

## SQUARE QUADRATIC PROXIMAL METHOD FOR NONLINEAR COMPLIMENTARITY PROBLEMS

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ABSTRACT. In this paper, we propose a new interior point method for solving nonlinear complementarity problems. In this method, we use a new profitable searching direction and instead of using the logarithmic quadratic term, we use a square root quadratic term. We prove the global convergence of the proposed method under the assumption that  $F$  is monotone. Some preliminary computational results are given to illustrate the efficiency of the proposed method.

### 1. Introduction

Let  $R$  stand for the real axis; and  $R_+ = \{x \in R; x \geq 0\}$ ,  $R_{++} = \{x \in R; x > 0\}$ , denote the positive half-axis and strict positive half-axis, respectively. Further, given  $n \in N$ , put

$$R_+^n = \{x = (x_1, \dots, x_n)^T; x_1, \dots, x_n \in R_+\}.$$

The nonlinear complementarity problem (NCP) is to find a vector  $x \in R^n$  such that

$$(1) \quad x \geq 0, \quad F(x) \geq 0 \quad \text{and} \quad x^T F(x) = 0,$$

where  $F$  is a nonlinear mapping from  $R^n$  into itself. Throughout this paper we assume that  $F$  is continuous and monotone with respect to  $R_+^n$ , that is,  $(F(x) - F(y))^T (x - y) \geq 0$  for all  $x, y \in R_+^n$  and the solution set of (1), denoted by  $\Omega^*$ , is nonempty.

NCP was introduced by Cottle in his PhD thesis in the early 1960's. Complementarity problems have attracted great attention of researchers and several works have been published to set up the fundamental theoretical results of this problem (see, e.g., [9, 10] and the references therein).

Many attempts have been made to develop implementable algorithms for the solution of NCP. A popular way to solve the NCP is to reformulate as finding

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Received March 19, 2018; Revised August 24, 2018; Accepted August 29, 2018.

2010 *Mathematics Subject Classification.* Primary 49J40, 65N30.

*Key words and phrases.* nonlinear complementarity problems, monotone operator, square root quadratic term, interior point method.

the zero point of the operator  $T(x) = F(x) + N_{R_+^n}(x)$ , i.e., find  $x^* \in R_+^n$  such that  $0 \in T(x^*)$ , where  $N_{R_+^n}(\cdot)$  is the normal cone operator to  $R_+^n$  defined by

$$N_{R_+^n}(x) = \begin{cases} \{y \in R^n : y^T(v - x) \leq 0, \forall v \in R_+^n\} & \text{if } x \in R_+^n, \\ \emptyset & \text{otherwise.} \end{cases}$$

A classical method to solve this problem is the proximal point algorithm (PPA), which starting with any vector  $x^0 \in R_+^n$  and  $\beta_k \geq \beta > 0$ , iteratively updates  $x^{k+1}$  conforming the following problem:

$$(2) \quad 0 \in \beta_k T(x) + \nabla_x q(x, x^k),$$

where

$$(3) \quad q(x, x^k) = \frac{1}{2} \|x - x^k\|^2$$

is a quadratic function of  $x$ .

Recently, many studies have been focused on some new interior point methods to tackle NCP. This type of methods have a common feature which used to force the iterates  $\{x^{k+1}\}$  to stay in the interior of the nonnegative orthant  $R_{++}^n$ . Auslender *et al.* [1] have proposed a new type of proximal interior algorithms via replacing the quadratic function (3) by  $d_\phi(x, x^k)$  which could be defined as

$$d_\phi(x, y) = \sum_{j=1}^n y_j^2 \phi(y_j^{-1} x_j).$$

Let  $\nu > \mu > 0$  be given fixed parameters, and  $\phi$  is defined by

$$\phi(t) = \begin{cases} \frac{\nu}{2}(t-1)^2 + \mu\varphi(t) & \text{if } t > 0, \\ +\infty & \text{otherwise.} \end{cases}$$

The following few examples of  $\varphi$  function enjoy many attractive properties for developing efficient algorithms to solve NCP.

$$\varphi_1(t) = t - \log(t) - 1,$$

$$\varphi_2(t) = t \log(t) - t + 1,$$

$$\varphi_3(t) = (\sqrt{t} - 1)^2.$$

In [2], Auslender *et al.* have used an logarithmic-quadratic proximal (LQP) method by using  $\varphi_1$  (with  $\nu = 2, \mu = 1$ ) for solving variational inequalities over polyhedra. Later, Bnouhachem [3] has proposed a new modified LQP method by using  $\varphi_2$  (with  $\nu = 1, \mu \in (0, 1)$ ). The interior point methods with logarithmic-quadratic proximal regularization, we quoted references [2–8, 11, 12, 15, 17].

