

BRACKET FUNCTIONS ON GROUPOIDS

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ABSTRACT. In this paper, we introduce an operation denoted by $[Br_e]$, a bracket operation, which maps an arbitrary groupoid $(X, *)$ on a set X to another groupoid $(X, \bullet) = [Br_e](X, *)$ which on groups corresponds to sending a pair of elements (x, y) of X to its commutator $xyx^{-1}y^{-1}$. When applied to classes such as d -algebras, BCK -algebras, a variety of results is obtained indicating that this construction is more generally useful than merely for groups where it is of fundamental importance.

1. Introduction

The notions of BCK -algebras and BCI -algebras were introduced by Y. Imai and K. Iséki ([5, 6]). The class of BCK -algebras is a proper subclass of the class of BCI -algebras. We refer useful textbooks for BCK -algebras and BCI -algebras to [4, 11, 16]. J. Neggers and H. S. Kim [13] introduced the notion of d -algebras which is another useful generalization of BCK -algebras, and then investigated several relations between d -algebras and BCK -algebras as well as several other relations between d -algebras and oriented digraphs. J. Neggers and H. S. Kim introduced and investigated a class of algebras, called a B -algebra [14], which is related to several classes of algebras of interest such as $BCH/BCI/BCK$ -algebras and which seems to have rather nice properties without being excessively complicated otherwise. H. S. Kim and J. Neggers [9] introduced the notion of $Bin(X)$ and obtained a semigroup structure. H. F. Fayoumi [3] introduced the notion of the center $ZBin(X)$ in the semigroup $Bin(X)$ of all binary systems on a set X , and showed that a groupoid $(X, \bullet) \in ZBin(X)$ if and only if it is a locally-zero groupoid.

In this paper, we introduce an operation denoted by $[Br_e]$, a bracket operation, which maps an arbitrary groupoid $(X, *)$ on a set X to another groupoid $(X, \bullet) = [Br_e](X, *)$ which on groups corresponds to sending a pair of elements (x, y) of X to its commutator $xyx^{-1}y^{-1}$. When applied to classes such as d -algebras, BCK -algebras, a variety of results is obtained indicating that

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this construction is more generally useful than merely for groups where it is of fundamental importance.

2. Preliminaries

A *d-algebra* [13] is a non-empty set X with a constant 0 and a binary operation “ $*$ ” satisfying the following axioms:

- (I) $x * x = 0$,
- (II) $0 * x = 0$,
- (III) $x * y = 0$ and $y * x = 0$ imply $x = y$

for all $x, y \in X$. For more information on *d*-algebras we refer to [2, 7, 10, 12].

A *B-algebra* [14] is a non-empty set X with a constant 0 and a binary operation “ $*$ ” satisfying the following axioms:

- (I) $x * x = 0$,
- (IV) $x * 0 = x$,
- (V) $(x * y) * z = x * (z * (0 * y))$

for all x, y, z in X . A *B*-algebra $(X, *, 0)$ is said to be *0-commutative* if $x * (0 * y) = y * (0 * x)$ for all $x, y \in X$. For more information on *B*-algebras, we refer to [8, 15].

Given a non-empty set X , two groupoids $(X, *)$, (X, \bullet) are said to be *Smarandache disjoint* [1] if X has both an $(X, *)$ -structure and an (X, \bullet) -structure, then $|X| = 1$. The notion of “Smarandache disjoint” means that, given a groupoid $(X, *)$, if we add another groupoid (X, \bullet) to it, then it becomes a trivial groupoid.

Given a non-empty set X , we let $Bin(X)$ denote the collection of all groupoids $(X, *)$, where $*$: $X \times X \rightarrow X$ is a map and where $*(x, y)$ is written in the usual product form. Given elements $(X, *)$ and (X, \bullet) of $Bin(X)$, define a product “ \square ” on these groupoids as follows:

$$(X, *) \square (X, \bullet) = (X, \square)$$

where

$$x \square y = (x * y) \bullet (y * x)$$

for any $x, y \in X$. Using that notion, H. S. Kim and J. Neggers proved the following theorem.

Theorem 2.1 ([9]). *$(Bin(X), \square)$ is a semigroup, i.e., the operation “ \square ” as defined in general is associative. Furthermore, the left-zero-semigroup is the identity for this operation.*

3. Bracket image algebras

Given a set X , let $[Br_e] : Bin(X) \rightarrow Bin(X)$ be defined by $[Br_e]((X, *)) := (X, \bullet)$, where for any $e, x, y \in X$ we have

$$(1) \quad x \bullet y := (x * (e * y)) * (y * (e * x)).$$

We denote $x \bullet y$ by $[x, y]_e$ for specific mention. We may have other bracket mappings. For example, $[x, y]_e = (x * y) * ((e * x) * (e * y))$, $[x, y]_e = (x * y) * ((e * y) * x)$ can be useful functions to investigate some properties in this fashion.

