

## A NOTE ON CONCIRCULAR STRUCTURE SPACE-TIMES

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ABSTRACT. In this note we show that Lorentzian Conircular Structure manifolds  $(LCS)_n$  coincide with Generalized Robertson-Walker space-times.

*Generalized Robertson-Walker* (GRW) space-times were introduced in 1995 by Alías, Romero and Sánchez [1] as the warped product  $-1 \times_{q^2} M^*$ , where  $(M^*, g^*)$  is a Riemannian submanifold. In other terms, they are Lorentzian manifolds characterised by a metric

$$(1) \quad g_{ij} dx^i dx^j = -(dt)^2 + q(t)^2 g_{\mu\nu}^*(x^1, \dots, x^{n-1}) dx^\mu dx^\nu.$$

They are interesting not only for geometry [2, 5, 9–11], but also for physics: they include relevant space-times such as Robertson-Walker, Einstein-de Sitter, static Einstein, de-Sitter, the Friedmann cosmological models. They are a wide generalization of space-times for cosmological models.

In 2003 A. A. Shaikh [12] introduced the notion of *Lorentzian Conircular Structure*  $(LCS)_n$ . It is a Lorentzian manifold endowed with a unit time-like concircular vector field, i.e.,  $u^i u_i = -1$  and

$$(2) \quad \nabla_k u_j = \varphi(u_k u_j + g_{kj}),$$

where  $\varphi \neq 0$  is a scalar function obeying

$$(3) \quad \nabla_j \varphi = \mu u_j$$

being  $\mu$  a scalar function. Various authors studied the properties of  $(LCS)_n$  manifolds [6, 13–15].

We show that GRW and  $(LCS)_n$  are the same space-times.

We recall few definitions that will be used in this note. The first ones are the definitions of “torse-forming” and “conircular” vector fields, by Yano:

**Definition 1** (Yano, [16, 17]). A vector field  $X_j$  is named torse-forming if  $\nabla_k X_j = \omega_k X_j + \varphi g_{kj}$ , being  $\varphi$  a scalar function and  $\omega_k$  a non vanishing one-form. It is named conircular if  $\omega_k$  is a gradient or locally a gradient of a scalar function.

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Fialkow gave a definition different from Yano's:

**Definition 2** (Fialkow [4]). A vector field  $X_j$  is named concircular if it satisfies  $\nabla_k X_j = \rho g_{kj}$ , being  $\rho$  a scalar function.

The following simple but deep result was recently proven:

**Theorem 3** (Bang-Yen Chen, [3]). *A  $n > 3$  dimensional Lorentzian manifold is a GRW space-time if and only if it admits a time-like concircular vector field (in the sense of Fialkow).*

It is worth noticing that for a unit time-like vector field, the torse-forming property by Yano becomes precisely Eq. (2), with generic scalar field  $\varphi$ . Based on Chen's theorem we proved:

**Proposition 4** (Mantica and Molinari, [7,8]). *A  $n > 3$  dimensional Lorentzian manifold is a GRW space-time if and only if it admits a unit time-like torse-forming vector, (2), that is also an eigenvector of the Ricci tensor.*

Now comes the equivalence: from (2) (holding either for GRW and  $(LCS)_n$  space-times) we evaluate

$$R_{jkl}{}^m u_m = [\nabla_j, \nabla_k] u_l = (h_{kl} \nabla_j - h_{jl} \nabla_k) \varphi - \varphi^2 (u_j g_{kl} - u_k g_{jl}),$$

where  $h_{kl} = u_k u_l + g_{kl}$ . Contraction with  $g^{jl}$  gives

$$(4) \quad R_k{}^m u_m = u_k [u^m \nabla_m \varphi + (n-1)\varphi^2] - (n-2)\nabla_k \varphi.$$

If (3) holds, then  $u_k$  is an eigenvector of the Ricci tensor, and we conclude that a  $(LCS)_n$  manifold is a GRW space-time.

If  $R_{km} u^m = \xi u_k$  it is  $(n-2)\nabla_k \varphi = \alpha u_k$  for some scalar field  $\alpha$ , i.e., (3) holds. Then we conclude that a GRW space-time is a  $(LCS)_n$  manifold.

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