Let  $\nu = \frac{1}{2}$  and  $\mu \in (0, 1)$ , in our proposed method, we consider the function  $\varphi_3$ , and we get

$$\phi(t) = \begin{cases} \frac{1}{4}(t-1)^2 + \mu(\sqrt{t}-1)^2 & \text{if } t > 0, \\ +\infty & \text{otherwise.} \end{cases}$$

Then, the problem (2) becomes for given  $x^k \in R_{++}^n$  and  $\beta_k \geq \beta > 0$ , the new iterate  $x^{k+1}$  is unique solution of the following set-valued equation:

$$(4) \quad 0 \in \beta_k T(x) + \nabla_x d_\phi(x, x^k),$$

where

$$d_\phi(x, x^k) = \begin{cases} \frac{1}{4} \|x - x^k\|^2 + \mu \sum_{j=1}^n \left( x_j^k x_j - 2(x_j^k)^2 \sqrt{\frac{x_j}{x_j^k}} + (x_j^k)^2 \right) & \text{if } x \in R_{++}^n, \\ +\infty & \text{otherwise.} \end{cases}$$

It is easy to see that

$$(5) \quad \begin{aligned} \nabla_x d_\phi(x, x^k) &= \frac{1}{2}(x - x^k) + \mu \sum_{j=1}^n \left( x_j^k - \frac{(x_j^k)^2}{\sqrt{x_j^k}} \frac{1}{\sqrt{x_j}} \right) \\ &= \frac{1}{2}(x - x^k) + \mu(x^k - X_k(\sqrt{x})^{-1}), \end{aligned}$$

where  $X_k = \text{diag}(\sqrt{x_1^k}, \dots, \sqrt{x_n^k})$  and  $\sqrt{x} = (\sqrt{x_1}, \dots, \sqrt{x_n})$ .

Since  $d_\phi(x, x^k)$  includes both square and quadratic terms, this method is called the Square-Quadratic Proximal (SQP) method.

Then the problem (4) is equivalent to the following systems of nonlinear equations

$$(6) \quad \beta_k F(x) + \frac{1}{2}(x - x^k) + \mu(x^k - X_k(\sqrt{x})^{-1}) = 0.$$

The aim of this paper is to develop an algorithm for solving nonlinear complementarity problems. More precisely, instead of using the logarithmic quadratic term, we used an square root quadratic term. It is more practical to find approximate solutions of a system of nonlinear equations (6) rather than exact solutions due to the fact in general that excludes some practical applications. Driven by the fact of eliminating this drawback, we presented a prediction-correction method to solve (6) approximately. Under certain conditions, the global convergence of the proposed method is proved. Our results can be viewed as significant extensions of the previously known results.

## 2. Preliminaries

The following lemma provides some basic properties of projection onto  $R_+^n$ . We denote by  $P_{R_+^n}(\cdot)$  the projection under  $R_+^n$ , that is,

$$P_{R_+^n}(z) = \operatorname{argmin}\{\|z - x\| : x \in R_+^n\}.$$

**Lemma 2.1** ([14]). *We have the following inequalities.*

$$(7) \quad (y - P_{R_+^n}(y))^T (P_{R_+^n}(y) - x) \geq 0, \quad \forall y \in R^n, \quad \forall x \in R_+^n.$$

$$(8) \quad \|P_{R_+^n}(v) - u\|^2 \leq \|v - u\|^2 - \|v - P_{R_+^n}(v)\|^2, \quad \forall v \in R^n, u \in R_+^n.$$

*Proof.* First, based on the definition of  $P_{R_+^n}$ , we have

$$\|y - P_{R_+^n}(y)\| \leq \|y - z\| \quad \forall z \in R_+^n.$$

Note that for any  $y \in R^n$ ,  $P_{R_+^n}(y) \in R_+^n$ , since  $R_+^n \subset R^n$  is convex and closed, then for any  $x \in R_+^n$  and  $\theta \in (0, 1)$ , it follows that

$$\theta x + (1 - \theta)P_{R_+^n}(y) = P_{R_+^n}(y) + \theta(x - P_{R_+^n}(y)) \in R_+^n$$

and

$$\|y - P_{R_+^n}(y)\|^2 \leq \|y - P_{R_+^n}(y) - \theta(x - P_{R_+^n}(y))\|^2.$$

Hence, we get

$$[y - P_{R_+^n}(y)]^T [x - P_{R_+^n}(y)] \leq \frac{\theta}{2} \|x - P_{R_+^n}(y)\|^2, \quad \forall y \in R_+^n \text{ and } \theta \in (0, 1).$$

