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ON A CERTAIN EXTENSION OF THE RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVE OPERATOR

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Abstract. The main aim of this present paper is to present a new extension of the fractional derivative operator by using the extension of beta function recently defined by Shadab et al. [19]. Moreover, we establish some results related to the newly defined modified fractional derivative operator such as Mellin transform and relations to extended hypergeometric and Appell's function via generating functions.

1. Introduction and preliminaries

We begin with the well-known Riemann-Liouville (R-L) fractional derivative of order μ is defined (see [5, 18]) by

$$\mathfrak{D}_x^{\mu}\{f(x)\} = \frac{1}{\Gamma(-\mu)}\int_0^x f(t)(x-t)^{-\mu-1}dt,\ \Re(\mu) > 0. \tag{1.1}$$

For the case $m-1 < \Re(\mu) < m$ where $m = 1, 2, \ldots$, it follows

$$\begin{aligned}\mathfrak{D}_x^{\mu}\{f(x)\} &= \frac{d^m}{dx^m}\mathfrak{D}_x^{\mu-m}\Big\{f(x)\Big\}\\ &= \frac{d^m}{dx^m}\Big\{\frac{1}{\Gamma(-\mu+m)}\int_0^x f(t)(x-t)^{-\mu+m-1}dt\Big\},\ \Re(\mu) > 0 \end{aligned}\tag{1.2}$$

and

$$\mathfrak{D}_x^{\mu}\{x^{\sigma}\} = \frac{\Gamma(\sigma+1)}{\Gamma(\sigma-\mu+1)}x^{\sigma-\mu},\ \Re(\sigma) > -1. \tag{1.3}$$

The researchers (see [7, 8, 11, 14, 21]) investigated the various extensions and generalization of fractional derivative operators.

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The extended R-L fractional derivative of order μ is defined in [12] by

$$\mathfrak{D}_x^{\mu}\{f(x);p\} = \frac{1}{\Gamma(-\mu)}\int_0^x f(t)(x-t)^{-\mu-1}\exp\left(-\frac{px^2}{t(x-t)}\right)dt,\ \Re(\mu)>0. \tag{1.4}$$

For the case $m-1<\Re(\mu)<m$ where $m=1,2,\ldots$, it follows

$$\begin{aligned} &\mathfrak{D}_x^{\mu}\{f(x);p\} \\ &= \frac{d^m}{dx^m}\mathfrak{D}_x^{\mu-m}\Big\{f(x);p\Big\} \\ &= \frac{d^m}{dx^m}\left\{\frac{1}{\Gamma(-\mu+m)}\int_0^x f(t)(x-t)^{-\mu+m-1}\exp\left(-\frac{px^2}{t(x-t)}\right)dt\right\},\ \Re(\mu)>0. \end{aligned} \tag{1.5}$$

An extension of fractional derivative operator established in [1] is given by

$$\begin{aligned} &\mathfrak{D}_x^{\mu}\{f(x);p,q\} \\ &= \frac{1}{\Gamma(-\mu)}\int_0^x f(t)(x-t)^{-\mu-1}\exp\left(-\frac{px}{t}-\frac{qx}{(x-t)}\right)dt,\ \Re(\mu)>0. \end{aligned} \tag{1.6}$$

For example,

$$\mathfrak{D}_x^{\mu}\left\{x^{\nu};p,q\right\}_{x=1} = \frac{B_{p,q}(\nu+1,-\mu)}{\Gamma(-\mu)},$$

where $B_{p,q}(x,y)$ is the extended beta function (see [9]) defined by

$$B_{p,q}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathcal{E}_{p,q}(t)dt,\ x,y,p,q\in\mathbb{C},\ \Re(p),\ \Re(q)>0,$$

where $\mathcal{E}_{p,q}(t) = e^{-\frac{p}{t}-\frac{q}{1-t}}$. For $p=q$ we denote $B_{p,q}$ by B_p and for $p=q=0$, we get classical beta function defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt,\ (\Re(x)>0,\ \Re(y)>0). \tag{1.7}$$

For the case $m-1<\Re(\mu)<m$ where $m=1,2,\ldots$, it follows

$$\begin{aligned} \mathfrak{D}_x^{\mu}\{f(x);p,q\} &= \frac{d^m}{dx^m}\mathfrak{D}_x^{\mu-m}\Big\{f(x);p,q\Big\} \\ &= \frac{d^m}{dx^m}\Big\{\frac{1}{\Gamma(-\mu+m)}\int_0^x f(t)(x-t)^{-\mu+m-1} \\ &\quad\times\exp\left(-\frac{px}{t}-\frac{qx}{(x-t)}\right)dt\Big\},\ \Re(\mu)>0. \end{aligned} \tag{1.8}$$

Recently, Rahman et al. [16] define an extension of extended R-L fractional derivative of order μ as

$$\mathfrak{D}_x^{\mu}\{f(x);p,q,\lambda,\rho\} = \frac{1}{\Gamma(-\mu)}\int_0^x f(t)(x-t)^{-\mu-1}$$

$$\times\, {}_1F_1\Big[\lambda;\rho;-\frac{px}{t}\Big]\, {}_1F_1\Big[\lambda;\rho;-\frac{qx}{(x-t)}\Big]dt,\ \Re(\mu)>0, \tag{1.9}$$

where ${}_1F_1$ is the confluent hypergeometric function.

For the case $m-1<\Re(\mu)<m$ where $m=1,2,\ldots$, it follows

(1.10)

$$\begin{aligned}&\mathfrak{D}_x^{\mu}\{f(x);p,q,\lambda,\rho\}\\ =&\frac{d^m}{dx^m}\mathfrak{D}_x^{\mu-m}\Big\{f(x);p,q,\lambda,\rho\Big\}\\ =&\frac{d^m}{dx^m}\Big\{\frac{1}{\Gamma(-\mu+m)}\int_0^x f(t)(x-t)^{-\mu+m-1}\,{}_1F_1\Big[\lambda;\rho;-\frac{px}{t}\Big]\,{}_1F_1\Big[\lambda;\rho;-\frac{qx}{(x-t)}\Big]dt\Big\},\end{aligned}$$

where $\Re(\mu)>0$, $\Re(p)>0$ and $\Re(q)>0$. It is clear that when $\lambda=\rho$, then (1.9) reduce to (1.6).

