the fibres of f. Since $\mathcal{T}_V$ is skew-symmetric, we get f has totally geodesic fibres if and only if $\mathcal{T} = 0$.

Let $(M, \phi, \xi, \eta, g_M)$ be a Lorentzian para Sasakian manifold and (N, g_N) be a Riemannian manifold. Let $f : M \to N$ be a smooth map. Then the second fundamental form of f is given by

$$(\nabla f_*)(X, Y) = \nabla^f_X f_* Y - f_*(\nabla_X Y) \text{ for } X, Y \in \Gamma(T_pM), \tag{2.17}$$

where we denote conveniently by ∇ the Levi-Civita connections of the metrics g_M and g_N and ∇^f is the pullback connection [5]. We also know that, f is said to be totally geodesic map if $(\nabla f_*)(X, Y) = 0$ for $X, Y \in \Gamma(TM)$.

Lemma 2. *Let $f : M \to N$ be a horizontal conformal submersion. Then, for any horizontal vector fields X, Y and vertical vector fields U, V, we have*

(i) $(\nabla f_*)(X, Y) = X(\ln \lambda) f_* Y + Y(\ln \lambda) f_* X - g_M(X, Y) f_*(grad \ln \lambda)$,
(ii) $(\nabla f_*)(U, V) = -f_*(\mathcal{T}_U V)$,
(iii) $(\nabla f_*)(X, U) = -f_*(\nabla^M_X U) = -f_*(\mathcal{A}_X U)$.

3. Conformal semi-slant submersions

In this section, we define and study conformal semi-slant submersion from Lorentzian para Sasakian manifolds.

Definition 1. Let $(M, \phi, \xi, \eta, g_M)$ be a Lorentzian para Sasakian manifold and (N, g_N) be a Riemannian manifold. A horizontal conformal submersion $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ is called conformal semi-slant submersion if there is a distribution $D_1 \subset (\ker f_*)$ such that

$$\ker f_* = D_1 \oplus D_2 \oplus \langle\xi\rangle, \phi(D_1) = D_1, \tag{3.1}$$

and the angle $\theta = \theta(X)$ between ϕX and the space $(D_2)_p$ is constant for non-zero vector field $X \in (D_2)_p$ and $p \in M$, where D_2, D_1 and are $\langle\xi\rangle$ mutually orthogonal in $(\ker f_*)$. As it is, the angle θ is called the semi-slant angle of the horizontally conformal submersions.

It is known that the distribution $\ker f_*$ is integrable. Hence above Definition 1 implies that the integral manifold (fiber) $f^{-1}(q), q \in N$ of $\ker f_*$ is a semi-slant submanifold. Let f be a conformal semi-slant submersion from Lorentzian para Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ onto Riemannian manifold (N, g_N). For $U \in \Gamma(\ker f_*)$, we have

$$U = PU + QU - \eta(U)\xi, \tag{3.2}$$

where $PU \in \Gamma(D_1)$ and $QU \in \Gamma(D_2)$.

For $V \in \Gamma(\ker f_*)$, we have

$$\phi V = \psi V + \omega V, \tag{3.3}$$

where ψV and ωV are vertical and horizontal components of ϕV respectively.

Also for $X \in \Gamma(\ker f_*)^\perp$, we have

$$\phi X = BX + CX, \tag{3.4}$$

where BX and CX are vertical and horizontal components of ϕX respectively.

Then, $\Gamma(\ker f_*)^\perp$ decomposed as

$$\Gamma(\ker f_*)^\perp = \omega D_2 \oplus \mu, \tag{3.5}$$

where μ is the orthogonal complement of ωD_2 in $\Gamma(\ker f_*)^\perp$ and it is invariant with respect to ϕ.

Let $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal semi-slant submersion from Lorentzian para Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ onto Riemannian manifold (N, g_N). Thus the using equations (2.3), (3.3) and (3.4), we get

$$g_M(\psi X, Y) = g_M(X, \psi Y),\ g_M(U, CV) = g_M(CU, V) \tag{3.6}$$

for all $X, Y \in \Gamma(\ker \pi_*)$ and $U, V \in \Gamma(\ker \pi_*)^\perp$.

Then the using equations (3.1), (3.3), (3.4) and (3.5), we get

$$\psi D_1 = D_1,\ \omega D_1 = 0,\ \psi D_2 \subset D_2,\ B(\Gamma(\ker f_*)^\perp) = D_2. \tag{3.7}$$

Lemma 3. *Let $(M, \phi, \xi, \eta, g_M)$ be a Lorentzian para Sasakian manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ is a conformal semi-slant submersion, then*

$$\psi^2 + B\omega = I + \eta \otimes \xi,\ \omega\psi + C\omega = 0,$$
$$\psi B + BC = 0,\ \omega B + C^2 = I.$$

We define the co-variant derivatives of ψ and ω as follows:

$$(\nabla_X \psi)Y = \widehat{\nabla}_X \psi Y - \psi \widehat{\nabla}_X Y, \tag{3.8}$$

$$(\nabla_X \omega)Y = \mathcal{H}\nabla_X \omega Y - \omega \widehat{\nabla}_X Y \tag{3.9}$$

for all $X, Y \in \Gamma(\ker f_*)$, where $\widehat{\nabla}_X Y = \mathcal{V}\widehat{\nabla}_X Y$.

Lemma 4. *Let $(M, \phi, \xi, \eta, g_M)$ be a Lorentzian para Sasakian manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ is a conformal semi-slant submersion, then*

(1) $(\nabla_X \psi)Y = B\mathcal{T}_X Y - \mathcal{T}_X \omega Y - \eta(Y)X - g_M(X,Y)\xi - 2\eta(X)\eta(Y)\xi,$
$(\nabla_X \omega)Y = C\mathcal{T}_X Y - \mathcal{T}_X \psi Y,$
for all $X, Y \in \Gamma(\ker f_*)$.

(2) $\mathcal{T}_X BV + \mathcal{H}\nabla_X CV = C\mathcal{H}\nabla_X V + \omega \mathcal{T}_X V,$
$\widehat{\nabla}_X BV + \mathcal{T}_X CV = B\mathcal{H}\nabla_X V + \psi \nabla_X V,$
for $X \in \Gamma(\ker f_*)$ *and* $V \in \Gamma(\ker f_*)^\perp$.

(3) $\mathcal{V}\nabla_V \psi X + \mathcal{A}_V \omega = B\mathcal{A}_V X + \psi \mathcal{V}\nabla_V X,$
$\mathcal{A}_V \psi X + \mathcal{H}\nabla_X \omega X + \eta(X)V = C\mathcal{A}_V X + \omega \mathcal{V}\nabla_V X,$
for $X \in \Gamma(\ker f_*)$ *and* $V \in \Gamma(\ker f_*)^\perp$.