Letting  $\theta \rightarrow 0_+$  and inequality (7) is proved. Moreover, using (7) we have

$$\begin{aligned} \|P_{R_+^n}(y) - x\|^2 &= \|(y - x) - (y - P_{R_+^n}(y))\|^2 \\ &= \|y - x\|^2 + 2(x - y)^T (y - P_{R_+^n}(y)) + \|y - P_{R_+^n}(y)\|^2 \\ &= \|y - x\|^2 + 2(x - P_{R_+^n}(y))^T (y - P_{R_+^n}(y)) - \|y - P_{R_+^n}(y)\|^2 \\ &\leq \|y - x\|^2 - \|y - P_{R_+^n}(y)\|^2, \end{aligned}$$

and inequality (8) is proved.  $\square$

We need the following result in the convergence analysis of the proposed method.

**Lemma 2.2.** *For given  $x^k > 0$  and  $q \in R^n$ , let  $x$  be the positive solution of the following equation:*

$$(9) \quad q + \frac{1}{2}(x - x^k) + \mu(x^k - X_k(\sqrt{x})^{-1}) = 0$$

where  $X_k = \text{diag}(\sqrt{x_1^k}, \dots, \sqrt{x_n^k})$  and  $\sqrt{x} = (\sqrt{x_1}, \dots, \sqrt{x_n})$ , then for any  $y \geq 0$  we have

$$(10) \quad (x - y)^T (-q) \geq \frac{1 + \mu}{4} (\|x - y\|^2 - \|x^k - y\|^2) + \frac{1 - \mu}{4} \|x^k - x\|^2.$$

*Proof.* For each  $t > 0$  we have  $\frac{1}{2}(1 - \frac{1}{t}) \leq 1 - \frac{1}{\sqrt{t}} \leq \frac{1}{2}(t - 1)$ , then we obtain after multiplication by  $y_j x_j^k \geq 0$  for each  $j = 1, \dots, n$ ,

$$y_j x_j^k \left(1 - \frac{\sqrt{x_j^k}}{\sqrt{x_j}}\right) \leq y_j x_j^k \frac{1}{2} \left(\frac{x_j}{x_j^k} - 1\right) = \frac{1}{2} y_j (x_j - x_j^k)$$

and after multiplication by  $x_j x_j^k \geq 0$  for each  $j = 1, \dots, n$ ,

$$-x_j x_j^k \left(1 - \frac{\sqrt{x_j^k}}{\sqrt{x_j}}\right) \leq x_j x_j^k \frac{1}{2} \left(\frac{x_j}{x_j^k} - 1\right) = \frac{1}{2} x_j^k (x_j^k - x_j),$$

adding the two inequalities, then we obtained

$$\begin{aligned} & (y_j - x_j) \left( \frac{1}{2}(x_j - x_j^k) + \mu \left( x_j^k - \left( \sqrt{x_j^k} \right)^3 (\sqrt{x_j})^{-1} \right) \right) \\ & \leq \frac{1}{2} \mu (y_j - x_j^k)(x_j - x_j^k) + \frac{1}{2} (x_j - x_j^k)(y_j - x_j). \end{aligned}$$

Using the identities

$$\begin{aligned} \frac{1}{2} (y_j - x_j^k)(x_j - x_j^k) &= \frac{1}{4} \left( (x_j - x_j^k)^2 - (x_j - y_j)^2 + (y_j - x_j^k)^2 \right), \\ \frac{1}{2} (x_j - x_j^k)(y_j - x_j) &= \frac{1}{4} \left( (y_j - x_j^k)^2 - (y_j - x_j)^2 - (x_j - x_j^k)^2 \right) \end{aligned}$$

and recalling (9), thus we obtained

$$(x_j - y_j)(-q_j) \geq \frac{1 + \mu}{4} \left( (x_j - y_j)^2 - (x_j^k - y_j)^2 \right) + \frac{1 - \mu}{4} (x_j^k - x_j)^2.$$

Summing over  $j = 1, \dots, n$ , encountered (10) and this completes the proof.  $\square$

### 3. Square quadratic proximal method

We propose the following SQP method for solving problem (1). For given  $x^1 > 0$ ,  $\mu \in (0, 1)$  and  $D_0 = 0$ , the proposed method consists of two steps, the first step offers  $\tilde{x}^k$ , and the second step produces the new iterate  $x^{k+1}$ .