The Gauss hypergeometric function which is defined (see [17]) as

$$ {}_2F_1(\sigma_1,\sigma_2;\sigma_3;z)=\sum_{n=0}^{\infty}\frac{(\sigma_1)_n(\sigma_2)_n}{(\sigma_3)_n}\frac{z^n}{n!},(|z|<1), \tag{1.11}$$

where $(\sigma)_n$ is the Pochhammer symbol and $\sigma_1,\sigma_2,\sigma_3\in\mathbb{C}$ and $\sigma_3\neq 0,-1,-2,-3,\ldots$. The integral representation of hypergeometric function is defined by

(1.12)

$$ {}_2F_1(\sigma_1,\sigma_2;\sigma_3;z)=\frac{\Gamma(\sigma_3)}{\Gamma(\sigma_2)\Gamma(\sigma_3-\sigma_2)}\int_0^1 t^{\sigma_2-1}(1-t)^{\sigma_3-\sigma_2-1}(1-zt)^{-\sigma_1}dt,$$

$(\Re(\sigma_3)>\Re(\sigma_2)>0,\ |\arg(1-z)|<\pi)$.

The Apell series or bivariate hypergeometric series and its integral representation is respectively defined by

$$F_1(\sigma_1,\sigma_2,\sigma_3;\sigma_4;x,y)=\sum_{m,n=0}^{\infty}\frac{(\sigma_1)_{m+n}(\sigma_2)_m(\sigma_3)_n x^m y^n}{(\sigma_4)_{m+n}m!n!} \tag{1.13}$$

for all $\sigma_1,\sigma_2,\sigma_3,\sigma_4\in\mathbb{C}$, $\sigma_4\neq 0,-1,-2,-3,\ldots$, $|x|<1,|y|<1$.

$$\begin{aligned}F_1\left(\sigma_1,\sigma_2,\sigma_3,\sigma_4;x,y\right)&=\frac{\Gamma(\sigma_4)}{\Gamma(\sigma_1)\Gamma(\sigma_4-\sigma_1)}\\ &\times\int_0^1 t^{\sigma_1-1}(1-t)^{\sigma_4-\sigma_1-1}(1-xt)^{-\sigma_2}(1-yt)^{-\sigma_3}dt,\end{aligned} \tag{1.14}$$

$\Re(\sigma_4)>\Re(\sigma_1)>0$, $|\arg(1-x)|<\pi$ and $|\arg(1-y)|<\pi$.

Chaudhry et al. [2] introduced the extended beta function is defined by

$$B(\sigma_1,\sigma_2;p)=B_p(\sigma_1,\sigma_2)=\int_0^1 t^{\sigma_1-1}(1-t)^{\sigma_2-1}e^{-\frac{p}{t(1-t)}}dt, \tag{1.15}$$

where $\Re(p)>0$, $\Re(\sigma_1)>0$, $\Re(\sigma_2)>0$, respectively. When $p=0$, then $B(\sigma_1,\sigma_2;0)=B(\sigma_1,\sigma_2)$.

The extended hypergeometric function introduced in [3] by using the definition of extended beta function $B_p(\delta_1, \delta_2)$ as follows:

$$F_p(\sigma_1, \sigma_2; \sigma_3; z) = \sum_{n=0}^{\infty} \frac{B_p(\sigma_2 + n, \sigma_3 - \sigma_2)}{B(\sigma_2, \sigma_3 - \sigma_2)} (\sigma_1)_n \frac{z^n}{n!}, \tag{1.16}$$

where $p \geq 0$ and $\Re(\sigma_3) > \Re(\sigma_2) > 0$, $|z| < 1$.

In the same paper, they defined the following integral representations of extended hypergeometric and confluent hypergeometric functions as

$$\begin{aligned} F_p(\sigma_1, \sigma_2; \sigma_3; z) &= \frac{1}{B(\sigma_2, \sigma_3 - \sigma_2)} \\ &\times \int_0^1 t^{\sigma_2 - 1}(1-t)^{\sigma_3 - \sigma_2 - 1}(1 - zt)^{-\sigma_1} \exp\left(\frac{-p}{t(1-t)}\right) dt, \end{aligned} \tag{1.17}$$

$$(p \geq 0,\ \Re(\sigma_3) > \Re(\sigma_2) > 0,\ |\arg(1-z)| < \pi).$$

The extended Appell's function is defined by (see [12])

$$F_1(\sigma_1, \sigma_2, \sigma_3; \sigma_4; x, y; p) = \sum_{n=0}^{\infty} \frac{B_p(\sigma_1 + m + n, \sigma_4 - \sigma_1)}{B(\sigma_1, \sigma_4 - \sigma_1)} (\sigma_2)_m (\sigma_3)_n \frac{x^m y^n}{m! n!}, \tag{1.18}$$

where $p \geq 0$ and $\Re(\sigma_4) > \Re(\sigma_1) > 0$ and $|x|, |y| < 1$.

They [12] defined its integral representation by

$$\begin{aligned} & F_1(\sigma_1, \sigma_2, \sigma_3; \sigma_4; z; p) \\ &= \frac{1}{B(\sigma_1, \sigma_4 - \sigma_1)} \\ &\quad \times \int_0^1 t^{\sigma_1 - 1}(1-t)^{\sigma_4 - \sigma_1 - 1}(1 - xt)^{-\sigma_2}(1 - yt)^{-\sigma_3} \exp\left(\frac{-p}{t(1-t)}\right) dt, \end{aligned} \tag{1.19}$$

$$(\Re(p) > 0,\ \Re(\sigma_4) > \Re(\sigma_1) > 0,\ |\arg(1-x)| < \pi,\ |\arg(1-y)| < \pi).$$

It is clear that when $p = 0$, then the equations (1.16)-(1.19) reduce to the well known hypergeometric, confluent hypergeometric and Appell's series and their integral representation respectively.

