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THE UNIT BALL OF THE SPACE OF BILINEAR FORMS ON $\mathbb{R}^3$ WITH THE SUPREMUM NORM

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Abstract. We classify all the extreme and exposed bilinear forms of the unit ball of $\mathcal{L}(^2 l_\infty^3)$ which leads to a complete formula of $\|f\|$ for every $f \in \mathcal{L}(^2 l_\infty^3)^*$. It follows from this formula that every extreme bilinear form of the unit ball of $\mathcal{L}(^2 l_\infty^3)$ is exposed.

1. Introduction

We denote by B_E the closed unit ball of a real Banach space E and also by E^* the dual space of E. A point $x \in B_E$ is called an *extreme point* of B_E if the equation $x = \frac{1}{2}(y+z)$ for some $y, z \in B_E$ implies $x = y = z$. A point $x \in B_E$ is called an *exposed point* of B_E if there is a $f \in E^*$ so that $f(x) = 1 = \|f\|$ and $f(y) < 1$ for every $y \in B_E \setminus \{x\}$. $x \in B_E$ is called a *smooth point* of B_E if there is a unique $f \in E^*$ so that $f(x) = 1 = \|f\|$. It is easy to see that every exposed point of B_E is an extreme point. We denote by $extB_E, expB_E$ and smB_E the set of extreme points, the set of exposed points and the set of smooth points of B_E, respectively. A mapping $P : E \to \mathbb{R}$ is a continuous 2-homogeneous polynomial if there exists a continuous bilinear form L on the product $E \times E$ such that $P(x) = L(x,x)$ for every $x \in E$. We denote by $\mathcal{L}(^2E)$ the Banach space of all continuous bilinear forms on E endowed with the norm $\|L\| = \sup_{\|x\|=\|y\|=1} |L(x,y)|$. The subspace of all continuous symmetric bilinear forms on E is denoted by $\mathcal{L}_s(^2E)$. We denote by $\mathcal{P}(^2E)$ the Banach space of all continuous 2-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$. For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [8].

In 1998, Choi *et al.* ([3, 7]) characterized the extreme points of the unit ball of $\mathcal{P}(^2 l_1^2)$ and $\mathcal{P}(^2 l_2^2)$. Kim classified the exposed 2-homogeneous polynomials on $\mathcal{P}(^2 l_p^2)$ $(1 \le p \le \infty)$ ([12]) and the extreme, exposed, smooth points of the unit ball of $\mathcal{P}(^2 d_*(1,w)^2)$ ([14,16,20]), where $d_*(1,w)^2 = \mathbb{R}^2$ with the octagonal

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norm of weight w. Kim ([15, 17–19, 21, 22]) also classified the extreme, exposed, smooth points of the unit balls of $\mathcal{L}_s(^2d_*(1,w)^2), \mathcal{L}(^2d_*(1,w)^2), \mathcal{L}_s(^2\mathbb{R}^2_{h(w)})$ and $\mathcal{L}(^2\mathbb{R}^2_{h(w)})$, where $\mathbb{R}^2_{h(w)}$ is the plane $\mathbb{R}^2$ with the hexagonal norm of weight w.

Let $n \geq 2$ and $l^n_\infty := \mathbb{R}^n$ with the supremum norm. Given $\{a_{ij}\}^n_{i,j=1} \subset \mathbb{R}$, let $T \in \mathcal{L}(^2l^n_\infty)$ be defined by the rule

$$T((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = \sum_{i,j=1}^{n} a_{ij} x_i y_j.$$

If $n = 2$, for simplicity, we will write

$$T = (a_{11}, a_{12}, a_{21}, a_{22})^t$$

as a 4×1 column vector. If $n = 3$, for simplicity, we will write

$$T = (a_{11}, a_{12}, a_{21}, a_{13}, a_{31}, a_{22}, a_{23}, a_{32}, a_{33})^t$$

as a 9×1 column vector. If $T \in \mathcal{L}_s(^2l^3_\infty)$, we will write

$$T = (a_{11}, 2a_{12}, 2a_{13}, a_{22}, 2a_{23}, a_{33})^t$$

as a 6×1 column vector.