(4) $\mathcal{A}_U BV + \mathcal{H}\nabla_U CU = C\mathcal{H}\nabla_U V + \omega \mathcal{A}_U V,$
$\mathcal{V}\nabla_U BV + \mathcal{A}_U CV = B\mathcal{H}\nabla_U V + \psi \mathcal{A}_U V,$
for $U, V \in \Gamma(\ker f_*)^\perp$.

Lemma 5. *Let* $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ *be a conformal semi-slant submersion from Lorentzian para Sasakian manifold* $(M, \phi, \xi, \eta, g_M)$ *onto Riemannian manifold* (N, g_N). *Then* f *is a proper conformal semi-slant submersion if and only if there exists a constant* $\lambda \in [0, 1]$ *such that*

$$\psi^2 X = \lambda X \text{ for all } X \in \Gamma(D_2),$$

where $\lambda = \cos^2 \theta$.

Proof. For any non-zero vector field $X \in \Gamma(D_2)$, we have

$$\cos\theta = \frac{\| \psi X \|}{\| \phi X \|}, \tag{3.10}$$

and

$$\cos\theta = \frac{g_M(\phi X, \psi X)}{\| \psi X \| \| \phi X \|}, \tag{3.11}$$

where $\theta(X)$ is the semi-slant angle.

Using equations (2.1), (3.3) and (3.11), we get

$$\cos\theta = \frac{g_M(X, \psi^2 X)}{\| \psi X \| \| \phi X \|}. \tag{3.12}$$

From equations (3.11) and (3.12), we have

$$\psi^2 X = \cos^2\theta . X.$$

If $\lambda = \cos^2\theta$, then

$$\psi^2 X = \lambda X$$

for $X \in \Gamma(D_2)$. □

From Lemma 5 and equations (3.3), (3.4) and (3.6), then we easily have:

Corollary 1. *Let $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal semi-slant submersion from a Lorentzian para Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then*

$$g_M(\psi X, \psi Y) = \cos^2\theta g_M(X, Y), \tag{3.13}$$

$$g_M(\omega X, \omega Y) = \sin^2\theta g_M(X, Y), \tag{3.14}$$

for $X, Y \in \Gamma(D_2)$.

Lemma 6. *Let $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal semi-slant submersion from a Lorentzian para Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) with the slant angle $\theta \in [0, \frac{\pi}{2}]$. If ω is parallel with respect to ∇ on D_2, then we have*

$$\mathcal{T}_{\psi X}\psi X = \cos^2\theta.\mathcal{T}_X X \quad \textit{for } X \in \Gamma(D_2).$$

Proof. If ω is parallel, then from Lemma 4, we have

$$C\mathcal{T}_X Y = \mathcal{T}_X \psi Y \quad \text{for } X, Y \in \Gamma(D_2). \tag{3.15}$$

Interchanging the role of X and Y, we have

$$C\mathcal{T}_Y X = \mathcal{T}_Y \psi X \quad \text{for } X, Y \in \Gamma(D_2) \tag{3.16}$$

Since $\mathcal{T}$ is symmetric, from equations (3.15) and (3.16), we get

$$\mathcal{T}_{\psi X}\psi X = \cos^2\theta.\mathcal{T}_X X \quad \text{for } X \in \Gamma(D_2).$$ □

Theorem 1. *Let f be a conformal semi-slant submersion from a Lorentzian para Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then the semi-slant distribution D_1 is integrable if and only if*

$$\frac{1}{\lambda^2} g_N((\nabla f_*)(X, \phi Y) - (\nabla f_*)(Y, \phi X), f_*(\omega V)) = g_M(\psi(\widehat{\nabla}_X \phi Y - \widehat{\nabla}_Y \phi X), V)$$

for $X, Y \in \Gamma(D_1)$ and $V \in \Gamma(D_2)$.

Proof. Let $X, Y \in \Gamma(D_1)$, consider

$$\nabla_X\{g_M(Y, \xi)\} = (\nabla_X g_M)(Y, \xi) + g_M(\nabla_X Y, \xi) + g_M(Y, \nabla_X \xi).$$

Since X and Y are orthogonal to ξ ie. $g_M(Y, \xi) = g_M(X, \xi) = 0$. Hence from above equation

$$g_M(\nabla_X Y, \xi) = -g_M(Y, \nabla_X \xi),$$

using this relation in the equation

$$g_M([X, Y], \xi) = g_M(\nabla_X Y, \xi) - g_M(\nabla_Y X, \xi),$$

and from equations (2.4) and (2.7), we have

$$g_M([X, Y], \xi) = 0.$$

Now we note that D_1 is integrable if and only if $g_M([X, Y], V) = 0$, $g_M([X, Y], \xi) = 0$ and $g_M([X, Y], W) = 0$ for $X, Y \in \Gamma(D_1), V \in \Gamma(D_2)$ and $W \in (\ker f_*)^\perp$. Since $\ker f_*$ is integrable, then $g_M([X, Y], W) = 0$ and we proved above $g_M([X,Y],\xi)=0$. Thus, D_1 is integrable if and only if $g_M([X,Y],V)=0$.

Now we show that, from equations (2.3), (2.5), (2.12) and (3.3), we have

$$\begin{aligned}&g_M([X,Y],V)\\ &=g_M(\widehat{\nabla}_X\phi Y-\widehat{\nabla}_Y\phi X,\psi V)+g_M(\mathcal{H}\nabla_X\phi Y,\omega V)-g_M(\mathcal{H}\nabla_Y\phi X,\omega V).\end{aligned}$$

Since f is conformal submersion, using equation (2.17), we have

$$\begin{aligned}&g_M([X,Y],V)\\ &=\frac{1}{\lambda^2}g_N((\nabla f_*)(X,\phi Y)-(\nabla f_*)(Y,\phi X),f_*(\omega V))\\ &\quad-g_M(\psi(\widehat{\nabla}_X\phi Y-\widehat{\nabla}_Y\phi X),V).\end{aligned}$$

Then,

$$\begin{aligned}D_1\text{ is integrable }\iff\ &\frac{1}{\lambda^2}g_N((\nabla f_*)(X,\phi Y)-(\nabla f_*)(Y,\phi X),f_*(\omega V))\\ &=g_M(\psi(\widehat{\nabla}_X\phi Y-\widehat{\nabla}_Y\phi X),V).\end{aligned}$$

□

Theorem 2. *Let f be a conformal semi-slant submersion from a Lorentzian para Sasakian manifold (M,ϕ,ξ,η,g_M) onto a Riemannian manifold (N,g_N). Then the semi-slant distribution D_2 is integrable if and only if*

$$\mathcal{T}_X\omega Y-\mathcal{T}_Y\omega X+\psi(\mathcal{T}_X\omega\psi Y-\mathcal{T}_Y\omega\psi X)\in\Gamma(D_2)$$

for $X,Y\in\Gamma(D_2)$.