Let (X, \cdot, e) be a group and $(X, *, e)$ be its associated B -algebra, where $x * y = x \cdot y^{-1}$. Then we determine

$$(2) \quad x \bullet y = (x(ey^{-1})^{-1})(y(ex^{-1})^{-1})^{-1} = xyx^{-1}y^{-1},$$

i.e., it is the standard *commutator* of the group theory.

The advantage of formula (1) is that it extends the notion of “commutator” quite enormously to arbitrary groupoids. In fact, we no longer need to deal with inverses directly at all if we use formula (1) as our point of departure. We consider $[Br_e]$ to be a *bracket function* on $Bin(X)$. Obviously, if $(X, *)$ is any groupoid whatsoever, we may define $[Br_e](x, y) = (x * (e * y)) * (y * (e * x)) = [x, y]_e = x \bullet y$ for any $e \in X$. Thus we may define a mapping $[Br_e] : Bin(X) \rightarrow Bin(X)$ as a mapping $[Br_e]((X, *)) = (X, \bullet)$ where $x \bullet y = [Br_e](x, y)$ as above. We call (X, \bullet) a *e-bracket image algebra* of a groupoid $(X, *)$.

Proposition 3.1. *The e-bracket image algebra (X, \bullet) of a left-zero-semigroup $(X, *)$ is the left-zero-semigroup $(X, *)$ itself for any $e \in X$, i.e., $[Br_e]((X, *)) = (X, *)$.*

Proof. For any $x, y \in X$, we have $x \bullet y = (x * (e * y)) * (y * (e * x)) = x * y = x$, proving that $[Br_e]((X, *)) = (X, *)$. \square

Proposition 3.2. *The e-bracket image algebra (X, \bullet) of a right-zero-semigroup $(X, *)$ is a left-zero-semigroup for any $e \in X$.*

Proof. For any $x, y \in X$, we have $x \bullet y = (x * (e * y)) * (y * (e * x)) = (x * y) * (y * x) = y * x = x$, showing that (X, \bullet) is a left-zero-semigroup. \square

Proposition 3.3. *The 0-bracket image algebra $[Br_e]((X, *)) = (X, \bullet)$ of a BCK-algebra $(X, *, 0)$ is the BCK-algebra $(X, *, 0)$ itself.*

Proof. Since $(X, *, 0)$ is a BCK-algebra, we have $x * 0 = x$ for all $x \in X$. It follows that $x \bullet y = (x * (0 * y)) * (y * (0 * x)) = (x * 0) * (y * 0) = x * y$, proving the proposition. \square

An algebra, e.g., one of the left-zero-semigroups or a BCK-algebra (when $e = 0$) as described above, is said to be a *bracket-fixed algebra*.

Theorem 3.4. *There is no non-trivial group (X, \cdot) with identity e which is also an e-bracket-fixed algebra.*

Proof. Assume that there is a non-trivial group (X, \cdot) with identity e which is also an e-bracket-fixed algebra. Then $x \cdot y = (x \cdot (e \cdot y)) \cdot (y \cdot (e \cdot x))$ and hence $y = y^2 \cdot x$ since (X, \cdot) is a group, which shows that $y \cdot x = e$ for all $x, y \in X$. If we take $y := e$, then $x = e \cdot x = e$ for all $x \in X$, proving that $X = 1$, a contradiction. \square

Theorem 3.4 showed that the class of e -bracket-fixed algebras and the class of groups with identity e are Smarandache disjoint.

Proposition 3.5. *If $(X, *, 0)$ is a d -algebra with $(x * 0) * (y * 0) = x * y$ for any $x, y \in X$, then its 0-bracket image algebra $[Br_0]((X, *)) = (X, \bullet)$ is a bracket-fixed algebra.*

Proof. If $(X, *, 0)$ is a d -algebra, then for any $x, y \in X$, we have $x \bullet y = (x * (0 * y)) * (y * (0 * x)) = (x * 0) * (y * 0) = x * y$. \square

If $(X, *, 0)$ is an edge d -algebra or a BCK -algebra, then $x * 0 = x$ for all $x \in X$, and hence Proposition 3.5 follows.

Proposition 3.6. *Let $(X, *, 0)$ be a d -algebra. If we assume that $(x * 0) * (y * 0) = 0$ implies $x * y = 0$ for any $x, y \in X$, then its 0-bracket image algebra $(X, \bullet) = [Br_0]((X, *))$ is also a d -algebra.*

Proof. Since $(X, *, 0)$ is a d -algebra, we have

$$(3) \quad x \bullet y = (x * (0 * y)) * (y * (0 * x)) = (x * 0) * (y * 0).$$

If we let $y := x$ in (3), then $x \bullet x = 0$, and if we let $x := 0, y := x$ in (3), then $0 \bullet x = 0$. Suppose that $x \bullet y = y \bullet x = 0$. Then $(x * 0) * (y * 0) = (y * 0) * (x * 0) = 0$ and hence $x * y = 0 = y * x$ by assumption. Since $(X, *, 0)$ is a d -algebra, we obtain $x = y$, proving that (X, \bullet) is a d -algebra. \square

Note that the condition “ $(x * 0) * (y * 0) = x * y$ ” holds for BCK -algebras.