Kim [26] and Cavalcante and Pellegrino [2] independently classified $extB_{\mathcal{L}(^2l^2_\infty)}$ and showed that $expB_{\mathcal{L}(^2l^2_\infty)} = extB_{\mathcal{L}(^2l^2_\infty)}$ as follows:

$$\begin{aligned} extB_{\mathcal{L}(^2l^2_\infty)} = \{&\pm {e_j}^t (1 \leq j \leq 4), \pm\tfrac{1}{2}(-e_1 + e_2 + e_3 + e_4)^t, \pm\tfrac{1}{2}(e_1 - e_2 + e_3 + e_4)^t \\ &\pm \tfrac{1}{2}(e_1 + e_2 - e_3 + e_4)^t, \pm\tfrac{1}{2}(e_1 + e_2 + e_3 - e_4)^t\}. \end{aligned}$$

We refer to ([1–7, 9–31] and references therein) for some recent work about extremal properties of multilinear mappings and homogeneous polynomials on some classical Banach spaces.

In this paper, we classify all the extreme and exposed bilinear forms of the unit ball of $\mathcal{L}(^2l^3_\infty)$ which leads to a complete formula of $\|f\|$ for every $f \in \mathcal{L}(^2l^3_\infty)^*$. It follows from this formula that every extreme bilinear form of the unit ball of $\mathcal{L}(^2l^3_\infty)$ is exposed.

2. The extreme points of the unit ball of $\mathcal{L}(^2l^3_\infty)$

Recently, Kim [23] showed the following: *Let*

$$\begin{aligned} \Omega = \{&(1,1,1,1,1,1), (1,0,1,-1,0,1), (1,1,0,1,0,-1), (-1,0,0,1,1,1), \\ &(1,-1,1,1,-1,1), (1,0,0,-1,1,-1), (-1,1,0,-1,0,1), \\ &(1,1,-1,1,-1,1), (-1,0,1,1,0,-1), (1,-1,-1,1,1,1)\} \end{aligned}$$

and

$$\begin{aligned} \Gamma = \{&[(1,1,1),(1,1,1)], [(1,1,1),(1,-1,1)], [(1,1,1),(1,1,-1)], \\ &[(1,1,1),(-1,1,1)], [(1,-1,1),(1,-1,1)], [(1,-1,1),(1,1,-1)], \\ &[(1,-1,1),(-1,1,1)], [(1,1,-1),(1,1,-1)], [(1,1,-1),(-1,1,1)], \\ &[(-1,1,1),(-1,1,1)]\}. \end{aligned}$$

Then, (a) *Let* $T = (a_{11}, 2a_{12}, 2a_{13}, a_{22}, 2a_{23}, a_{33})^t \in \mathcal{L}_s(^2l_\infty^3)$ *with* $\|T\| = 1$. *Then,* $T \in extB_{\mathcal{L}_s(^2l_\infty^3)}$ *if and only if there exist at least 6 linearly independent vectors* $W_1, \ldots, W_6 \in \Omega$ *and* $Z_1, \ldots, Z_6 \in \Gamma$ *such that*

$$W_j \cdot S = S(Z_j) \text{ for all } S \in \mathcal{L}_s(^2l_\infty^3) \text{ and } |T(Z_j)| = 1 \text{ for } j = 1, \ldots, 6.$$

In this case, A *is an invertible* 6×6 *matrix with* $AS = (S(Z_1), \ldots, S(Z_6))^t$ *for all* $S \in \mathcal{L}_s(^2l_\infty^3)$ *when we consider the* j*-row vector of* A *as* $[Row(A)]_j = W_j$ *for* $j = 1, \ldots, 6$.

(b) $expB_{\mathcal{L}_s(^2l_\infty^3)} = extB_{\mathcal{L}_s(^2l_\infty^3)}$.

In order to classify the extreme points of $B_{\mathcal{L}(^2l_\infty^3)}$, we need a complete formula of $\|T\|$ for $T \in \mathcal{L}(^2l_\infty^3)$.