Proof. Similarly as Theorem 1, we can show

$$g_M([X,Y],\xi)=0\ \text{ for } X,Y\in\Gamma(D_2),$$

and we note that D_2 is integrable if and only if $g_M([X,Y],V)=0$, $g_M([X,Y],\xi)=0$ and $g_M([X,Y],W)=0$ for $X,Y\in\Gamma(D_2),V\in\Gamma(D_1)$ and $W\in(\ker f_*)^\perp$. Since $\ker f_*$ is integrable then $g_M([X,Y],W)=0$ and we proved above $g_M([X,Y],\xi)=0$ Thus, D_2 is integrable if and only if $g_M([X,Y],V)=0$.

From equations (2.3), (2.5) and (3.3), we have

$$\begin{aligned}g_M([X,Y],\phi V)&=g_M([X,Y],\phi V),\\ &=g_M(\nabla_X\psi Y,V)+g_M(\nabla_X\omega Y,V)-g_M(\nabla_Y\psi X,V)\\ &\quad-g_M(\nabla_Y\omega X,V),\end{aligned}$$

Next, using equation (2.13) and Lemma 5, we have

$$\sin^2\theta g_M([X,Y],\phi V)=g_M(\mathcal{T}_X\omega Y-\mathcal{T}_Y\omega X,V)+g_M(\psi(\mathcal{T}_X\omega\psi Y-\mathcal{T}_Y\omega\psi X),V).$$

Then,

$$D_2\text{ is integrable }\iff\mathcal{T}_X\omega Y-\mathcal{T}_Y\omega X+\psi(\mathcal{T}_X\omega\psi Y-\mathcal{T}_Y\omega\psi X)\in\Gamma(D_2).$$

□

Theorem 3. *Let f be a conformal semi-slant submersion from a Lorentzian para Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then the distribution $(\ker f_*)^{\perp}$ is integrable if and only if*

$$\begin{aligned}
&\frac{1}{\lambda^2}\{g_N(\nabla_Y f_*(CX), f_*(\omega V)) - g_N(\nabla_X f_*(CY), f_*(\omega V))\}\\
&= g_M(\mathcal{V}\nabla_X BY + \mathcal{A}_Y CY - \mathcal{V}\nabla_Y BX - \mathcal{A}_Y CY, \psi V)\\
&\quad + g_M(\mathcal{A}_X BY - \mathcal{A}_Y BX - CY(\ln\lambda)X + CX(\ln\lambda)Y\\
&\quad + 2g_M(X, CY) grad \ln\lambda, \omega V))
\end{aligned}$$

for $X, Y \in \Gamma(\ker f_*)^{\perp}$ and $V \in \Gamma(\ker f_*)$.

Proof. For $X, Y \in \Gamma(\ker f_*)^{\perp}$ and $V \in \Gamma(\ker f_*)$, using equations (2.3), (2.5), (2.14), (2.15), (3.3) and (3.4), we get

$$\begin{aligned}
g_M([X,Y],V) &= g_M(\nabla_X BY, \phi V) + g_M(\nabla_X CY, \phi V) - g_M(\nabla_X BY, \phi V)\\
&\quad - g_M(\nabla_X CY, \phi V) - \eta(V) g_M([X,Y],\xi),\\
&= g_M(\mathcal{V}\nabla_X BY + \mathcal{A}_X CY - \mathcal{V}\nabla_Y BX - \mathcal{A}_Y CY, \psi V)\\
&\quad + g_M(\mathcal{A}_X BY, \omega V) + g_M(\mathcal{H}\nabla_X CY, \omega V) - g_M(\mathcal{A}_Y BX, \omega V)\\
&\quad - g_M(\mathcal{H}\nabla_Y CX, \omega V) - \eta(V) g_M([X,Y],\xi).
\end{aligned}$$

Since f is conformal semi slant submersion and using equation (2.17) and Lemma 2, we get

$$\begin{aligned}
&g_M([X,Y],V)\\
&= g_M(\mathcal{V}\nabla_X BY + \mathcal{A}_X CY - \mathcal{V}\nabla_Y BX - \mathcal{A}_Y CY, \psi V)\\
&\quad + \frac{1}{\lambda^2} g_N(\nabla_X f_*(CY), f_*(\omega V)) - \frac{1}{\lambda^2} g_N(\nabla_Y f_*(CX), f_*(\omega V))\\
&\quad + \frac{1}{\lambda^2} g_N(f_*(\mathcal{A}_X BY), f_*(\omega V)) - \frac{1}{\lambda^2} g_N(f_*(\mathcal{A}_Y BX), f_*(\omega V))\\
&\quad - \frac{1}{\lambda^2} g_N(X(\ln\lambda) f_*(CY) + CY(\ln\lambda) f_*(X)\\
&\quad - g_M(X, CY) f_*(grad \ln\lambda), f_*(\omega V))\\
&\quad + \frac{1}{\lambda^2} g_N(Y(\ln\lambda) f_*(CX) + CX(\ln\lambda) f_*(Y)\\
&\quad - g_M(Y, CX) f_*(grad \ln\lambda), f_*(\omega V)) - \eta(V) g_M([X,Y],\xi).
\end{aligned}$$

$$\begin{aligned}
&g_M([X,Y],V)\\
&= g_M(\psi(\mathcal{V}\nabla_X BY + \mathcal{A}_X CY - \mathcal{V}\nabla_Y BX - \mathcal{A}_Y CY), V)\\
&\quad + g_M(A_X BY - A_Y BX - CY(\ln\lambda)X + CX(\ln\lambda)Y\\
&\quad + 2g_M(X, CY) grad \ln\lambda, \omega V) + \frac{1}{\lambda^2} g_N(\nabla_X f_*(CY), f_*(\omega V))\\
&\quad - \frac{1}{\lambda^2} g_N(\nabla_Y f_*(CX), f_*(\omega V)).
\end{aligned} \tag{3.17}$$

□

Theorem 4. *Let f be a conformal semi-slant submersion from a para Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then any two conditions below imply the third*:

(i) $(\ker f_*)^\perp$ *is integrable.*

(ii) f *is a horizontally homothetic map.*

(iii)
$$\frac{1}{\lambda^2}\{g_N(\nabla_Y f_*(CX), f_*(\omega V)) - g_N(\nabla_X f_*(CY), f_*(\omega V))\}$$
$$= g_M(\mathcal{V}\nabla_X BY + \mathcal{A}_Y CY - \mathcal{V}\nabla_Y BX - \mathcal{A}_Y CY, \psi V)$$
$$+ g_M(A_X BY - A_Y BX, \omega V))$$
for $X, Y \in \Gamma(\ker f_*)^\perp$ *and* $V \in \Gamma(\ker f_*)$.