*Prediction step:* Find an approximate solution  $\tilde{x}^k$  of (6), called predictor, such that

$$(11) \quad 0 \approx \beta_k F(x) + \frac{1}{2}(x - x^k) + \mu(x^k - X_k(\sqrt{x})^{-1}) = \xi^k$$

and  $\xi^k := \beta_k(F(\tilde{x}^k) - F(x^k))$  satisfies

$$(12) \quad \|\xi^k\| \leq \eta \|x^k - \tilde{x}^k\|, \quad 0 < \eta < \frac{1}{2}.$$

*Correction step:* Compute

$$(13) \quad d_k = \frac{1}{2}(x^k - \tilde{x}^k) + \frac{1}{1 + \mu} \xi^k$$

and

$$(14) \quad D_k = d_k + \theta_k D_{k-1},$$

where

$$(15) \quad \theta_k = \max \left( 0, \frac{-d_k^T D_{k-1}}{\|D_{k-1}\|^2} \right).$$

The new iterate  $x^{k+1}(\alpha_k)$  is defined by

$$(16) \quad x^{k+1}(\alpha_k) = \rho x^k + (1 - \rho) P_{R_+^n} \left[ x^k - \alpha_k D_k \right], \quad \rho \in (0, 1),$$

where

$$(17) \quad \alpha_k = \frac{\psi(x^k)}{\|D_k\|^2} \quad \text{and} \quad \psi(x^k) = \frac{1}{2(1+\mu)} \|x^k - \tilde{x}^k\|^2 + \frac{1}{1+\mu} (x^k - \tilde{x}^k)^T \xi^k.$$

*Remark 3.1.* It follows from (12) that

$$(18) \quad \psi(x^k) \geq \frac{1-2\eta}{2(1+\mu)} \|x^k - \tilde{x}^k\|^2.$$

We need the following results to study the convergence analysis of the proposed method.

**Lemma 3.1.** *Using the definitions of  $d_k$  and  $D_k$ , then*

$$(19) \quad \|D_k\| \leq \|d_k\|.$$

*Proof.* It follows from (14) that

$$\begin{aligned} \|D_k\|^2 &= \|d_k + \theta_k D_{k-1}\|^2 \\ &= \|d_k\|^2 - \frac{(d_k^T D_{k-1})^2}{\|D_{k-1}\|^2} \\ &\leq \|d_k\|^2 \end{aligned}$$

which implies that

$$\|D_k\| \leq \|d_k\|. \quad \square$$

**Theorem 3.1.** *Let  $x^*$  be any solution of (1). For given  $x^k \in R_{++}^n$  and  $\beta_k > 0$ , let  $\tilde{x}^k$  and  $\xi^k$  satisfied the condition (12), then it holds*

$$(20) \quad (x^k - x^*)^T d_k \geq \psi(x^k).$$

*Proof.* For given  $x^*$  be any solution of (1),  $x^k \in R_{++}^n$  and  $\beta_k > 0$ , let  $\tilde{x}^k$  and  $\xi_k$  be obtained by (11). By setting  $q = \beta_k F(\tilde{x}^k) - \xi_k$  in (10), we obtain

$$(21) \quad \begin{aligned} &(\tilde{x}^k - x^*)^T (\xi_k - \beta_k F(\tilde{x}^k)) \\ &\geq \frac{1+\mu}{4} (\|\tilde{x}^k - x^*\|^2 - \|x^k - x^*\|^2) + \frac{1-\mu}{4} \|x^k - \tilde{x}^k\|^2, \end{aligned}$$

then

$$(22) \quad \begin{aligned} (\tilde{x}^k - x^*)^T \xi_k &\geq \frac{1+\mu}{4} (\|\tilde{x}^k - x^*\|^2 - \|x^k - x^*\|^2) \\ &\quad + \frac{1-\mu}{4} \|x^k - \tilde{x}^k\|^2 + \beta_k (\tilde{x}^k - x^*)^T F(\tilde{x}^k). \end{aligned}$$

Since  $F$  is monotone and  $x^*$  is solution of (1), we get

$$(23) \quad (\tilde{x}^k - x^*)^T F(\tilde{x}^k) \geq (\tilde{x}^k - x^*)^T F(x^*) \geq 0.$$

Substituting (23) in (22), we obtain

$$\frac{1}{1+\mu} (x^k - x^*)^T \xi_k \geq \frac{1}{4} (\|\tilde{x}^k - x^*\|^2 - \|x^k - x^*\|^2)$$

$$(24) \quad + \frac{1-\mu}{4(1+\mu)} \|x^k - \tilde{x}^k\|^2 + \frac{1}{1+\mu} (x^k - \tilde{x}^k)^T \xi_k.$$