Theorem 2.1 ([27])**.** *Let* $T = (a_{11}, a_{12}, a_{21}, a_{13}, a_{31}, a_{22}, a_{23}, a_{32}, a_{33})^t \in \mathcal{L}(^2l_\infty^3)$. *Then,*

$$\begin{aligned}\|T\| = \max\{&|a_{11} + a_{12} + a_{13}| + |a_{21} + a_{22} + a_{23}| + |a_{31} + a_{32} + a_{33}|,\\ &|-a_{11} + a_{12} + a_{13}| + |-a_{21} + a_{22} + a_{23}| + |-a_{31} + a_{32} + a_{33}|,\\ &|a_{11} - a_{12} + a_{13}| + |a_{21} - a_{22} + a_{23}| + |a_{31} - a_{32} + a_{33}|,\\ &|a_{11} + a_{12} - a_{13}| + |a_{21} + a_{22} - a_{23}| + |a_{31} + a_{32} - a_{33}|\}.\end{aligned}$$

Note that if $\|T\| = 1$, *then* $|a_{ij}| \le 1$ *for* $i, j = 1, 2, 3$.

To detect the extreme points of $B_{\mathcal{L}(^2l_\infty^3)}$, we first consider sets Ω_1 and Γ_1; Ω_1 is the set of vectors

$$\begin{aligned}&(1,1,1,1,1,1,1,1,1), (1,-1,1,1,1,-1,1,-1,1),\\ &(1,1,-1,1,1,-1,-1,1,1), (1,1,1,-1,1,1,-1,1,-1),\\ &(1,1,1,1,-1,1,1,-1,-1), (-1,1,-1,1,-1,1,1,1,1),\\ &(-1,-1,1,-1,1,1,1,1,1), (1,-1,-1,1,1,1,-1,-1,1),\\ &(1,1,-1,-1,1,-1,1,1,-1), (1,-1,1,1,-1,-1,1,1,-1),\\ &(-1,1,1,1,-1,-1,-1,1,1), (-1,1,1,-1,1,-1,1,-1,1),\\ &(1,1,1,-1,-1,1,-1,-1,1), (-1,1,-1,1,1,1,1,-1,-1),\\ &(-1,-1,1,1,1,1,-1,1,-1), (1,-1,-1,-1,-1,1,1,1,1)\end{aligned}$$

and Γ_1 is the set of vectors

$$\begin{aligned}&[(1,1,1),(1,1,1)], [(1,1,1),(1,-1,1)], [(1,-1,1),(1,1,1)],\\ &[(1,1,1),(1,1,-1)], [(1,1,-1),(1,1,1)], [(1,1,1),(-1,1,1)],\\ &[(-1,1,1),(1,1,1)], [(1,-1,1),(1,-1,1)], [(1,-1,1),(1,1,-1)],\\ &[(1,1,-1),(1,-1,1)], [(1,-1,1),(-1,1,1)], [(-1,1,1),(1,-1,1)],\\ &[(1,1,-1),(1,1,-1)], [(1,1,-1),(-1,1,1)], [(-1,1,1),(1,1,-1)],\\ &[(-1,1,1),(-1,1,1)].\end{aligned}$$

Theorem 2.2. *Let* $T = (a_{11}, a_{12}, a_{21}, a_{13}, a_{31}, a_{22}, a_{23}, a_{32}, a_{33})^t \in \mathcal{L}(^2 l_\infty^3)$ *with* $\|T\| = 1$. *Then,* $T \in extB_{\mathcal{L}(^2 l_\infty^3)}$ *if and only if there exist at least 9 linearly independent vectors* $W_1, \ldots, W_9 \in \Omega_1$ *and* $Z_1, \ldots, Z_9 \in \Gamma_1$ *such that*

$$W_j \cdot S = S(Z_j) \text{ for all } S \in \mathcal{L}(^2 l_\infty^3) \text{ and } |T(Z_j)| = 1 \text{ for } j = 1, \ldots, 9.$$

In this case, A *is an invertible* 9×9 *matrix with* $AS = (S(Z_1), \ldots, S(Z_9))^t$ *for all* $S \in \mathcal{L}(^2 l_\infty^3)$ *when we consider the* j*-row vector of* A *as* $[Row(A)]_j = W_j$ *for* $j = 1, \ldots, 9$.