Proof. From equation (3.18), we have

$$\begin{aligned} g_M([X,Y],V) &= g_M(\psi(\mathcal{V}\nabla_X BY + \mathcal{A}_X CY - \mathcal{V}\nabla_Y BX - \mathcal{A}_Y CY), V) \\ &+ g_M(\mathcal{A}_X BY - \mathcal{A}_Y BX - CY(\ln\lambda)X + CX(\ln\lambda)Y \\ &+ 2g_M(X, CY)grad\ln\lambda, \omega V) + \frac{1}{\lambda^2}g_N(\nabla_X f_*(CY), f_*(\omega V)) \\ &- \frac{1}{\lambda^2}g_N(\nabla_Y f_*(CX), f_*(\omega V)) \end{aligned}$$

for $X, Y \in \Gamma(\ker f_*)^\perp$ and $V \in \Gamma(\ker f_*)$. Now, if we have (i) and (iii), then we have

$$g_M(grad\ln\lambda, CY)g_M(X, \omega V) = g_M(grad\ln\lambda, CX)g_M(Y, \omega V) + 2g_M(X, CY)g_M(grad\ln\lambda, \omega V). \tag{3.18}$$

Now, putting $Y = \omega V$ for $V \in \Gamma(D_2)$ in equation (3.19), we have

$$g_M(grad\ln\lambda, CX)g_M(\omega V, \omega V) = 0.$$

Thus, λ is a constant on $\Gamma(\mu)$. On the other hand, taking $Y = CX$ for $X \in \Gamma(\mu)$ in equation (3.19), we get

$$2g_M(X, C^2X)g_M(grad\ln\lambda, \omega V) = 2g_M(X, X)g_M(grad\ln\lambda, \omega V) = 0. \quad \square$$

Theorem 5. *Let f be a conformal semi-slant submersion from a para Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemnnian manifold (N, g_N). Then the distribution $(\ker f_*)$ is a totally geodesic foliation on M if and only if*

$$\begin{aligned} &\frac{1}{\lambda^2}g_2(\nabla_{\omega V} f_*(\phi CX), f_*(\omega U)) \\ &= g_M(\omega(\widehat{\nabla}_U \psi V + \mathcal{T}_U \omega V), X) + g_M(\mathcal{T}_U \psi V, CX) - g_M(\mathcal{A}_{\omega V}\phi CX, \psi U) \\ &\quad + g_M(\omega U, \omega V)g_M(grad\ln\lambda, \phi CX) + \eta(V)g_M(U, BX) \end{aligned}$$

for $U, V \in \Gamma(\ker f_*)$ *and* $X \in \Gamma(\ker f_*)^\perp$.

Proof. For $U, V \in \Gamma(\ker f_*)$ and $X \in \Gamma(\ker f_*)^\perp$, using equations (2.3), (2.5), (3.3) and (3.4), we get

$$g_M(\nabla_U V, X) = g_M(\nabla_U \psi V, BX) + g_M(\nabla_U \omega V, BX) + g_M(\nabla_U \psi V, CX)$$

$$+ g_M(\nabla_U \omega V, CX) + \eta(V) g_M(U, BX).$$

Since f is conformal submersion, using equations (2.12), (2.13), (2.17) and Lemma 2, we get

$$\begin{aligned} & g_M(\nabla_U V, X) \\ = & g_M(\omega(\widehat{\nabla}_U \psi V + \mathcal{T}_U \omega V), X) + g_M(\mathcal{T}_U \psi V, CX) \\ & - g_M(\mathcal{A}_{\omega V} \phi CX, \psi V) - \frac{1}{\lambda^2} g_N(\nabla_{\omega V} f_*(\phi CX), f_*(\omega U)) \\ & + \frac{1}{\lambda^2} g_N(\omega V(\ln \lambda) f_*(\phi CX) + \phi CX(\ln \lambda) f_*(\omega V) \\ & - g_M(\omega V, \phi CX) f_*(grad \ln \lambda), f_*(\omega U)) + \eta(V) g_M(U, BX). \end{aligned}$$

Hence we have

$$\begin{aligned} & g_M(\nabla_U V, X) \\ = & g_M(\omega(\widehat{\nabla}_U \psi V + \mathcal{T}_U \omega V), X) + g_M(\mathcal{T}_U \psi V, CX) \\ & - g_M(\mathcal{A}_{\omega V} \phi CX, \psi V) - \frac{1}{\lambda^2} g_N(\nabla_{\omega V} f_*(\phi CX), f_*(\omega U)) \\ & + g_M(\omega U, \omega V) g_M(\mathcal{H} grad \ln \lambda, \phi CX) + \eta(V) g_M(U, BX). \end{aligned} \tag{3.19}$$

□

Theorem 6. *Let f be a conformal semi-slant submersion from a para Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then, any two conditions below imply third*:

(i) $\ker f_*$ *is a totally geodesic foliation on* M,
(ii) λ *is a constant on* $\Gamma(D_1)$,
(iii)
$$\begin{aligned} & \frac{1}{\lambda^2} g_N(\nabla_{\omega V} f_*(\phi CX), f_*(\omega U)) \\ = & - g_M(C(\widehat{\nabla}_U \psi V + \mathcal{T}_U \omega V), X) + g_M(\mathcal{T}_U \psi V, CX) \\ & - g_M(\mathcal{A}_{\omega V} \phi CX, \psi U) \end{aligned}$$
for $U, V \in \Gamma(\ker f_*)$ *and* $X \in \Gamma(\ker f_*)^\perp$.