Theorem 3.7. *Let $(K, +, \cdot)$ be a field of characteristic zero. Define a binary operation “ $*$ ” on K by $x * y := x^n(x - y), \forall x, y \in K$, where n is even (≥ 2). Then its 0-bracket image algebra $(K, \bullet) = [Br_0]((K, *))$ is a d -algebra.*

Proof. We claim that $(K, *, 0)$ is a d -algebra. In fact, $x * x = x^n(x - x) = 0$ and $0 * x = 0^n(0 - x) = 0$ for any $x \in K$. Assume $x * y = 0 = y * x$ and $x \neq y$. Then $x^n(x - y) = 0 = y^n(y - x)$. It follows that $x^n = y^n = 0$. Since K is of characteristic zero, we obtain $x = y$, a contradiction. This shows that $(K, *, 0)$ is a d -algebra.

Since $x \bullet y = (x * (0 * y)) * (y * (0 * x)) = x^{n(n+1)}(x^{n+1} - y^{n+1})$ for any $x, y \in K$, we have $x \bullet x = 0 \bullet x = 0$ for all $x \in K$. Assume $x \bullet y = y \bullet x = 0$. Then $x^{n(n+1)}[x^{n+1} - y^{n+1}] = 0 = y^{n(n+1)}[y^{n+1} - x^{n+1}]$. Since K is of characteristic zero, we obtain $x = 0 = y$, proving that $(K, \bullet, 0)$ is a d -algebra. \square

The assumption “ $x^{n+1} = y^{n+1}, \forall x, y \in K \implies x = y$ ” does not hold in general. If we define a binary operation “ $*$ ” on the set \mathbf{R} of all real numbers by $x * y := x(x - y)$, for any $x, y \in \mathbf{R}$, then (\mathbf{R}, \bullet) is not a d -algebra, since $x^2 = y^2$ does not imply $x = y$.

In Proposition 3.3, it is shown that every BCK -algebra is a 0-bracket-fixed algebra. We introduce a bracket-fixed d -algebra, which is not a BCK -algebra.

Example 3.8. Consider $X := [0, \infty)$ and $x*y := \sqrt{x}|\sqrt{x}-\sqrt{y}|$, where $x, y \in X$. Then $x*x = 0*x = 0$. Also, if $x*y = y*x = 0$, then $\sqrt{x} \neq \sqrt{y}$ means $\sqrt{x} = \sqrt{y} = 0$, and $x = y$. Thus $(X, *, 0)$ is a d -algebra. Since $x*0 = x$, we have $(x*0)*(y*0) = x*y$. By Proposition 3.3, it is a 0-bracket-fixed algebra. If we let $x := 9, y := 4$ in $(x*(x*y))*y$, then $(9*(9*4))*4 = \sqrt{9-3\sqrt{3}}|\sqrt{9-3\sqrt{3}}-2| \neq 0$, so that the d -algebra is not a BCK -algebra.

Note that not every d -algebra is a 0-bracket-fixed algebra, even if its 0-bracket image algebra is a d -algebra.

Example 3.9. Let \mathbf{R} be the set of all real numbers and let $x*y := x^2(x-y)$ for any $x, y \in \mathbf{R}$. Then $x \bullet y = (x*0)*(y*0) = x^3*y^3 = x^6(x^3-y^3)$ yields a d -algebra $(X, \bullet, 0)$ which is not the same as the original d -algebra.

4. B -algebras and e -bracket abelian groupoids

From the connection between B -algebras and groups, it is evident that the condition (X, \bullet, e) is an abelian group is tantamount to saying that if $(X, *, e)$ is an e -commutative B -algebra, then we shall prove that $x \bullet y = (x*(e*y))* (y*(e*x)) = e$ for all $x, y \in X$, i.e., (X, \bullet) is a trivial groupoid (See Proposition 4.2). Thus we consider $(X, *)$ to be e -bracket-abelian if $[Br_e]((X, *)) = (X, \bullet)$ is e -trivial, i.e., $x \bullet y = e$ for all $x, y \in X$. A groupoid $(X, *)$ is said to be e -bracket-almost-abelian if $[Br_e]((X, *)) = (X, \bullet)$ is trivial, i.e., for some $t \in X$, $x \bullet y = t$ for all $x, y \in X$. An e -bracket-almost-abelian groupoid becomes an e -bracket-abelian groupoid when $t = e$.