Proof. It follows from Theorem 3.1 of [27]. □

We are position to classify the extreme points of $B_{\mathcal{L}(^2 l_\infty^3)}$.

Theorem 2.3. $extB_{\mathcal{L}(^2 l_\infty^3)}$ *is the set consisting of the following bilinear forms on* $\mathbb{R}^3$:

$$\begin{aligned}
&\pm {e_j}^t \ (1 \le j \le 9), \pm\tfrac{1}{2}(-e_1 + e_2 + e_3 + e_6)^t, \pm\tfrac{1}{2}(e_1 - e_2 + e_3 + e_6)^t,\\
&\pm \tfrac{1}{2}(e_1 + e_2 - e_3 + e_6)^t, \pm\tfrac{1}{2}(e_1 + e_2 + e_3 - e_6)^t, \pm\tfrac{1}{2}(-e_1 + e_4 + e_5 + e_9)^t,\\
&\pm \tfrac{1}{2}(e_1 - e_4 + e_5 + e_9)^t, \pm\tfrac{1}{2}(e_1 + e_4 - e_5 + e_9)^t, \pm\tfrac{1}{2}(e_1 + e_4 + e_5 - e_9)^t,\\
&\pm \tfrac{1}{2}(-e_6 + e_7 + e_8 + e_9)^t, \pm\tfrac{1}{2}(e_6 - e_7 + e_8 + e_9)^t, \pm\tfrac{1}{2}(e_6 + e_7 - e_8 + e_9)^t,\\
&\pm \tfrac{1}{2}(e_6 + e_7 + e_8 - e_9)^t, \pm\tfrac{1}{2}(-e_3 + e_5 + e_6 + e_8)^t, \pm\tfrac{1}{2}(e_3 - e_5 + e_6 + e_8)^t,\\
&\pm \tfrac{1}{2}(e_3 + e_5 - e_6 + e_8)^t, \pm\tfrac{1}{2}(e_3 + e_5 + e_6 - e_8)^t, \pm\tfrac{1}{2}(-e_2 + e_4 + e_6 + e_7)^t,\\
&\pm \tfrac{1}{2}(e_2 - e_4 + e_6 + e_7)^t, \pm\tfrac{1}{2}(e_2 + e_4 - e_6 + e_7)^t, \pm\tfrac{1}{2}(e_2 + e_4 + e_6 - e_7)^t,\\
&\pm \tfrac{1}{2}(-e_1 + e_3 + e_4 + e_7)^t, \pm\tfrac{1}{2}(e_1 - e_3 + e_4 + e_7)^t, \pm\tfrac{1}{2}(e_1 + e_3 - e_4 + e_7)^t,\\
&\pm \tfrac{1}{2}(e_1 + e_3 + e_4 - e_7)^t, \pm\tfrac{1}{2}(-e_1 + e_2 + e_5 + e_8)^t, \pm\tfrac{1}{2}(e_1 - e_2 + e_5 + e_8)^t,\\
&\pm \tfrac{1}{2}(e_1 + e_2 - e_5 + e_8)^t, \pm\tfrac{1}{2}(e_1 + e_2 + e_5 - e_8)^t, \pm\tfrac{1}{2}(-e_2 + e_4 + e_6 + e_7)^t,\\
&\pm \tfrac{1}{2}(e_2 - e_4 + e_6 + e_7)^t, \pm\tfrac{1}{2}(e_2 + e_4 - e_6 + e_7)^t, \pm\tfrac{1}{2}(e_2 + e_4 + e_6 - e_7)^t,\\
&\pm \tfrac{1}{2}(-e_3 + e_5 + e_6 + e_8)^t, \pm\tfrac{1}{2}(e_3 - e_5 + e_6 + e_8)^t, \pm\tfrac{1}{2}(e_3 + e_5 - e_6 + e_8)^t,\\
&\pm \tfrac{1}{2}(e_3 + e_5 + e_6 - e_8)^t, \pm\tfrac{1}{2}(-e_3 + e_5 + e_7 + e_9)^t, \pm\tfrac{1}{2}(e_3 - e_5 + e_7 + e_9)^t,\\
&\pm \frac{1}{2}(e_3 + e_5 - e_7 + e_9)^t, \pm\tfrac{1}{2}(e_3 + e_5 + e_7 - e_9)^t, \pm\tfrac{1}{2}(-e_2 + e_4 + e_8 + e_9)^t,\\
&\pm \tfrac{1}{2}(e_2 - e_4 + e_8 + e_9)^t, \pm\tfrac{1}{2}(e_2 + e_4 - e_8 + e_9)^t, \pm\tfrac{1}{2}(e_2 + e_4 + e_8 - e_9)^t.
\end{aligned}$$