Proof. From equation (3.19), we have

$$\begin{aligned} g_M(\nabla_U V, X) = & g_M(\omega(\widehat{\nabla}_U \psi V + \mathcal{T}_U \omega V), X) + g_M(\mathcal{T}_U \psi V, CX) \\ & - g_M(\mathcal{A}_{\omega V} \phi CX, \psi V) - \frac{1}{\lambda^2} g_N(\nabla_{\omega V} f_*(\phi CX), f_*(\omega U)) \\ & + g_M(\omega U, \omega V) g_M(\mathcal{H} grad \ln \lambda, \phi CX) + \eta(V) g_M(U, BX) \end{aligned}$$

for $U, V \in \Gamma(\ker f_*)$ and $X \in \Gamma(\ker f_*)^\perp$. Now, if we have (i) and (ii), then we obtain

$$\begin{aligned} g_M(\nabla_U V, X) = & g_M(\omega(\widehat{\nabla}_U \psi V + \mathcal{T}_U \omega V), X) + g_M(\mathcal{T}_U \psi V, CX) \\ & - g_M(\mathcal{A}_{\omega V} \phi CX, \psi V) - \frac{1}{\lambda^2} g_N(\nabla_{\omega V} f_*(\phi CX), f_*(\omega U)) \\ & + \eta(V) g_M(U, BX). \end{aligned}$$

From above equation, we get (iii). Similarly, one can obtain the other assertions. □

Definition 2 (Totally geodesicess of the conformal semi-slant submersions)**.** At this part, we shall examine the totally geodesicess of conformal semi-slant submersion. First, we give necessary and sufficient condition for a conformal semi-slant submersion to be totally geodesic map. Remember that a smooth map f from a para Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemnnian manifold (N, g_N) is called totally geodesic if $(\nabla f_*)(X, Y) = 0$ for any $X, Y \in \Gamma(TM)$ ([10, 11]).

Theorem 7. *Let f be a conformal semi-slant submersion from a para Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemnnian manifold (N, g_N). Then, f is a totally geodesic map if*

$$\begin{aligned}\nabla_Z f_*(Y_2) = f_*(&C(\mathcal{A}_Z\psi Y_1 + \mathcal{H}\nabla_Z\omega Y_1 + \mathcal{A}_Z BY_2 + \nabla_Z CY_2))\\ &+ \omega(\mathcal{V}\nabla_Z\psi Y_1 + \mathcal{A}_Z\omega Y_1 + \mathcal{V}\nabla_Z BY_2 + \mathcal{A}_Z CY_2)\end{aligned}$$

for any $Z \in \Gamma(\ker f_*)^\perp$ and $Y = Y_1 + Y_2 \in \Gamma(TM)$, where $Y_1 \in \Gamma(\ker f_*)$ and $Y_2 \in \Gamma(\ker f_*)^\perp$.

Proof. For $Z \in \Gamma(\ker f_*)^\perp$ and $Y = Y_1 + Y_2 \in \Gamma(TM)$, where $Y_1 \in \Gamma(\ker f_*)$ and $Y \in \Gamma(\ker f_*)^\perp$, from equations (2.1), (2.5), (2.17), (3.3) and (3.4), we get

$$\begin{aligned}&(\nabla f_*)(Z, Y)\\ &= \nabla_Z f_*(Y) - f_*((\phi^2 - \eta \oplus \xi)(\nabla_Z Y)),\\ &= \nabla_Z f_*(Y) - f_*(\phi(\nabla_Z\psi Y_1 + \nabla_Z\omega Y_1 + \nabla_Z BY_2 + \nabla_Z CY_2)\\ &\quad + \eta(Y_1)\phi Z - \eta(\nabla_Z Y)\xi).\end{aligned}$$

Again, using equations (2.12), (2.13), (2.15), (3.3) and (3.4), we obtain

$$\begin{aligned}&(\nabla f_*)(Z, Y)\\ &= \nabla_Z f_*(Y) - f_*((B\mathcal{A}_Z\psi Y_1 + C\mathcal{A}_Z\psi Y_1 + \psi\mathcal{V}\nabla_Z\psi Y_1 + \omega\mathcal{V}\nabla_Z\psi Y_1\\ &\quad + B\mathcal{H}\nabla_Z\omega Y_1 + C\mathcal{H}\nabla_Z\omega Y_1 + \psi\mathcal{A}_Z\omega Y_1 + \omega\mathcal{A}_Z\omega Y_1 + B\mathcal{A}_Z BY_2\\ &\quad + C\mathcal{A}_Z BY_2 + \psi\mathcal{V}\nabla_Z BY_2 + \omega\mathcal{V}\nabla_Z BY_2 + B\mathcal{H}\nabla_Z CY_2 + C\mathcal{H}\nabla_Z CY_2\\ &\quad + \psi\mathcal{A}_Z CY_2 + \omega\mathcal{A}_Z CY_2) + \eta(Y_1)BZ + \eta(Y_1)CZ - \eta(\nabla_Z Y)\xi).\end{aligned}$$

Thus taking into account the vertical parts, we get

$$\begin{aligned}(\nabla f_*)(Z, Y) = \nabla_Z f_*(Y) - f_*(&C(\mathcal{A}_Z\psi Y_1 + \mathcal{H}\nabla_Z\omega Y_1 + \mathcal{A}_Z BY_2 + \mathcal{H}\nabla_Z CY_2)\\ &+ \omega(\mathcal{V}\nabla_Z\psi Y_1 + \mathcal{A}_Z\omega Y_1 + \mathcal{V}\nabla_Z BY_2 + \omega\mathcal{A}_Z CY_2) + \eta(Y_1)CZ).\end{aligned}$$

□

Theorem 8. *Let f be a conformal semi-slant submersion from a Lorentzian para Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemnnian manifold (N, g_N). Then, f is a totally geodesic map if and only if*

(i) $\frac{1}{\lambda^2}\{g_N((\nabla f_*)(U,\omega\psi V), f_*(Z)) - g_N(\nabla_{\omega V} f_*(\omega U), f_*(\phi CX))\}$
$= g_M(\mathcal{T}_U\omega V, BZ) - g_M(\mathcal{A}_{\omega V}\psi U, \phi CZ)$
$- g_M(grad\ln\lambda, \omega V)g_M(\omega U, \phi CZ),$

(ii) $g_M(\widehat{\nabla}_{U_1}\phi V_1, BZ) = g_M(\mathcal{T}_{U_1}CZ, V_1),$

(iii) $\frac{1}{\lambda^2}\{g_N((\nabla f_*)(W, CX), f_*(CY)) + g_N((\nabla f_*)(W, \omega BX), f_*(Y))\}$
$= g_M(\mathcal{T}_W\psi BX, Y) + g_M(\mathcal{T}_W CX, BY),$

(iv) *f is a horizontally homothetic map, for $U_1, V_1 \in \Gamma(D_1)$, $U, V \in \Gamma(D_2)$, $W \in \Gamma(\ker f_*)$ and $X, Y, Z \in \Gamma(\ker f_*)^\perp$.*