Using the following identity

$$(25) \quad \frac{1}{2} (x^k - x^*)^T (x^k - \tilde{x}^k) = \frac{1}{4} (\|x^k - x^*\|^2 - \|\tilde{x}^k - x^*\|^2) + \frac{1}{4} \|x^k - \tilde{x}^k\|^2$$

and (17), (24), we have

$$\begin{aligned} (x^k - x^*)^T d_k &= \frac{1}{2} (x^k - x^*)^T (x^k - \tilde{x}^k) + \frac{1}{1+\mu} (x^k - x^*)^T \xi_k \\ &\geq \frac{1}{2(1+\mu)} \|x^k - \tilde{x}^k\|^2 + \frac{1}{1+\mu} (x^k - \tilde{x}^k)^T \xi_k \\ &= \psi(x^k). \end{aligned}$$

Therefore, the assertion of this lemma is proved.  $\square$

**Lemma 3.2.** *For any  $k \geq 1$ , we have*

$$D_{k-1}^T (x^k - x^*) \geq 0.$$

*Proof.* Note that this is trivially true for  $k = 1$  since  $D_0 = 0$ . By induction, consider any  $k \geq 2$  and assume that

$$D_{k-2}^T (x^{k-1} - x^*) \geq 0.$$

Using the definition of  $D_{k-1}$ , we have

$$\begin{aligned} D_{k-1}^T (x^k - x^*) &= D_{k-1}^T (x^{k-1} - x^*) + D_{k-1}^T (x^k - x^{k-1}) \\ &= d_{k-1}^T (x^{k-1} - x^*) + \theta_{k-1} D_{k-2}^T (x^{k-1} - x^*) + D_{k-1}^T (x^k - x^{k-1}) \\ &\geq d_{k-1}^T (x^{k-1} - x^*) + D_{k-1}^T (x^k - x^{k-1}) \\ &\geq \psi(x^{k-1}) - \|D_{k-1}\| \|PR_+^n [x^{k-1} - \alpha_{k-1} D_{k-1}] - x^{k-1}\| \\ &\geq \psi(x^{k-1}) - \|D_{k-1}\| \|\alpha_{k-1} D_{k-1}\| \\ &= 0, \end{aligned}$$

where the second inequality follows from (20).  $\square$

*Remark 3.2.* From Theorem 3.1 and Lemma 3.2 it is easy to prove that

$$(26) \quad (x^k - x^*)^T D_k \geq \psi(x^k).$$

To ensure that  $x^{k+1}(\alpha_k)$  is closer to the solution set than  $x^k$ . For this purpose, we define

$$(27) \quad \Theta(\alpha_k) = \|x^k - x^*\|^2 - \|x^{k+1}(\alpha_k) - x^*\|^2.$$

The following theorem provides a unified framework for proving the convergence of the proposed algorithm.

**Theorem 3.2.** *Let  $D_k$  and  $\psi$  be defined by (14) and (17) respectively. Then for any  $x^* \in \Omega^*$  and  $\alpha_k > 0$ , we have*

$$(28) \quad \Theta(\alpha_k) \geq (1 - \rho)\Phi(\alpha_k),$$

where

$$(29) \quad \Phi(\alpha_k) = \alpha_k \psi(x^k).$$

*Proof.* Since  $x^* \in \Omega^* \subset R_+^n$  and let  $x_*^k(\alpha_k) = P_{R_+^n} [x^k - \alpha_k D_k]$ , it follows from (8) that

$$(30) \quad \|x_*^k(\alpha_k) - x^*\|^2 \leq \|x^k - \alpha_k D_k - x^*\|^2 - \|x^k - \alpha_k D_k - x_*^k(\alpha_k)\|^2.$$

On the other hand, we have

$$\begin{aligned} \|x^{k+1}(\alpha_k) - x^*\|^2 &= \|\rho(x^k - x^*) + (1 - \rho)(x_*^k(\alpha_k) - x^*)\|^2 \\ &= \rho^2 \|x^k - x^*\|^2 + (1 - \rho)^2 \|x_*^k(\alpha_k) - x^*\|^2 \\ &\quad + 2\rho(1 - \rho)(x^k - x^*)^T (x_*^k(\alpha_k) - x^*). \end{aligned}$$