Proof. Claim: $T = \frac{1}{2}(-e_1 + e_2 + e_3 + e_6)^t \in extB_{\mathcal{L}(^2 l_\infty^3)}$

Let

$$\begin{aligned}
T_1 = [&(-\tfrac{1}{2} + \epsilon_{11})e_1 + (\tfrac{1}{2} + \epsilon_{12})e_2 + (\tfrac{1}{2} + \epsilon_{21})e_3 + (\tfrac{1}{2} + \epsilon_{22})e_6\\
&+ \epsilon_{13}e_4 + \epsilon_{31}e_5 + \epsilon_{23}e_7 + \epsilon_{32}e_8 + \epsilon_{33}e_9]^t
\end{aligned}$$

and

$$\begin{aligned}
T_2 = [&(-\tfrac{1}{2} - \epsilon_{11})e_1 + (\tfrac{1}{2} - \epsilon_{12})e_2 + (\tfrac{1}{2} - \epsilon_{21})e_3 + (\tfrac{1}{2} - \epsilon_{22})e_6\\
&- \epsilon_{13}e_4 - \epsilon_{31}e_5 - \epsilon_{23}e_7 - \epsilon_{32}e_8 - \epsilon_{33}e_9]^t
\end{aligned}$$

with $\|T_1\| = \|T_2\| = 1$ for some $\epsilon_{ij} \in \mathbb{R}$ for $i, j = 1, 2, 3$. By Theorem 2.1, it follows that

$$\begin{aligned}
0 &= \epsilon_{11} + \epsilon_{12} + \epsilon_{13},\\
0 &= \epsilon_{21} + \epsilon_{22} + \epsilon_{23},\\
0 &= \epsilon_{31} + \epsilon_{32} + \epsilon_{33},\\
0 &= -\epsilon_{11} + \epsilon_{12} + \epsilon_{13},\\
0 &= -\epsilon_{21} + \epsilon_{22} + \epsilon_{23},\\
0 &= -\epsilon_{31} + \epsilon_{32} + \epsilon_{33},\\
0 &= \epsilon_{11} - \epsilon_{12} + \epsilon_{13},\\
0 &= \epsilon_{21} - \epsilon_{22} + \epsilon_{23},\\
0 &= \epsilon_{31} - \epsilon_{32} + \epsilon_{33},\\
0 &= \epsilon_{11} + \epsilon_{12} - \epsilon_{13},\\
0 &= \epsilon_{21} + \epsilon_{22} - \epsilon_{23},\\
0 &= \epsilon_{31} + \epsilon_{32} - \epsilon_{33},
\end{aligned}$$

which show that $\epsilon_{ij} = 0$ for $i, j = 1, 2, 3$. Hence, $T \in extB_{\mathcal{L}(^2l^3_\infty)}$. Similarly, we conclude that the other 104 bilinear forms in the list of Theorem 2.3 are extreme.

Let $T = (a_{11}, a_{12}, a_{21}, a_{13}, a_{31}, a_{22}, a_{23}, a_{32}, a_{33})^t \in \mathcal{L}(^2l^3_\infty)$ with $\|T\| = 1$. By Theorem 2.2, $T \in extB_{\mathcal{L}(^2l^3_\infty)}$ if and only if $T = A^{-1}(T(Z_1), \ldots, T(Z_9))^t$ for some $Z_1, \ldots, Z_9 \in \Gamma_1$ with $|T(Z_j)| = 1$ $(j = 1, \ldots, 9)$. Therefore, we may classify all the extreme points of $B_{\mathcal{L}(^2l^3_\infty)}$ by the following steps: First, among 11440 cases, find $W_1, \ldots, W_9 \in \Omega_1$ such that the corresponding matrix A with rows $W_1, \ldots, W_9$ is invertible, and next solve A^{-1} and using Theorem 2.1, obtain $T = A^{-1}(b_1, \ldots, b_9)^t$ satisfying

$$\|A^{-1}(b_1, \ldots, b_9)^t\| = 1$$

for some $b_1, \ldots, b_9 = \pm 1$.