Proof. (i) For $U, V \in \Gamma(D_2)$ and $Z \in \Gamma(\ker f_*)^\perp$, using equations (2.3), (2.5), (2.8), (2.17) and (3.3), we get

$$\begin{aligned}&\frac{1}{\lambda^2}g_N((\nabla f_*)(U,V), f_*(Z))\\ =&\ -g_M(\nabla_U\psi^2V, Z) - g_M(\nabla_U\omega\psi V, Z) - g_M(\nabla_U\omega V, \phi Z).\end{aligned}$$

From equation (2.3), (2.5), (2.13), (3.4) and Lemma 5, we get

$$\begin{aligned}&\sin^2\theta\frac{1}{\lambda^2}g_N((\nabla f_*)(U,V), f_*(Z))\\ =&\ -g_M(\mathcal{H}\nabla_U\omega\psi V, Z) - g_M(\mathcal{T}_U\omega V, BZ) + g_M(\nabla_{\omega V}\phi U, \phi CZ).\end{aligned}$$

Since f is conformal submersion, using equations (2.17) and Lemma 2, we get

$$\begin{aligned}&\sin^2\theta\frac{1}{\lambda^2}g_N((\nabla f_*)(U,V), f_*(Z))\\ =&\ -g_M(\mathcal{T}_U\omega V, BZ) + \frac{1}{\lambda^2}g_N((\nabla f_*)(U,\omega\psi V), f_*(Z))\\ &+ g_M(\mathcal{A}_{\omega V}\psi U, \phi CZ) + \frac{1}{\lambda^2}g_M(\nabla_{\omega V} f_*(\omega U), f_*(\phi CZ))\\ &- g_M(\omega V(\ln\lambda)f_*(\omega U) + \omega U(\ln\lambda)f_*(\omega V)\\ &- g_M(\omega V, \omega U)f_*(grad\ln\lambda), f_*(\phi CX)).\end{aligned}$$

Hence, we get

$$\begin{aligned}&\sin^2\theta\frac{1}{\lambda^2}g_N((\nabla f_*)(U,V), f_*(Z))\\ =&\ -g_M(\mathcal{T}_U\omega V, BZ) + \frac{1}{\lambda^2}g_N((\nabla f_*)(U,\omega\psi V), f_*(Z))\\ &+ g_M(\mathcal{A}_{\omega V}\psi U, \phi CZ) + \frac{1}{\lambda^2}g_M(\nabla_{\omega V} f_*(\omega U), f_*(\phi CZ))\\ &+ g_M(\omega V, \omega U)g_M(grad\ln\lambda, \phi CX).\end{aligned}$$

(ii) For $U_1, V_1 \in \Gamma(\ker f_*)^\perp$, using equations (2.3), (2.5), (2.8), (2.17) and (3.3), we get

$$\frac{1}{\lambda^2}g_N((\nabla f_*)(U_1,V_1), f_*(Z)) = -g_M(\nabla_{U_1}\phi V_1, BZ) - g_M(\nabla_{U_1}\phi V_1, CZ).$$

Again using equations (2.12) and (2.13), we get

$$\frac{1}{\lambda^2}g_N((\nabla f_*)(U_1,V_1),f_*(Z)) = -g_M(\widehat{\nabla}_{U_1}\phi V_1, BZ) + g_M(\phi V_1, T_1 CZ).$$

(iii) For $W \in \Gamma(\ker f_*)$ and $X, Y \in \Gamma(\ker f_*)^\perp$, using equations (2.3), (2.5), (2.8) and (2.17), we get

$$\frac{1}{\lambda^2}g_N((\nabla f_*)(X,Y),f_*(W)) = -g_M(\nabla_X \phi Y, \phi W).$$

Again using equations (2.3), (2.5), (3.3), (3.4), (2.12) and (2.13), we get

$$\begin{aligned}\frac{1}{\lambda^2}g_N((\nabla f_*)(W,X),f_*(Y)) = & -g_M(\mathcal{T}_W \psi BX, Y) - g_M(\mathcal{T}_W CX, BY) \\ & - g_M(\mathcal{H}\nabla_W \omega BX, Y) - g_M(\mathcal{H}\nabla_W CX, CY).\end{aligned}$$

Since f is conformal submersion and using equation (2.17), we have

$$\begin{aligned}&\frac{1}{\lambda^2}g_N((\nabla f_*)(W,X),f_*(Y)) \\ = & -g_M(\mathcal{T}_W \psi BX, Y) - g_M(\mathcal{T}_W CX, BY) \\ & + \frac{1}{\lambda^2}\{g_N((\nabla f_*)(W,\omega BX),f_*(Y)) - g_N((\nabla f_*)(W,CX),f_*(CY))\}.\end{aligned}$$

(iv) $X_1, X_2 \in \Gamma(\mu)$, from Lemma 2, we have

$$(\nabla f_*)(X_1,X_2) = X_1(\ln\lambda)f_*(X_2) + X_2(\ln\lambda)f_*(X_1) - g_M(X_1,X_2)f_*(grad\ln\lambda).$$

From above equation putting $X_2 = \phi X_1$ for $X_1 \in \Gamma(\mu)$, we get

$$\begin{aligned}&(\nabla f_*)(X_1,\phi X_1) \\ = & X_1(\ln\lambda)f_*(\phi X_1) + \phi X_1(\ln\lambda)f_*(X_1) - g_M(X_1,\phi X_1)f_*(grad\ln\lambda), \\ = & X_1(\ln\lambda)f_*(\phi X_1) + \phi X_1(\ln\lambda)f_*(X_1).\end{aligned}$$

If $(\nabla f_*)(X_1,\phi X_1) = 0$, the we have

$$X_1(\ln\lambda)f_*(\phi X_1) + \phi X_1(\ln\lambda)f_*(X_1) = 0. \tag{3.20}$$

Taking inner product in equation (3.20) with $f_*(\phi X_1)$ and since f is conformal submersion, we have

$$\begin{aligned}&\frac{1}{\lambda^2}\{g_M(grad\ln\lambda, X_1)g_N(f_*(\phi X_1),f_*(\phi X_1)) \\ &\quad + g_M(grad\ln\lambda,\phi X_1)g_N(f_*(X_1),f_*(\phi X_1))\} = 0.\end{aligned}$$

From above equation, it follows that λ is a constant $\Gamma(\mu)$. In similar way, for $U_2, V_2 \in \Gamma(\ker f_*)$, using Lemma 2, we have

$$\begin{aligned}&(\nabla f_*)(\omega U_2,\omega V_2) \\ = & \omega U_2(\ln\lambda)f_*(\omega V_2) + \omega V_2(\ln\lambda)f_*(\omega U_2) - g_M(\omega U_2,\omega V_2)f_*(grad\ln\lambda),\end{aligned}$$