Very recently Shadab et al. [19] introduced a new and modified extension of beta function as:

$$B_p^{\alpha}(\sigma_1, \sigma_2) = \int_0^1 t^{\sigma_1 - 1}(1-t)^{\sigma_2 - 1} E_{\alpha}\left(-\frac{p}{t(1-t)}\right) dt, \tag{1.20}$$

where $\Re(\sigma_1) > 0$, $\Re(\sigma_2) > 0$ and $E_{\alpha}(\cdot)$ is Mittag-Leffler function defined by

$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}. \tag{1.21}$$

Obviously, when $\alpha = 1$ then $B_p^1(x, y) = B_p(x, y)$ is the extended beta function (see [2]). Similarly, when $\alpha = 1$ and $p = 0$, then $B_0^1(x, y) = B_0(x, y)$ is the classical beta function.

They [19] also defined extended hypergeometric function and its integral representation

$$\begin{aligned} F_p^{\alpha}(\sigma_1,\sigma_2;\sigma_3;z) &= {}_2F_1\left(\sigma_1,\sigma_2;\sigma_3;z;p,\alpha\right)\\ &= \sum_{n=0}^{\infty}(\sigma_1)_n\frac{B_p^{\alpha}(\sigma_2+n,\sigma_3-\sigma_2)}{B(\sigma_2,\sigma_3-\sigma_2)}\frac{z^n}{n!}\\ &= \sum_{n=0}^{\infty}(\sigma_1)_n\frac{B(\sigma_2+n,\sigma_3-\sigma_2;p,\alpha)}{B(\sigma_2,\sigma_3-\sigma_2)}\frac{z^n}{n!}, \end{aligned} \tag{1.22}$$

where $p,\alpha\geq 0$, $\sigma_1,\sigma_2,\sigma_3\in\mathbb{C}$ and $|z|<1$.

$$\begin{aligned} F_p^{\alpha}(\sigma_1,\sigma_2;\sigma_3;z) &= \frac{1}{B(\sigma_2;\sigma_3-\sigma_2)}\\ &\times\int_0^1 t^{\sigma_2-1}(1-t)^{\sigma_3-\sigma_2-1}(1-tz)^{-\sigma_1}E_{\alpha}\left(-\frac{p}{t(1-t)}\right)dt, \end{aligned} \tag{1.23}$$

where $\Re(p)>0$, $\Re(\alpha)>0$, $\Re(\sigma_3)>\Re(\sigma_2)>0$. Obviously when $\alpha=1$, then the hypergeometric function (1.23) will reduce to the extended hypergeometric function (1.17) and similarly when $\alpha=1$ and $p=0$ then the hypergeometric function (1.23) will reduce to the hypergeometric function (1.12).

For various extensions and generalizations of beta function and hypergeometric functions the interested readers may refer to the recent work of researchers (see e.g., [4, 10, 12, 13]).

2. Extension of Appell's functions and its integral representations

We start the section by deriving the relation of (1.20) with multi index Mittag-Leffler function [6] as follows:

Proposition 2.1. *For $\Re(p),\Re(\sigma_1),\Re(\sigma_2),\Re(\alpha)>0$, $|p|<1$ the following relation holds true*:

$$B_p^{\alpha}(\sigma_1,\sigma_2)=\frac{\pi\sin\pi(\sigma_1+\sigma_2)}{(\sin\pi\sigma_1)(\sin\pi\sigma_2)}E_{(\alpha,1),(1,1-\sigma_1),(1,1-\sigma_2)}^{(2,1-\sigma_1-\sigma_2)}(-p), \tag{2.1}$$

where $E_{(\alpha,1),(1,1-\sigma_1),(1,1-\sigma_2)}^{(2,1-\sigma_1-\sigma_2)}(-p)$ is the multi index Mittag-Leffler function [6].

Proof. Using the definition of beta function and reduction theorem of gamma function, we get

$$\begin{aligned} B_p^{\alpha}(\sigma_1,\sigma_2) &= \sum_{n=0}^{\infty}\frac{(-p)^n}{\Gamma(\alpha n+1)}\int_0^1 t^{\sigma_1-1}(1-t)^{\sigma_2-1}\frac{1}{t^n(1-t)^n}dt\\ &= \sum_{n=0}^{\infty}\frac{(-p)^n}{\Gamma(\alpha n+1)}\int_0^1 t^{\sigma_1-n-1}(1-t)^{\sigma_2-n-1}dt\\ &= \sum_{n=0}^{\infty}\frac{(-p)^n}{\Gamma(\alpha n+1)}B(\sigma_1-n,\sigma_2-n) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{(-p)^n}{\Gamma(\alpha n+1)} \frac{\Gamma(\sigma_1-n)\Gamma(\sigma_2-n)}{\Gamma(\sigma_1+\sigma_2-2n)}\\
&= \frac{\Gamma(-\sigma_1)\Gamma(1+\sigma_1)\Gamma(-\sigma_2)\Gamma(1+\sigma_2)}{\Gamma(-\sigma_1-\sigma_2)\Gamma(1+\sigma_1+\sigma_2)}\\
&\times \sum_{n=0}^{\infty} \frac{\Gamma(2n+1-\sigma_1-\sigma_2)(-p)^n}{\Gamma(\alpha n+1)\Gamma(n+1-\sigma_1)\Gamma(n+1-\sigma_2)}.
\end{aligned}
$$

Now, using the Euler's reflection formula on gamma function,

$$
\Gamma(r)\Gamma(1-r) = \frac{\pi}{\sin(\pi r)}, \tag{2.2}
$$

we get the desired result. □

Next, we used the definition (1.20) and consider the following modified extension Appell's functions.

Definition 2.1. The modified extended Appell's function F_1 is defined by

$$
\begin{aligned}
F^{\alpha}_{1,p}(\sigma_1,\sigma_2,\sigma_3;\sigma_4;x,y) &= F_1(\sigma_1,\sigma_2,\sigma_3;\sigma_4;x,y;p,\alpha)\\
&= \sum_{m,n=0}^{\infty} (\sigma_2)_m(\sigma_3)_n \frac{B^{\alpha}_p(\sigma_1+m+n,\sigma_4-\sigma_1)}{B(\sigma_1,\sigma_4-\sigma_1)} \frac{x^m}{m!}\frac{y^n}{n!}\\
&= \sum_{m,n=0}^{\infty} (\sigma_2)_m(\sigma_3)_n \frac{B(\sigma_1+m+n,\sigma_4-\sigma_1;p,\alpha)}{B(\sigma_1,\sigma_4-\sigma_1)} \frac{x^m}{m!}\frac{y^n}{n!}, \qquad (2.3)
\end{aligned}
$$

where $p,\alpha \geq 0$, $\sigma_1,\sigma_2,\sigma_3,\sigma_4 \in \mathbb{C}$ and $|x|<1$, $|y|<1$.