Using the following identity

$$2(a + b)^T b = \|a + b\|^2 - \|a\|^2 + \|b\|^2$$

for  $a = x^k - x_*^k(\alpha_k)$ ,  $b = x_*^k(\alpha_k) - x^*$  and (30), and using  $0 < \rho < 1$ , we obtain

$$\begin{aligned} \|x^{k+1}(\alpha_k) - x^*\|^2 &= \rho^2 \|x^k - x^*\|^2 + (1 - \rho)^2 \|x_*^k(\alpha_k) - x^*\|^2 \\ &\quad + \rho(1 - \rho)\{\|x^k - x^*\|^2 - \|x^k - x_*^k(\alpha_k)\|^2 \\ &\quad + \|x_*^k(\alpha_k) - x^*\|^2\} \\ &= \rho \|x^k - x^*\|^2 + (1 - \rho) \|x_*^k(\alpha_k) - x^*\|^2 \\ &\quad - \rho(1 - \rho) \|x^k - x_*^k(\alpha_k)\|^2 \\ &\leq \rho \|x^k - x^*\|^2 + (1 - \rho) \|x^k - \alpha_k D_k - x^*\|^2 \\ &\quad - (1 - \rho) \|x^k - \alpha_k D_k - x_*^k(\alpha_k)\|^2 \\ &\quad - \rho(1 - \rho) \|x^k - x_*^k(\alpha_k)\|^2 \\ &= \|x^k - x^*\|^2 - (1 - \rho) \{\|x^k - x_*^k(\alpha_k) - \alpha_k D_k\|^2 \\ &\quad + \rho \|x^k - x_*^k(\alpha_k)\|^2 - \alpha_k^2 \|D_k\|^2 + 2\alpha_k (x^k - x^*)^T D_k\} \\ &\leq \|x^k - x^*\|^2 - (1 - \rho) \{2\alpha_k \psi(x^k) - \alpha_k^2 \|D_k\|^2\}, \end{aligned}$$

where the second inequality follows from (26). Using the definition of  $\Theta(\alpha_k)$  and  $\Phi(\alpha_k)$ , then (28) is proved.  $\square$

In the next theorem, we show that  $\alpha_k$  and  $\Phi(\alpha_k)$  are bounded away from zero, and it is one of the keys to prove the global convergence results.



**Theorem 3.3.** For given  $x^k \in R_+^n$  and  $\beta_k > 0$ , let  $\tilde{x}^k$  and  $\xi^k$  satisfied the condition (12), then we have the following,

$$(31) \quad \alpha_k \geq \frac{1-2\eta}{4(1+\mu)}$$

and

$$(32) \quad \Phi(\alpha_k) \geq \frac{(1-2\eta)^2}{8(1+\mu)^2} \|x^k - \tilde{x}^k\|^2.$$

*Proof.* If  $(x^k - \tilde{x}^k)^T \xi^k \leq 0$ , since  $\mu > 0$  it follows from (12), (13), (14) and (19) that

$$(33) \quad \begin{aligned} \|D_k\|^2 &\leq \|d_k\|^2 \\ &\leq \frac{1}{4} \|x^k - \tilde{x}^k\|^2 + \frac{1}{(1+\mu)^2} \|\xi^k\|^2 \\ &\leq \|x^k - \tilde{x}^k\|^2 + \|\xi^k\|^2 \\ &\leq 2\|x^k - \tilde{x}^k\|^2, \end{aligned}$$

from (18) and (33), we obtain

$$\alpha_k = \frac{\psi(x^k)}{\|D_k\|^2} \geq \frac{1-2\eta}{4(1+\mu)}.$$

Otherwise, if  $(x^k - \tilde{x}^k)^T \xi^k \geq 0$ , it follows from  $0 < \mu < 1$  and (12) that

$$\begin{aligned} \psi(x^k) &= \frac{1}{2(1+\mu)} \|x^k - \tilde{x}^k\|^2 + \frac{1}{1+\mu} (x^k - \tilde{x}^k)^T \xi^k \\ &\geq \frac{1}{1+\mu} \left\{ \frac{1}{4} \|x^k - \tilde{x}^k\|^2 + \frac{1}{1+\mu} (x^k - \tilde{x}^k)^T \xi^k \right. \\ &\quad \left. + \frac{1}{4} \|x^k - \tilde{x}^k\|^2 \right\} \\ &\geq \frac{1}{1+\mu} \left\{ \frac{1}{16} \|x^k - \tilde{x}^k\|^2 + \frac{1}{4(1+\mu)} (x^k - \tilde{x}^k)^T \xi^k \right. \\ &\quad \left. + \frac{1}{4(1+\mu)^2} \|\xi^k\|^2 \right\} \\ &= \frac{1}{4(1+\mu)} \|d_k\|^2 \\ &\geq \frac{1}{4(1+\mu)} \|D_k\|^2 \end{aligned}$$