From above equation putting $V_2 = U_2$, we have

$$(\nabla f_*)(\omega U_2,\omega V_2) \tag{3.21}$$

$$= 2\omega U_2(\ln\lambda)f_*(\omega U_2) - g_M(\omega U_2, \omega U_2)f_*(grad\ln\lambda).$$

Taking inner product in equation (3.21) with $f_*(\omega U_2)$ and since f is conformal submersion, we get

$$g_M(\omega U_2, \omega U_2)g_M(grad\ln\lambda, \omega U_2) = 0.$$

From above equation, it is follows that λ is a constant on $\Gamma(\omega(\ker f_*))$. So λ is a constant on $\Gamma(\ker f_*)^\perp$. On the other hand, if f is a horizontally homothetic map, it is obvious that $(\nabla f_*)(X, Y) = 0$. □

Definition 3. Let f be a conformal semi-slant submersion from a Lorentzian para Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N). Then f is called a $(\omega D_2, \mu)$totally geodesic map if

$$(\nabla f)(\omega V, X) = 0$$

for $V \in \Gamma(D_2)$ and $X \in \Gamma(\mu)$.

Theorem 9. *Let f be a conformal semi-slant submersion from a Lorentzian para Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then, f is called a $(\omega D_2, \mu)$-totally geodesic map if and only if f is horizontally homothetic map.*

Proof. For $U \in \Gamma(D_2)$ and $X \in \Gamma(\mu)$, from Lemma 2, we get

$$(\nabla f_*)(\omega U, X) = \omega U(\ln\lambda)f_*(X) + X(\ln\lambda)f_*(\omega U) - g_M(\omega U, X)f_*(grad\ln\lambda).$$

If f is a horizontally homothetic then, $(\nabla f_*)(\omega U, X) = 0$. Conversely if $(\nabla f_*)(\omega U, X) = 0$, we get

$$\omega U(\ln\lambda)f_*(X) + X(\ln\lambda)f_*(\omega U) = 0. \tag{3.22}$$

Since f is a conformal semi-slant submersion and taking inner product in equation (3.22) with $f_*(\omega U)$, we get

$$g_M(\omega U, \omega U)g_M(grad\ln\lambda, X) = 0.$$

Above equation implies that λ is constant on $\Gamma(\mu)$.

Again, since f is a conformal semi-slant submersion and taking inner product in equation (3.22) with $f_*(X)$, we get

$$g_M(X, X)g_M(grad\ln\lambda, \omega U) = 0.$$

From above equation, it follows that λ is constant on $\Gamma(\omega D_2)$. Thus, λ is a constant on $\Gamma(\ker f_*)^\perp$. □

Theorem 10. *Let f be a conformal semi-slant submersion from a Lorentzian para Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemnnian manifold (N, g_N). Then f is a totally geodesic map if and only if*

(a) $C\mathcal{T}_{U_1}\psi V_1 + \omega\widehat{\nabla}_{U_1}\omega V_1 = 0$ *for* $U_1, V_1 \in \Gamma(D_1)$,
(b) $C(\mathcal{T}_{U_1}\psi U_2 + \mathcal{A}_{\omega U_2}U_1) + \omega(\widehat{\nabla}_{U_1}\psi U_2 + \mathcal{T}_{U_1}\omega U_2) = 0$ *for* $U_1 \in \Gamma(D_1)$, $U_2 \in \Gamma(D_2)$,

(C) $C(\mathcal{T}_U BX + \mathcal{A}_{CX}U) + \omega(\widehat{\nabla}_U BX + \mathcal{T}_U CX) = 0$ *for* $U \in \Gamma(\ker f_*), X \in \Gamma(\ker f_*)^\perp$.

Proof. For $U_1, V_1 \in \Gamma(D_{1,})$, from equation (2.1), (2.5), (2.17) and (3.3), we get

$$(\nabla f_*)(U_1, V_1) = -f_*(\phi(\nabla_{U_1}\psi V_1) - \eta(\nabla_{U_1}V_1)\xi).$$

Next, from equations (2.12), (3.3) and (3.4), we get

$$\begin{aligned}&(\nabla f_*)(U_1, V_1)\\ =& - f_*(B\mathcal{T}_{U_1}\psi V_1 + C\mathcal{T}_{U_1}\psi V_1 + \psi\widehat{\nabla}_{U_1}\psi V_1 + \omega\widehat{\nabla}_{U_1}\psi V_1 - \eta(\nabla_{U_1}V_1)\xi).\end{aligned}$$

Since $B\mathcal{T}_{U_1}\psi V_1 + \psi\widehat{\nabla}_{U_1}\psi V_1 - \eta(\nabla_{U_1}V_1)\xi \in \Gamma(\ker f_*)$, we have

$$(\nabla f_*)(U_1, V_1) = -f_*(C\mathcal{T}_{U_1}\psi V_1 + \omega\widehat{\nabla}_{U_1}\psi V_1).$$

Then, since f is a linear isomorphism between $(\ker f_*)^\perp$ and TM, $(\nabla f_*)(U_1, V_1) = 0 \Leftrightarrow (C\mathcal{T}_{U_1}\psi V_1 + \omega\widehat{\nabla}_{U_1}\psi V_1) = 0$.

(b) For $U_1 \in \Gamma(D_1)$, $U_2 \in \Gamma(D_2)$, from equations (2.1), (2.5), (2.17) and (3.3), we get

$$(\nabla f_*)(U_1, U_2) = -f_*(\phi\nabla_{U_1}\psi U_2 + \phi\nabla_{U_1}\omega U_2 - \eta(\nabla_{U_1}U_2)\xi).$$

Again using equation (2.12), (2.13), (3.3) and (3.4), we get

$$\begin{aligned}(\nabla f_*)(U_1, U_2) =& - f_*(B\mathcal{T}_{U_1}\psi U_2 + C\mathcal{T}_{U_1}\psi U_2 + \psi\widehat{\nabla}_{U_1}\psi U_2 + \omega\widehat{\nabla}_{U_1}\psi U_2\\ &+ B\mathcal{A}_{\omega U_2}U_1 + C\mathcal{A}_{\omega U_2}U_1 + \psi\mathcal{T}_{U_1}\omega U_2 + \omega\mathcal{T}_{U_1}\omega U_2\\ &- \eta(\nabla_{U_1}U_2)\xi).\end{aligned}$$

Since $B\mathcal{T}_{U_1}\psi U_2 + \psi\widehat{\nabla}_{U_1}\psi U_2 + B\mathcal{A}_{\omega U_2}U_1 + \psi\mathcal{T}_{U_1}\omega U_2 - \eta(\nabla_{U_1}U_2)\xi \in \Gamma(\ker f_*)$, we have

$$(\nabla f_*)(U_1, U_2) = -f_*(C(\mathcal{T}_{U_1}\psi U_2 + \mathcal{A}_{\omega U_2}U_1) + \omega(\widehat{\nabla}_{U_1}\psi U_2 + \mathcal{T}_{U_1}\omega U_2)).$$

Since f is a linear isomorphism between $(\ker f_*)^\perp$ and TN, $(\nabla f_*)(U_1, U_2) = 0 \Leftrightarrow C(\mathcal{T}_{U_1}\psi U_2 + \mathcal{A}_{\omega U_2}U_1) + \omega(\widehat{\nabla}_{U_1}\psi U_2 + \mathcal{T}_{U_1}\omega U_2) = 0$.