Such $T = A^{-1}(b_1, \ldots, b_9)^t$ are all the extreme points of $B_{\mathcal{L}(^2l^3_\infty)}$. With the aid of Wolfram Mathematica 8, the conclusion follows. □

Note that $|extB_{\mathcal{L}(^2l^3_\infty)}| = 106$.

3. The exposed points of the unit ball of $\mathcal{L}(^2l^3_\infty)$

Using Theorem 2.3, we present a complete formula of $\|f\|$ for every $f \in \mathcal{L}(^2l^3_\infty)^*$.

Theorem 3.1. *Let $f \in \mathcal{L}(^2l^3_\infty)^*$ with $\alpha_{ij} := f(x_i y_j)$ for $i, j = 1, 2, 3$. Then,*

$$\|f\| = \frac{1}{2} \max E,$$

where E is the set of the values listed below:

$$\begin{aligned}
&2|\alpha_{11}|, 2|\alpha_{22}|, 2|\alpha_{33}|, |-\alpha_{11}+\alpha_{22}+\alpha_{12}+\alpha_{21}|, |\alpha_{11}-\alpha_{22}+\alpha_{12}+\alpha_{21}|,\\
&|\alpha_{11}+\alpha_{22}-\alpha_{12}+\alpha_{21}|, |\alpha_{11}+\alpha_{22}+\alpha_{12}-\alpha_{21}|, |-\alpha_{11}+\alpha_{33}+\alpha_{13}+\alpha_{31}|,\\
&|\alpha_{11}-\alpha_{33}+\alpha_{13}+\alpha_{31}|, |\alpha_{11}+\alpha_{33}-\alpha_{13}+\alpha_{31}|, |\alpha_{11}+\alpha_{33}+\alpha_{13}-\alpha_{31}|,\\
&|-\alpha_{22}+\alpha_{33}+\alpha_{23}+\alpha_{32}|, |\alpha_{22}-\alpha_{33}+\alpha_{23}+\alpha_{32}|, |\alpha_{22}+\alpha_{33}-\alpha_{23}+\alpha_{32}|,\\
&|\alpha_{22}+\alpha_{33}+\alpha_{23}-\alpha_{32}|, |-\alpha_{22}+\alpha_{21}+\alpha_{31}+\alpha_{32}|, |\alpha_{22}-\alpha_{21}+\alpha_{31}+\alpha_{32}|,\\
&|\alpha_{22}+\alpha_{21}-\alpha_{31}+\alpha_{32}|, |\alpha_{22}+\alpha_{21}+\alpha_{31}-\alpha_{32}|, |-\alpha_{22}+\alpha_{12}+\alpha_{13}+\alpha_{23}|,\\
&|\alpha_{22}-\alpha_{12}+\alpha_{13}+\alpha_{23}|, |\alpha_{22}+\alpha_{12}-\alpha_{13}+\alpha_{23}|, |\alpha_{22}+\alpha_{12}+\alpha_{13}-\alpha_{23}|,\\
&|-\alpha_{11}+\alpha_{21}+\alpha_{13}+\alpha_{23}|, |\alpha_{11}-\alpha_{21}+\alpha_{13}+\alpha_{23}|, |\alpha_{11}+\alpha_{21}-\alpha_{13}+\alpha_{23}|,\\
&|\alpha_{11}+\alpha_{21}+\alpha_{13}-\alpha_{23}|, |-\alpha_{11}+\alpha_{12}+\alpha_{31}+\alpha_{32}|, |\alpha_{11}-\alpha_{12}+\alpha_{31}+\alpha_{32}|,\\
&|\alpha_{11}+\alpha_{12}-\alpha_{31}+\alpha_{32}|, |\alpha_{11}+\alpha_{12}+\alpha_{31}-\alpha_{32}|, |-\alpha_{22}+\alpha_{12}+\alpha_{13}+\alpha_{23}|,\\
&|\alpha_{22}-\alpha_{12}+\alpha_{13}+\alpha_{23}|, |\alpha_{22}+\alpha_{12}-\alpha_{13}+\alpha_{23}|, |\alpha_{22}+\alpha_{12}+\alpha_{13}-\alpha_{23}|,\\
&|-\alpha_{22}+\alpha_{21}+\alpha_{31}+\alpha_{32}|, |\alpha_{22}-\alpha_{21}+\alpha_{31}+\alpha_{32}|, |\alpha_{22}+\alpha_{21}-\alpha_{31}+\alpha_{32}|,\\
&|\alpha_{22}+\alpha_{21}+\alpha_{31}-\alpha_{32}|, |-\alpha_{33}+\alpha_{21}+\alpha_{31}+\alpha_{23}|, |\alpha_{33}-\alpha_{21}+\alpha_{31}+\alpha_{23}|,\\
&|\alpha_{33}+\alpha_{21}-\alpha_{31}+\alpha_{23}|, |\alpha_{33}+\alpha_{21}+\alpha_{31}-\alpha_{23}|, |-\alpha_{33}+\alpha_{12}+\alpha_{13}+\alpha_{32}|,\\
&|\alpha_{33}-\alpha_{12}+\alpha_{13}+\alpha_{32}|, |\alpha_{33}+\alpha_{12}-\alpha_{13}+\alpha_{32}|, |\alpha_{33}+\alpha_{12}+\alpha_{13}-\alpha_{32}|.
\end{aligned}$$