Remark 2.1. Setting $\alpha = 1$ in (2.3), then we get the extended Appell's functions (see [12]).

Now, we derive the following proposition.

Proposition 2.2. *For $p,q>0$, $0<x,y<\frac{1}{2}$, the following inequality holds true:*

$$
B_{p,q}(x,y) \leq (2p)^{\frac{2x-1}{2}}(2q)^{\frac{2y-1}{2}}\sqrt{\Gamma(-2x+1,2p)\Gamma(-2y+1,2q)}. \tag{2.4}
$$

Proof. Applying the Cauchy-Schwarz integral inequality, we obtain:

$$
\begin{aligned}
B_{p,q}(x,y) &= \int_0^1 t^{x-1}(1-t)^{y-1}e^{-\frac{p}{t}}e^{-\frac{q}{1-t}}dt\\
&\leq \left(\int_0^1 (t^{x-1}e^{-p/t})^2dt\right)^{1/2}\left(\int_0^1 [(1-t)^{y-1}e^{-q/(1-t)}]^2dt\right)^{1/2}\\
&= \left(\int_1^{\infty} t^{-2x}e^{-2pt}dt\right)^{1/2}\left(\int_1^{\infty} t^{-2y}e^{-2qt}dt\right)^{1/2}\\
&= \left[(2p)^{2x-1}\Gamma(-2x+1,2p)\right]^{1/2}\left[(2q)^{2y-1}\Gamma(-2y+1,2q)\right]^{1/2}
\end{aligned}
$$

$$= (2p)^{\frac{2x-1}{2}}(2q)^{\frac{2y-1}{2}}\sqrt{\Gamma(-2x+1,2p)\Gamma(-2y+1,2q)},$$

where $\Gamma(x,y)$ is the incomplete gamma function. □

Theorem 2.1. *The following integral representation holds true for* (2.3)

$$\begin{aligned}(2.5)\qquad & F_1(\sigma_1,\sigma_2,\sigma_3;\sigma_4;x,y;p,\alpha)\\ &= \frac{1}{B(\sigma_1;\sigma_4-\sigma_1)}\int_0^1 t^{\sigma_1-1}(1-t)^{\sigma_4-\sigma_1-1}(1-tx)^{-\sigma_2}(1-ty)^{-\sigma_3}\\ &\quad\times E_\alpha\left(-\frac{p}{t(1-t)}\right)dt,\end{aligned}$$

where $\Re(p)>0$, $\Re(\alpha)>0$, $\Re(\sigma_4)>\Re(\sigma_1)>0$.

Proof. Using the definition (1.20) in (2.3), we have

$$\begin{aligned}(2.6)\qquad & F_1(\sigma_1,\sigma_2,\sigma_3;\sigma_4;x,y;p,\alpha)\\ &= \frac{1}{B(\sigma_1;\sigma_4-\sigma_1)}\int_0^1 t^{\sigma_1-1}(1-t)^{\sigma_4-\sigma_1-1}\\ &\quad\times E_\alpha\left(-\frac{p}{t(1-t)}\right)\left(\sum_{m,n=0}^{\infty}\frac{(\sigma_2)_m(\sigma_3)_n(tx)^m(ty)^n}{m!n!}\right)dt.\end{aligned}$$

Since

$$(2.7)\qquad \sum_{m,n=0}^{\infty}\frac{(\sigma_2)_m(\sigma_3)_n(tx)^m(ty)^n}{m!n!} = (1-tx)^{-\sigma_2}(1-ty)^{-\sigma_3}.$$

Thus by using (2.7) in (2.6), we get the desired result. □

3. Extension of fractional derivative operator

In this section, we define a new and modified extension of Riemann-Liouville fractional derivative and obtain its related results.

Definition 3.1.

$$(3.1)\quad \mathfrak{D}_{z;p}^{\mu;\alpha}\{f(z)\} = \frac{1}{\Gamma(-\mu)}\int_0^z f(t)(z-t)^{-\mu-1}E_\alpha\left(-\frac{pz^2}{t(z-t)}\right)dt,\ \Re(\mu)>0.$$

For the case $m-1<\Re(\mu)<m$ where $m=1,2,\ldots$, it follows

(3.2)

$$\begin{aligned}&\mathfrak{D}_{z;p}^{\mu;\alpha}\{f(z)\}\\ &= \frac{d^m}{dz^m}\mathfrak{D}_{z;p}^{\mu-m;\alpha}\Big\{f(z)\Big\}\\ &= \frac{d^m}{dz^m}\Big\{\frac{1}{\Gamma(-\mu+m)}\int_0^z f(t)(z-t)^{-\mu+m-1}E_\alpha\left(-\frac{pz^2}{t(z-t)}\right)dt\Big\},\ \Re(\mu)>0.\end{aligned}$$

Remark 3.1. Obviously if $\alpha = 1$, then (3.1) and (3.2) respectively reduces to the extended fractional derivative (1.4) and (1.5). Similarly, if we set $\alpha = 1$ and $p = 0$ we get (1.1) and (1.2).

Now, we prove some theorems involving the modified extension of fractional derivative operator.

Theorem 3.1. *The following formula hold true,*

$$\mathfrak{D}_z^{\mu}\{z^{\eta}; p, \alpha\} = \frac{B_p^{\alpha}(\eta + 1, -\mu)}{\Gamma(-\mu)} z^{\eta-\mu}, \ \Re(\mu) > 0. \tag{3.3}$$

Proof. From (3.1), we have

$$\mathfrak{D}_z^{\mu}\{z^{\eta}; p, \alpha\} = \frac{1}{\Gamma(-\mu)} \int_0^z t^{\eta}(z - t)^{-\mu-1} E_{\alpha}\left(-\frac{pz^2}{t(z-t)}\right) dt. \tag{3.4}$$