(c) For $U \in \Gamma(\ker f_*)$ and $X \in \Gamma(\ker f_*)^\perp$, from equation (2.1), (2.3), (2.17) and (3.4), we obtain

$$(\nabla f_*)(U, X) = -f_*(\phi(\nabla_U BX + \nabla_U CX) - \eta(\nabla_U X)\xi).$$

Using equations (2.12), (2.13), (3.3) and (3.4), we have

$$\begin{aligned}(\nabla f_*)(U, X) =& - f_*(B\mathcal{T}_U BX + C\mathcal{T}_U BX + \psi\widehat{\nabla}_U BX + \omega\widehat{\nabla}_U BX\\ &+ \psi\mathcal{T}_U CX + \omega\mathcal{T}_U CX + B\mathcal{A}_U CX + C\mathcal{A}_U CX) - \eta(\nabla_U X)\xi).\end{aligned}$$

Since $B\mathcal{T}_U BX + \psi\widehat{\nabla}_U BX + \psi\mathcal{T}_U CX + B\mathcal{A}_U CX - \eta(\nabla_U X)\xi \in \Gamma(\ker f_*)^\perp$, we have

Since f is a linear isomorphism between $(\ker f_*)^\perp$ and TN, $(\nabla f_*)(U, X) = 0 \Leftrightarrow C(\mathcal{T}_U BX + \mathcal{A}_U CX) + \omega(\widehat{\nabla}_U BX + \mathcal{T}_U CX) = 0$. □

4. Examples

Note that given an Euclidean space R^{2m+1} with coordinates $\{(x^1, x^2, \ldots, x^m, y^1, y^2, \ldots, y^m, z) : x^i, y^i, z \in R\}$. Consider the base field $\{E_i, E_{m+i}, \xi\}$ where $E_i = \frac{\partial}{\partial y^i}, E_{m+i} = \frac{\partial}{\partial x^i} + y^i \frac{\partial}{\partial z}$ and contravariant vector field $\xi = \frac{\partial}{\partial z}$. Define Lorentzian almost para contact structure on R^{2m+1} as follows:

$$\phi(\sum_{i=1}^{m}(X_i \frac{\partial}{\partial x^i} + Y_i \frac{\partial}{\partial y^i}) + Z \frac{\partial}{\partial z}) = -\sum_{i=1}^{m} Y_i \frac{\partial}{\partial x^i} - \sum_{i=1}^{m} X_i \frac{\partial}{\partial y^i} + \sum_{i=1}^{m} Y_i y^i \frac{\partial}{\partial z},$$

$$\xi = \frac{\partial}{\partial z},$$

$$\eta = -(dz - \sum_{i=1}^{m} y^i dx^i),$$

$$g = -\eta \otimes \eta + (\sum_{i=1}^{m} dx^i \otimes dx^i + \sum_{i=1}^{m} dy^i \otimes dy^i).$$

Then $(R^{2m+1}, \phi, \xi, \eta, g)$ is Lorentzian para Sasakian manifold.

Example 2. Every semi-slant submersion from Lorentzian para Sasakian manifold to a Riemannian manifold is a conformal semi-slant submersion with $\lambda = I$, where I denotes the identity function. We say that a conformal semi-slant submersion is proper if $\lambda \neq I$.

Example 3. Consider the Euclidean space R^7 with coordinates $(x^1, x^2, x^3, y^1, y^2, y^3, z)$ and base field $\{E_i, E_{3+i}, \xi\}$ where $E_i = \frac{\partial}{\partial y^i}, E_{3+i} = \frac{\partial}{\partial x^i} + y^i \frac{\partial}{\partial z}$, $i = 1, 2, 3$ and contravariant vector field $\xi = \frac{\partial}{\partial z}$. Define Lorentzian almost para contact structure on R^7 as follows:

$$\phi(\sum_{i=1}^{3}(X_i \frac{\partial}{\partial x^i} + Y_i \frac{\partial}{\partial y^i}) + Z \frac{\partial}{\partial z}) = -\sum_{i=1}^{3} Y_i \frac{\partial}{\partial x^i} - \sum_{i=1}^{3} X_i \frac{\partial}{\partial y^i} + \sum_{i=1}^{3} Y_i y^i \frac{\partial}{\partial z},$$

$$\xi = \frac{\partial}{\partial z},$$

$$\eta = -(dz - \sum_{i=1}^{3} y^i dx^i),$$

$$g = -\eta \otimes \eta + (\sum_{i=1}^{3} dx^i \otimes dx^i + \sum_{i=1}^{3} dy^i \otimes dy^i).$$

Then $(R^7, \phi, \xi, \eta, g)$ is Lorentzian para Sasakian manifold.

Let F be a submersion defined by

$$F : R^7 \to R^2$$

$$F(x^1, x^2, x^3, y^1, y^2, y^3, z) = (e^{y^1} \cos y^3, e^{y^1} \sin y^3)$$

where $y^3 \in R - \{k\frac{\pi}{2}, k\pi\}$, $k \in R$. Then it follows that

$$\ker F_* = span\{V_1 = \partial x^1, V_2 = \partial x^2, V_3 = \partial x^3, V_4 = \partial y^2, V_5 = \partial z\}$$

and

$$(\ker F_*)^\perp = span\{X_1 = e^{y^1} \cos y^3 \partial y^1 - e^{y^1} \sin y^3 \partial y^3, \quad X_2 = e^{y^1} \sin y^3 \partial y^1 + e^{y^1} \cos y^3 \partial y^3\}.$$

Hence, we have

$$\overline{g}(F_* X_1, F_* X_1) = (e^{y^1})^2 g(X_1, X_1), \overline{g}(F_* X_2, F_* X_2) = (e^{y^1})^2 g(X_2, X2),$$

where $\overline{g}$ denote the standard metric (Euclidean metric) on R^2. Thus F is a conformal semi-slant submersion with $\lambda = e^{y^1}$.

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