and thus

$$\alpha_k \geq \frac{1}{4(1+\mu)} \geq \frac{1-2\eta}{4(1+\mu)}.$$

Using (29), (31) and (18), directly we obtain (32).  $\square$

For fast convergence, we take a relaxation factor  $\gamma \in [1, 2)$  and the step-size  $\alpha_k$  in (16) by  $\alpha_k = \gamma\alpha_k$ . Through simple manipulation we obtain

$$\begin{aligned} \Phi(\gamma\alpha_k) &= 2\gamma\alpha_k\psi(x^k) - (\gamma^2\alpha_k)(\alpha_k\|D_k\|^2) \\ &= (2\gamma\alpha_k - \gamma^2\alpha_k)\psi(x^k) \\ (34) \quad &= \gamma(2 - \gamma)\Phi(\alpha_k). \end{aligned}$$

It follows from Theorem 3.2 and Theorem 3.3 that there is a constant  $c > 0$  such that

$$(35) \quad \|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 - c\|x^k - \tilde{x}^k\|^2 \quad \forall x^* \in \Omega^*.$$

Now, the convergence of the proposed method could be proved as follows:

**Theorem 3.4** ([3]). *If  $\inf_{k=0}^{\infty} \beta_k = \beta > 0$ , then the sequence  $\{x^k\}$  generated by the proposed method converges to some  $x^\infty$  which is a solution of the NCP.*

#### 4. Preliminary computational results

For numerical experimental we need to find the value of the approximate solution  $\tilde{x}^k$ , in the special case  $\xi^k = \beta_k(F(\tilde{x}^k) - F(x^k))$ , then (11) is equivalent to the following system of nonlinear equations

$$(36) \quad \beta_k F(x^k) + \frac{1}{2}(\tilde{x}^k - x^k) + \mu(x^k - X_k(\sqrt{\tilde{x}^k})^{-1}) = 0,$$

hence

$$\frac{1}{2}\tilde{x}_j^k - \mu \frac{(\sqrt{x_j^k})^3}{\sqrt{\tilde{x}_j^k}} + \left( \beta_k F_j(x^k) - \frac{1}{2}x_j^k + \mu x_j^k \right) = 0, \quad j = 1, \dots, n.$$

Then

$$(37) \quad \frac{1}{2}\tilde{x}_j^k - \mu \frac{(\sqrt{x_j^k})^3}{\sqrt{\tilde{x}_j^k}} + \left( \beta_k F_j(x^k) - \frac{1}{2}x_j^k + \mu \frac{(\sqrt{x_j^k})^3}{\sqrt{x_j^k}} \right) = 0, \quad j = 1, \dots, n.$$

The recursion of classical Newton method for the above problem is

$$\tilde{x}_j^k := x_j^k - \frac{2\beta_k}{1 + \mu} F_j(x^k).$$

The solution of (37) is  $\tilde{x}^k > 0$ , to avoid the non-positive value  $\tilde{x}_j^k$  in the iteration process, we take

$$\tilde{x}_j^k := \rho_1 x_j^k + (1 - \rho_1) \max\{x_j^k - \frac{2\beta_k}{1 + \mu} F_j(x^k), 0\}, \quad j = 1, \dots, n, \quad \rho_1 \in (0, 1).$$

We consider a network [13, 16, 17] shown in Fig.1 which consisted of 7 nodes, 11 links, we use the same notations as [17]. The traffic equilibrium problems can be described as follows:

Let  $t = \{t_a, a \in L\}$  be the row vector of link costs, with  $t_a$  denoting the user cost of travelling link  $a$  which is given by

$$(38) \quad t_a(f_a) = t_a^0 \left[ 1 + 0.15 \left( \frac{f_a}{C_a} \right)^4 \right],$$

where  $t_a^0$  is the free-flow travel cost on link  $a$  and  $C_a$  is designed capacity of link  $a$ . Let  $P$  denote the set of all the paths concerned. Let  $\theta = \{\theta_p, p \in P\}$  be the vector of (path) travel cost. For given link travel cost vector  $t$ ,  $\theta$  is a mapping of the path-flow  $u$ , which is given by

$$\theta(u) = At(u) = At(A^T u).$$

Associated with every O/D pair  $\omega$ , there is a travel disutility  $\lambda_\omega(d)$ , which is defined as following

$$(39) \quad \lambda_\omega(d) = -m_\omega \log(d_\omega) + q_\omega.$$

Note that both the path costs and the travel disutilities are functions of the flow pattern  $u$ . The traffic network equilibrium problem is to seek the path flow pattern  $u^*$ , which induces a demand pattern  $d^* = d(u^*)$ , for every O/D pair  $\omega$  and each path  $p \in P_\omega$ ,

$$T_p(u) = \theta_p(u) - \lambda_\omega(d(u)).$$

The problem can be reduced to a variational inequality in the space of path-flow pattern  $u$ :

$$(40) \quad \text{Find } u^* \geq 0 \text{ such that } (u - u^*)^T T(u^*) \geq 0, \quad \forall u \geq 0.$$