Proposition 4.1. *Let $(X, *, e)$ be an e -commutative B -algebra. Then it is e -bracket-abelian.*

Proof. If $(X, *, e)$ is a 0-commutative B -algebra, then, for any $x, y \in X$, we have $x \bullet y = (x*(e*y))* (y*(e*x)) = (x*(e*y))* (x*(e*y)) = e$, which shows that (X, \bullet) is a trivial groupoid, and hence $(X, *)$ is e -bracket-abelian. \square

Proposition 4.2. *If a B -algebra $(X, *, e)$ is e -bracket-almost-abelian, then $(X, *)$ is e -bracket-abelian and its associated group (X, \odot) is abelian.*

Proof. If a B -algebra $(X, *, e)$ is e -bracket-almost-abelian, then there is an element $t \in X$ such that $x \bullet y = t$ for any $x, y \in X$, i.e.,

$$(4) \quad x \bullet y = (x*(e*y))* (y*(e*x)) = t.$$

If we let $y := x$ in (4), then $t = e$, i.e., $(X, *, e)$ is e -bracket-abelian. If (X, \odot, e) is its associated group, then $e*y = e \odot y^{-1} = y^{-1}$, $x*(e*y) = x \odot (y^{-1})^{-1} = x \odot y$ and hence $e = t = x \bullet y = (x*(e*y))* (y*(e*x)) = (x \odot y) * (y \bullet x) = (x \bullet y) \bullet (y \bullet x)^{-1}$. It follows that $x \bullet y = y \bullet x$ for any $x, y \in X$, proving that (X, \odot) is commutative. \square

5. Mappings $[Br_\alpha](x, y)$

Given a groupoid $(X, *, e)$, we consider a mapping $[Br_\alpha] : X \times X \rightarrow X$ defined by

$$(5) \quad [Br_\alpha](x, y) := (x * (\alpha * y)) * (y * (\alpha * x)),$$

where $\alpha \in X$.

Proposition 5.1. *If $(X, *, e)$ is a B -algebra, then $[Br_e](x, y) = [Br_\alpha](x, y)$ for any $\alpha \in X$ where $x, y \in X$.*

Proof. If we use the group structure associated with $(X, *, e)$, then $x * (\alpha * y) = x \cdot (\alpha \cdot y^{-1})^{-1} = x \cdot y \cdot \alpha^{-1}$, and $(x * (\alpha * y)) * (y * (\alpha * x)) = (x \cdot y \cdot \alpha^{-1}) \cdot (y \cdot x \cdot \alpha^{-1})^{-1} = (xy\alpha^{-1}) \cdot (\alpha x^{-1}y^{-1}) = x \cdot y \cdot x^{-1} \cdot y^{-1} = (x * (e * y)) * (y * (e * x))$, i.e., we have $Br_e(x, y) = Br_\alpha(x, y)$. \square

By Proposition 5.1, we can see that the class of groupoids $(X, *)$ such that $[Br_e](x, y) = [Br_f](x, y)$ for all $e, f \in X$ where $x, y \in X$, contains the class of B -algebras. We note next that this class of groupoids is Smarandache disjoint from the class of d -algebras.

Theorem 5.2. *The d -algebras and the groupoids $(X, *)$ such that $[Br_e](x, y) = [Br_f](x, y)$ for all $e, f \in X$ where $x, y \in X$ are Smarandache disjoint.*

Proof. Let $(X, *, 0)$ be a d -algebra with $[Br_e](x, y) = [Br_f](x, y)$ for all $e, f \in X$ where $x, y \in X$. Then we have, for all $x, y \in X$,

$$(6) \quad (x * (e * y)) * (y * (e * x)) = (x * (f * y)) * (y * (f * x)).$$

If we let $e := 0$ in (6), then

$$(7) \quad (x * 0) * (y * 0) = (x * (f * y)) * (y * (f * x)).$$

If we let $y := 0$ in (7), then

$$(8) \quad (x * 0) * 0 = (x * (f * 0)) * 0.$$

If we let $x := f * 0$ in (8), then

$$(9) \quad ((f * 0) * 0) * 0 = 0 * 0 = 0$$

so that $(f * 0) * 0 = 0$, $f * 0 = 0$ and $f = 0$. But f is arbitrary, we conclude $|X| = 1$. This proves the theorem. \square

6. Comments

In this paper we have developed a theory of what we have called bracket functions as a generalization of the theory of commutators on groups. The initial results prove to be interesting and highly non-trivial for several classes of algebras other than groups, viz., the class of d -algebras (containing the class of BCK -algebras and the class of B -algebras which again behaves somewhat differently than the class of groups as (indirectly) illustrated by Theorem 3.4.

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