Substituting $t = uz$ in (3.4), we get

$$\begin{aligned}\mathfrak{D}_z^{\mu}\{z^{\eta}; p, \alpha\} &= \frac{1}{\Gamma(-\mu)} \int_0^1 (uz)^{\eta}(z - uz)^{-\mu-1} E_{\alpha}\left(-\frac{pz^2}{uz(z-uz)}\right) dt \\ &= \frac{z^{\eta-\mu}}{\Gamma(-\mu)} \int_0^1 u^{\eta}(1 - u)^{-\mu-1} E_{\alpha}\left(-\frac{p}{u(1-u)}\right) dt.\end{aligned}$$

In view of (1.20) to the above equation, we get the required result. □

Theorem 3.2. *Let* $\Re(\mu) > 0$ *and assume that the function* $f(z)$ *is analytic at the origin with its Maclaurin expansion given by* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *where* $|z| < \delta$ *for some* $\delta \in \mathbb{R}^+$*. Then*

$$\begin{aligned}\mathfrak{D}_{z;p}^{\mu;\alpha}\{f(z)\} = \mathfrak{D}_z^{\mu}\{f(z); p, \alpha\} &= \sum_{n=0}^{\infty} a_n \mathfrak{D}_z^{\mu}\{z^n; p, \alpha\} \\ &= \frac{1}{\Gamma(-\mu)} \sum_{n=0}^{\infty} a_n B_p^{\alpha}(n + 1, -\mu) z^{n-\mu}.\end{aligned} \tag{3.5}$$

Proof. Using the series expansion of the function $f(z)$ in (3.1) gives

$$\mathfrak{D}_z^{\mu}\{f(z); p, \alpha\} = \frac{1}{\Gamma(-\mu)} \int_0^z \sum_{n=0}^{\infty} a_n t^n (z - t)^{-\mu-1} E_{\alpha}\left(-\frac{pz^2}{t(z-t)}\right) dt.$$

The series is uniformly convergent on any closed disk centered at the origin with its radius smaller than δ, so does on the line segment from 0 to a fixed z for $|z| < \delta$. Thus it guarantee terms by terms integration as follows

$$\begin{aligned}\mathfrak{D}_z^{\mu}\{f(z); p, \alpha\} &= \sum_{n=0}^{\infty} a_n \Big\{ \frac{1}{\Gamma(-\mu)} \int_0^z t^n (z - t)^{-\mu-1} E_{\alpha}\left(-\frac{pz^2}{uz(z-uz)}\right) dt \\ &= \sum_{n=0}^{\infty} a_n \mathfrak{D}_z^{\mu}\{z^n; p, \alpha\}.\end{aligned}$$

Now, applying Theorem 3.1, we get

$$\mathfrak{D}_z^{\mu}\{f(z);p,\alpha\} = \frac{1}{\Gamma(-\mu)}\sum_{n=0}^{\infty} a_n B_p^{\alpha}(n+1,-\mu)z^{n-\mu},\ \Re(\mu)>0,$$

which is the required proof. □

Example 3.1. The following result holds true:

$$\mathfrak{D}_z^{\mu}\{e^z;p,\alpha\} = \frac{z^{-\mu}}{\Gamma(-\mu)}\sum_{n=0}^{\infty} B_p^{\alpha}(n+1,-\mu)\frac{z^n}{n!}. \tag{3.6}$$

Using the power series of $\exp(z)$ and applying Theorem 3.2, we have

$$\mathfrak{D}_z^{\mu}\{e^z;p,\alpha\} = \sum_{n=0}^{\infty}\frac{1}{n!}\mathfrak{D}_z^{\mu}\{z^n;p,\alpha\}. \tag{3.7}$$

Now, applying Theorem 3.1, we get the desired result.

Theorem 3.3. *The following formula holds true*:

$$\mathfrak{D}_z^{\eta-\mu}\{z^{\eta-1}(1-z)^{-\beta};p,\alpha\} = \frac{\Gamma(\eta)}{\Gamma(\mu)}z^{\mu-1}\,{}_2F_{1;p}^{\alpha}\left(\beta,\mu;\eta;z\right), \tag{3.8}$$

where $\Re(\mu)>\Re(\eta)>0$ *and* $|z|<1$.

Proof. Apply the definition (1.4), we have

$$\begin{aligned}&\mathfrak{D}_z^{\eta-\mu}\{z^{\eta-1}(1-z)^{-\beta};p,\alpha\}\\ &= \frac{1}{\Gamma(\mu-\eta)}\int_0^z t^{\eta-1}(1-t)^{-\beta}(z-t)^{\mu-\eta-1}E_{\alpha}\left(-\frac{pz^2}{t(z-t)}\right)dt\\ &= \frac{z^{\mu-\eta-1}}{\Gamma(\mu-\eta)}\int_0^z t^{\eta-1}(1-t)^{-\beta}(1-\frac{t}{z})^{\mu-\eta-1}E_{\alpha}\left(-\frac{pz^2}{t(z-t)}\right)dt.\end{aligned}$$

Substituting $t=zu$ in the above equation, we get

$$\begin{aligned}&\mathfrak{D}_z^{\eta-\mu}\{z^{\eta-1}(1-z)^{-\beta};p,\alpha\}\\ &= \frac{z^{\mu-1}}{\Gamma(\mu-\eta)}\int_0^1 u^{\eta-1}(1-uz)^{-\beta}(1-u)^{\mu-\eta-1}E_{\alpha}\left(-\frac{pz^2}{uz(z-uz)}\right)du\\ &= \frac{z^{\mu-1}}{\Gamma(\mu-\eta)}\int_0^1 u^{\eta-1}(1-uz)^{-\beta}(1-u)^{\mu-\eta-1}E_{\alpha}\left(-\frac{p}{u(1-u)}\right)du.\end{aligned}$$

Using (1.23) and after simplification we get the required proof. □

Theorem 3.4. *The following formula holds true*:

$$\begin{aligned}&\mathfrak{D}_z^{\eta-\mu}\{z^{\eta-1}(1-az)^{-\alpha}(1-bz)^{-\beta};p,\alpha\}\\ &= \frac{\Gamma(\eta)}{\Gamma(\mu)}z^{\mu-1}F_1\left(\eta,\alpha,\beta;\mu;az,bz;p,\alpha\right),\end{aligned} \tag{3.9}$$

where $\Re(\mu)>\Re(\eta)>0$, $\Re(\alpha)>0$, $\Re(\beta)>0$, $|az|<1$ *and* $|bz|<1$.