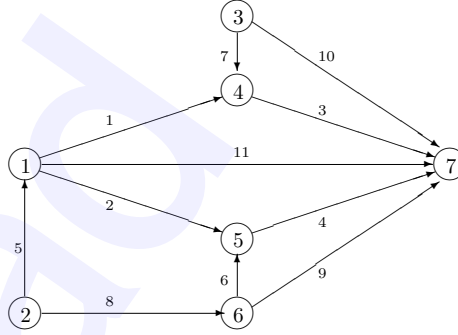


FIGURE 1. The network used for the numerical test.

The free-flow travel cost and the designed capacity of links (38) are given in Table 1, the O/D pairs and the coefficient  $m$  and  $q$  in the disutility function

(39) are given in Table 2. For this example, there are together 12 paths for the 4 given O/D pairs listed in Table 4.

TABLE 1. The free-flow cost and the designed capacity of links in (38)

Link	Free-flow travel time $t_a^0$	Capacity $C_a$	Link	Free-flow travel time $t_a^0$	Capacity $C_a$
1	6	200	7	5	150
2	5	200	8	10	150
3	6	200	9	11	200
4	16	200	10	11	200
5	6	100	11	15	200
6	1	100	—	—	—

TABLE 2. The O/D pairs and the coefficient  $m$  and  $q$  in (39)

No. of the pair	O/D pair	$m_\omega$	$q_\omega$
1	(1, 7)	25	$25 \log 600$
2	(2, 7)	33	$33 \log 500$
3	(3, 7)	20	$20 \log 500$
4	(6, 7)	20	$20 \log 400$

TABLE 3. Numerical results for different  $\varepsilon$

Different $\varepsilon$	The method in [17]		The proposed method	
	No. It.	CPU(Sec.)	No. It.	CPU(Sec.)
$10^{-4}$	231	0.031	182	0.023
$10^{-5}$	284	0.037	235	0.032
$10^{-6}$	338	0.041	287	0.038
$10^{-7}$	391	0.046	340	0.042
$10^{-8}$	444	0.051	393	0.048

In all tests we take the logarithmic proximal parameter  $\mu = 0.9$ ,  $\rho = \rho_1 = 0.01$ ,  $\gamma = 1.9$  and  $\eta = 0.45$ . All iterations start with  $x^1 = (0.5, \dots, 0.5)^T$  and  $\beta_1 = 1$ , and stopped whenever  $\|\min(x^k, F(x^k))\|_\infty \leq \varepsilon$ . The iteration numbers and the computational time for the proposed method and the method in [17] for different  $\varepsilon$  are reported in Table 3. For the case  $\varepsilon = 10^{-8}$ , the optimal path flow and link flow are given in Tables 4 and 5, respectively.

Table 3 shows that the proposed method is more flexible and efficient for the problem tested. Moreover, it demonstrates computationally that the new method is more effective than that in [17] in the sense that the new method needs fewer iterations and less computational time.

TABLE 4. The optimal path follow

O/D pair	Path no.	Link of path	Optimal path-flow
O/D pair (1,7)	1	(1, 3)	165.3145
	2	(2, 4)	0
	3	(11)	138.5735
	4	(5, 1, 3)	82.5281
O/D pair (2,7)	5	(5, 2, 4)	0
	6	(5, 11)	55.7871
	7	(8, 6, 4)	0
	8	(8, 9)	87.0260
O/D pair (3,7)	9	(7, 3)	19.7549
	10	(10)	229.9747
O/D pair (6,7)	11	(9)	178.5600
	12	(6, 4)	0

TABLE 5. The optimal link flow

Link no.	Link flow	Link no.	Link flow	Link no.	Link flow	Link no.	Link flow
1	247.8426	4	0	7	19.7549	10	229.9747
2	0	5	138.3152	8	87.0260	11	194.3606
3	267.5974	6	0	9	265.5860	—	—

### 5. Concluding remarks

In this paper, we proposed a new class of predictor-corrector algorithms for solving nonlinear complementarity problems by using an square root quadratic proximal term. Global convergence of the proposed method is proved under mild assumptions. The numerical results showed that our algorithm is efficient for the problem tested, well as the computational results emphasis that its very significant.

**Acknowledgements.** The authors are very grateful to the referees for their careful reading, comments, and suggestions, which help us improve the presentation of this paper.

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