Proof. Since $\|f\| = \sup_{T\in extB_{\mathcal{L}(^2l^3_\infty)}} |f(T)|$, it follows from Theorem 2.3 and the Krein-Milman Theorem. □

Note that if $\|f\| = 1$, then $|\alpha_{ij}| \leq 1$ for $i, j = 1, 2, 3$.

Theorem 3.2 ([18]). *Let E be a real finite dimensional Banach space such that $extB_E$ is finite. Suppose that $x \in extB_E$ satisfies that there exists $f \in E^*$ with $f(x) = 1 = \|f\|$ and $|f(y)| < 1$ for every $y \in extB_E\backslash\{\pm x\}$. Then, $x \in expB_E$.*

Theorem 3.3. $expB_{\mathcal{L}(^2l^3_\infty)} = extB_{\mathcal{L}(^2l^3_\infty)}$.

Proof. Claim: $T = e_1 \in expB_{\mathcal{L}(^2l^3_\infty)}$

Let $f = (\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{13}, \alpha_{31}, \alpha_{22}, \alpha_{23}, \alpha_{32}, \alpha_{33}) = (1, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, \frac{1}{4}, \frac{1}{4}, 0) \in \mathcal{L}(^2l^3_\infty)^*$. Then, by Theorems 3.1 and 3.2, $\|f\| = 1 = f(T)$ and $|f(S)| < 1$ for $S \in extB_{\mathcal{L}(^2l^3_\infty)}\backslash\{\pm T\}$. Similarly, $-e_1, \pm e_k$ for $2 \leq k \leq 9$ are exposed.

Claim: $T = \frac{1}{2}(-e_1 + e_2 + e_3 + e_6) \in expB_{\mathcal{L}(^2l^3_\infty)}$

Let $f = (\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{13}, \alpha_{31}, \alpha_{22}, \alpha_{23}, \alpha_{32}, \alpha_{33}) = (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, \frac{1}{2}, 0, 0, 0) \in \mathcal{L}(^2l^3_\infty)^*$. Then, by Theorems 3.1 and 3.2, $\|f\| = 1 = f(T)$ and $|f(S)| < 1$ for $S \in extB_{\mathcal{L}(^2l^3_\infty)}\backslash\{\pm T\}$. Similar arguments in the above claims show that all the other extreme bilinear forms in $\mathcal{L}(^2l^3_\infty)$ are exposed. We complete the proof. □

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