Proof. Consider the following power series expansion

$$(1-az)^{-\alpha}(1-bz)^{-\beta}=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\alpha)_m(\beta)_n\frac{(az)^m}{m!}\frac{(bz)^n}{n!}.$$

Now, applying Theorem 3.3, we obtain

$$\begin{aligned}&\mathfrak{D}_z^{\eta-\mu}\{z^{\eta-1}(1-az)^{-\alpha}(1-bz)^{-\beta};p,\alpha\}\\ &=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\alpha)_m(\beta)_n\frac{(a)^m}{m!}\frac{(b)^n}{n!}\mathfrak{D}_z^{\eta-\mu}\{z^{\eta+m+n-1};p,\alpha\}.\end{aligned}$$

Using Theorem 3.1, we have

$$\begin{aligned}&\mathfrak{D}_z^{\eta-\mu}\{z^{\eta-1}(1-az)^{-\alpha}(1-bz)^{-\beta};p,\alpha\}\\ &=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\alpha)_m(\beta)_n\frac{(a)^m}{m!}\frac{(b)^n}{n!}\frac{B_p^{\alpha}(\eta+m+n,\mu-\eta)}{\Gamma(\mu-\eta)}z^{\mu+m+n-1}.\end{aligned}$$

Now, applying (2.3), we get

$$\mathfrak{D}_z^{\eta-\mu}\{z^{\eta-1}(1-az)^{-\alpha}(1-bz)^{-\beta};p,\alpha\}=\frac{\Gamma(\eta)}{\Gamma(\mu)}z^{\mu-1}F_1\left(\eta,\alpha,\beta;\mu;az,bz;p,\alpha\right).$$

□

Theorem 3.5. *The following Mellin transform formula holds true:*

$$M\Big\{\mathfrak{D}_{z;p}^{\mu;\alpha}(z^{\eta});p\to r\Big\}=\frac{\pi}{\sin(\pi r)}\frac{z^{\eta-\mu}}{\Gamma(-\mu)\Gamma(1-r\alpha)}B(\eta+r+1,-\mu+r), \tag{3.10}$$

where $\Re(\eta)>-1$, $\Re(\mu)>0$, $\Re(r)>0$, $\Re(r)>0$.

Proof. Applying the Mellin transform on definition (3.1), we have

$$\begin{aligned}&M\Big\{\mathfrak{D}_{z;p}^{\mu;\alpha}(z^{\eta});p\to r\Big\}\\ &=\int_0^{\infty}p^{r-1}\mathfrak{D}_{z;p}^{\mu;\alpha}(z^{\eta})dp\\ &=\frac{1}{\Gamma(-\mu)}\int_0^{\infty}p^{r-1}\Big\{\int_0^z t^{\eta}(z-t)^{-\mu-1}E_{\alpha}\left(-\frac{pz^2}{t(z-t)}\right)dt\Big\}dp\\ &=\frac{z^{-\mu-1}}{\Gamma(-\mu)}\int_0^{\infty}p^{r-1}\Big\{\int_0^z t^{\eta}(1-\frac{t}{z})^{-\mu-1}E_{\alpha}\left(-\frac{pz^2}{t(z-t)}\right)dt\Big\}dp\\ &=\frac{z^{\eta-\mu}}{\Gamma(-\mu)}\int_0^{\infty}p^{r-1}\Big\{\int_0^1 u^{\eta}(1-u)^{-\mu-1}E_{\alpha}\left(-\frac{p}{u(1-u)}\right)du\Big\}dp.\end{aligned}$$

From the uniform convergence of the integral, the order of integration can be interchanged. Thus, we have

$$\begin{aligned}&M\Big\{\mathfrak{D}_{z;p}^{\mu;\alpha}(z^{\eta});p\to r\Big\}\\ &=\frac{z^{\eta-\mu}}{\Gamma(-\mu)}\int_0^1 u^{\eta}(1-u)^{-\mu-1}\left(\int_0^{\infty}p^{r-1}E_{\alpha}\left(-\frac{p}{u(1-u)}\right)dp\right)du.\end{aligned} \tag{3.11}$$

Letting $v = \frac{p}{u(1-u)}$, (3.11) reduces to

$$M\Big\{\mathfrak{D}^{\mu;\alpha}_{z;p}(z^\eta); p \to r\Big\} = \frac{z^{\eta-\mu}}{\Gamma(-\mu)} \int_0^1 u^{\eta+r}(1-u)^{-\mu+r-1} \left(\int_0^\infty v^{r-1}\, E_\alpha\,(-v)\, dv\right) du.$$

By using the following formula,

$$\int_0^\infty v^{r-1} E^{\delta}_{\alpha,\gamma}(-wv)dv = \frac{\Gamma(r)\Gamma(\delta-r)}{\Gamma(\delta)w^r\Gamma(\gamma-r\alpha)} \tag{3.12}$$

for $\gamma = \delta = 1$ and $w = 1$, we have

$$\begin{aligned} M\Big\{\mathfrak{D}^{\mu;\alpha}_{z;p}(z^\eta); p \to r\Big\} &= \frac{z^{\eta-\mu}\Gamma(r)\Gamma(1-r)}{\Gamma(-\mu)\Gamma(1-r\alpha)} \int_0^1 u^{\eta+r}(1-u)^{-\mu+r-1} du \\ &= \frac{z^{\eta-\mu}\Gamma(r)\Gamma(1-r)}{\Gamma(-\mu)\Gamma(1-r\alpha)} B(\eta+r+1, -\mu+r). \end{aligned}$$

Now, using (2.2) we get the desired result. □

Theorem 3.6. *The following Mellin transform formula holds true*:

$$\begin{aligned} M\Big\{\mathfrak{D}^{\mu;\alpha}_{z;p}((1-z)^\alpha); p \to r\Big\} &= z^{-\mu} \frac{\pi}{\sin(\pi r)} \frac{B(1+r, -\mu+r)}{\Gamma(-\mu)\Gamma(1-r\alpha)} \\ &\quad \times {}_2F_1\,(\lambda, r+1; 1-\mu+2r; z)\,, \end{aligned} \tag{3.13}$$

where $\Re(p) > 0$, $\Re(\mu) < 0$, $\Re(r) > 0$.

Proof. Applying Theorem 3.5 with $\eta = n$, we can write

$$\begin{aligned} &M\Big\{\mathfrak{D}^{\mu;\alpha}_{z;p}((1-z)^\lambda); p \to r\Big\} \\ &= \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} M\Big\{\mathfrak{D}^{\mu;\alpha}_{z;p}(z^n); p \to r\Big\} \\ &= \frac{\Gamma(r)\Gamma(1-r)}{\Gamma(-\mu)\Gamma(1-r\alpha)} \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} B(n+r+1, -\mu+r) z^{n-\mu} \\ &= z^{-\mu} \frac{\Gamma(r)\Gamma(1-r)}{\Gamma(-\mu)\Gamma(1-r\alpha)} \sum_{n=0}^{\infty} B(n+r+1, -\mu+r) \frac{(\lambda)_n z^n}{n!}. \end{aligned}$$

In view of (2.2), we obtain the required result. □

4. Generating relations

In this section, we derive generating relations of linear and bilinear type for the extended hypergeometric functions.

Theorem 4.1. *The following generating relation holds true*:

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,{}_2F^{\alpha}_{1;p}\left(\lambda+n,\beta;\gamma;z\right)t^n=(1-t)^{-\lambda}\,{}_2F^{\alpha}_{1;p}\left(\lambda,\beta;\gamma;\frac{z}{1-t}\right),\tag{4.1}$$

where $|z|<\min(1,|1-t|)$, $\Re(\alpha)>0$, $\Re(\gamma)>\Re(\beta)>0$.

Proof. Consider the following series identity

$$[(1-z)-t]^{-\lambda}=(1-t)^{-\lambda}[1-\frac{z}{1-t}]^{-\lambda}.$$

Thus, the power series expansion yields

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}(1-z)^{-\lambda}\left(\frac{t}{1-z}\right)^n=(1-t)^{-\lambda}[1-\frac{z}{1-t}]^{-\lambda}.\tag{4.2}$$

Multiplying both sides of (4.2) by $z^{\beta-1}$ and then applying the operator $\mathfrak{D}^{\beta-\gamma;\alpha}_{z;p}$ on both sides, we have

$$\mathfrak{D}^{\beta-\gamma;\alpha}_{z;p}\Big[\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}(1-z)^{-\lambda}\left(\frac{t}{1-z}\right)^n z^{\beta-1}\Big]$$
$$=(1-t)^{-\lambda}\mathfrak{D}^{\beta-\gamma;\alpha}_{z;p}\left[z^{\beta-1}\left(1-\frac{z}{1-t}\right)^{-\lambda}\right].$$

Interchanging the order of summation and the operator $\mathfrak{D}^{\beta-\gamma;\alpha}_{z;p}$, we have

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\mathfrak{D}^{\beta-\gamma;\alpha}_{z;p}\left[z^{\beta-1}(1-z)^{-\lambda-n}\right]t^n$$
$$=(1-t)^{-\lambda}\mathfrak{D}^{\beta-\gamma;\alpha}_{z;p}\left[z^{\beta-1}\left(1-\frac{z}{1-t}\right)^{-\lambda}\right].$$

Thus by applying Theorem 3.3, we obtain the required result. □

Theorem 4.2. *The following generating relation holds true*:

$$\sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}\,{}_2F^{\alpha}_{1;p}\left(\delta-n,\beta;\gamma;z\right)t^n=(1-t)^{-\beta}F_1\left(\lambda,\delta,\beta;\gamma;-\frac{zt}{1-t};p,\alpha\right),\tag{4.3}$$

where $|z|<\frac{1}{1+|t|}$, $\Re(\delta)>0$, $\Re(\beta)>0$, $\Re(\gamma)>\Re(\lambda)>0$.

Proof. Consider the series identity

$$[1-(1-z)t]^{-\beta}=(1-t)^{-\beta}\left[1+\frac{zt}{1-t}\right]^{-\beta}.$$

Using the power series expansion to the left sides, we have

$$\sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}(1-z)^n t^n=(1-t)^{-\beta}\left[1-\frac{-zt}{1-t}\right]^{-\beta}.\tag{4.4}$$

Multiplying both sides of (4.4) by $z^{\alpha-1}(1-z)^{-\delta}$ and applying the operator $\mathfrak{D}_{z;p}^{\lambda-\gamma;\alpha}$ on both sides, we have

$$\mathfrak{D}_{z;p}^{\lambda-\gamma;\alpha}\Big[\sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}z^{\alpha-1}(1-z)^{-\delta+n}t^n\Big]$$
$$=(1-t)^{-\beta}\mathfrak{D}_{z;p}^{\lambda-\gamma;\alpha}\Big[z^{\lambda-1}(1-z)^{-\delta}\left(1-\frac{-zt}{1-t}\right)^{-\beta}\Big],$$

where $\Re(\lambda)>0$ and $|zt|<|1-t|$, thus by Theorem 2.1, we have

$$\sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}\mathfrak{D}_{z;p}^{\lambda-\gamma;\alpha}\Big[z^{\lambda-1}(1-z)^{-\delta+n}\Big]t^n$$
$$=(1-t)^{-\beta}\mathfrak{D}_{z;p}^{\lambda-\gamma;\alpha}\Big[z^{\lambda-1}(1-z)^{-\delta}\left(1-\frac{-zt}{1-t}\right)^{-\beta}\Big].$$

Applying Theorem 3.4 on both sides, we get the required result. □

Theorem 4.3. *The following result holds true*:

$$\mathfrak{D}_{z;p}^{\eta-\mu;\alpha}\Big[z^{\eta-1}E_{\gamma,\delta}^{\mu}(z)\Big]=\frac{z^{\mu-1}}{\Gamma(\mu-\eta)}\sum_{n=0}^{\infty}\frac{(\mu)_n}{\Gamma(\gamma n+\delta)}B_p^{\alpha}(\eta+n,\mu-\eta)\frac{z^n}{n!},\tag{4.5}$$

where $\gamma,\delta,\mu,\alpha\in\mathbb{C}$, $\Re(p)>0$, $\Re(\mu)>\Re(\eta)>0$, $\Re(\lambda)>0$, $\Re(\rho)>0$ *and* $E_{\gamma,\delta}^{\mu}(z)$ *is Mittag-Leffler function* (*see* [15]) *defined as*:

$$E_{\gamma,\delta}^{\mu}(z)=\sum_{n=0}^{\infty}\frac{(\mu)_n}{\Gamma(\gamma n+\delta)}\frac{z^n}{n!}.\tag{4.6}$$

Proof. Using (4.6) in (4.5), we have

$$\mathfrak{D}_{z;p}^{\eta-\mu;\alpha}\Big[z^{\eta-1}E_{\gamma,\delta}^{\mu}(z)\Big]=\mathfrak{D}_{z;p}^{\eta-\mu;\alpha}\Big[z^{\eta-1}\Big\{\sum_{n=0}^{\infty}\frac{(\mu)_n}{\Gamma(\gamma n+\delta)}\frac{z^n}{n!}\Big\}\Big].$$

By Theorem 2.1, we have

$$\mathfrak{D}_{z;p}^{\eta-\mu;\alpha}\Big[z^{\eta-1}E_{\gamma,\delta}^{\mu}(z)\Big]=\sum_{n=0}^{\infty}\frac{(\mu)_n}{\Gamma(\gamma n+\delta)}\Big\{\mathfrak{D}_{z;p}^{\eta-\mu;\alpha}\Big[z^{\eta+n-1}\Big]\Big\}.$$

Applying Theorem 3.1, we get the required proof. □

Remark 4.1. In [9] Mehrez and Tomovski introduced a new p-Mittag-Leffler function defined by

$$E_{\lambda,\gamma,\delta;p}^{(\mu,\eta,\omega)}(z)=\sum_{k=0}^{\infty}\frac{(\mu)_k}{[\Gamma(\gamma k+\delta)]^{\lambda}}\frac{B_p(\eta+k,\omega-\eta)}{B(\eta,\omega-\eta)}\frac{z^k}{k!},(z\in\mathbb{C})\tag{4.7}$$
$$(\mu,\,\eta,\,\omega,\,\gamma,\,\delta,\,\lambda>0,\,\Re(p)>0).$$

Hence we get the following corollary.

Corollary 4.1. *The following formula holds true for* $E^{\mu}_{\gamma,\delta}(z)$:

$$\mathfrak{D}_z^{\eta-\mu}\Big\{z^{\eta-1}E^{\mu}_{\gamma,\delta}(z);p\Big\} = z^{\mu-1}\frac{\Gamma(\eta)}{\Gamma(\mu)}E^{(\mu,\eta,\mu)}_{1,\gamma,\delta;p}(z), \tag{4.8}$$

$$(\mu,\eta,\,\gamma,\,\delta>0,\,\Re(p)>0).$$

Theorem 4.4. *The following result holds true*:

$$\mathfrak{D}_{z;p}^{\eta-\mu;\alpha}\Big\{z^{\eta-1}\,{}_m\Psi_n\left[\begin{array}{c}(\alpha_i,A_i)_{1,m};\\ \\ (\beta_j,B_j)_{1,n};\end{array}\Big|z\right]\Big\} \tag{4.9}$$
$$=\frac{z^{\mu-1}}{\Gamma(\mu-\eta)}\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{m}\Gamma(\alpha_i+A_ik)}{\prod_{j=1}^{n}\Gamma(\beta_j+B_jk)}B_p^{\alpha}(\eta+k,\mu-\eta)\frac{z^k}{k!},$$

where $\Re(p)>0$, $\Re(\alpha)>0$, $\Re(\mu)>\Re(\eta)>0$, $\Re(\lambda)>0$, $\Re(\rho)>0$ *and* ${}_m\Psi_n(z)$ *represents the Fox-Wright function* (*see* [5, pp. 56–58])

$${}_m\Psi_n(z) = {}_m\Psi_n\left[\begin{array}{c}(\alpha_i,A_i)_{1,m};\\ \\ (\beta_j,B_j)_{1,n};\end{array}\Big|z\right]=\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{m}\Gamma(\alpha_i+A_ik)}{\prod_{j=1}^{n}\Gamma(\beta_j+B_jk)}\frac{z^k}{k!}. \tag{4.10}$$

Proof. Applying Theorem 3.1 and followed the same procedure used in Theorem 4.3, we get the desired result. □

Remark 4.2. In [20] Sharma and Devi introduced extended Wright generalized hypergeometric function defined by

$${}_{m+1}\Psi_{n+1}(z;p) = {}_{m+1}\Psi_{n+1}\left[\begin{array}{c}(\alpha_i,A_i)_{1,m},(\gamma,1);\\ \\ (\beta_j,B_j)_{1,n};(c,1)\end{array}\Big|(z;p)\right]$$
$$=\frac{1}{\Gamma(c-\gamma)}\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{m}\Gamma(\alpha_i+A_ik)}{\prod_{j=1}^{n}\Gamma(\beta_j+B_jk)}\frac{B_p(\gamma+k,c-\gamma)z^k}{k!}, \tag{4.11}$$

$$(\Re(p)>0,\,\Re(c)>\Re(\gamma)>0).$$

Hence we get the following corollary.

Corollary 4.2. *The following results holds true for the Fox-Wright hypergeometric function*:

$$\mathfrak{D}_z^{\eta-\mu}\Big\{z^{\eta-1}\,{}_m\Psi_n\left[\begin{array}{c}(\alpha_i,A_i)_{1,m};\\ \\ (\beta_j,B_j)_{1,n};\end{array}\Big|z\right];p\Big\} \tag{4.12}$$
$$=z^{\mu-1}{}_{m+1}\Psi_{n+1}\left[\begin{array}{c}(\alpha_i,A_i)_{1,m},(\eta,1);\\ \\ (\beta_j,B_j)_{1,n};(\mu,1)\end{array}\Big|(z;p)\right],$$

$$(\Re(p)>0,\,\Re(\mu)>\Re(\eta)>0).$$

5. Concluding remarks

In this paper, we established a modified extension of Riemnn-Liouville fractional derivative operator. We conclude that when $\alpha = 1$, then all the results established in this paper will reduce to the results associated with classical Riemnn-Liouville derivative operator (see [12]). Similarly, if we letting $\alpha = 1$ and $p = 0$, then all the results established in this paper will reduce to the results associated with classical Riemnn-Liouville fractional derivative operator (see [5]).

Remark 5.1. The preprint of this paper is available at ‘https://arxiv.org/abs/1801.05001’.

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