

ON COEFFICIENT PROBLEMS FOR STARLIKE FUNCTIONS RELATED TO VERTICAL STRIP DOMAINS

OH SANG KWON AND YOUNG JAE SIM

ABSTRACT. In the present paper, we find the sharp bound for the fourth coefficient of starlike functions f which are normalized by $f(0) = 0 = f'(0) - 1$ and satisfy the following two-sided inequality:

$$1 + \frac{\gamma - \pi}{2 \sin \gamma} < \Re \left\{ \frac{zf'(z)}{f(z)} \right\} < 1 + \frac{\gamma}{2 \sin \gamma}, \quad z \in \mathbb{D},$$

where $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ is the unit disk and γ is a real number such that $\pi/2 \leq \gamma < \pi$. Moreover, the sharp bound for the fifth coefficient of f defined above with γ in a subset of $[\pi/2, \pi)$ also will be found.

1. Introduction

Let \mathcal{H} be the class of analytic functions in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and let \mathcal{A} be its subclass of f normalized by $f(0) = 0 = f'(0) - 1$. That is, f has the form

$$(1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}.$$

Let the parameters α and β be real numbers such that $0 \leq \alpha < 1 < \beta$. A function $p \in \mathcal{H}$ is said to belong to the class $\mathcal{P}(\alpha, \beta)$ if p satisfies $p(0) = 1$ and the inequality $\alpha < \Re\{p(z)\} < \beta$ for all $z \in \mathbb{D}$. In [4], the authors introduced the class $\mathcal{S}(\alpha, \beta)$ of functions $f \in \mathcal{A}$ for which satisfies $zf'(z)/f(z) \in \mathcal{P}(\alpha, \beta)$ for all $z \in \mathbb{D}$. They obtained for functions $f \in \mathcal{S}(\alpha, \beta)$ which have the form given by (1), the following inequality holds.

$$(2) \quad |a_n| \leq \prod_{k=2}^n \frac{k - 2 + \frac{2(\beta - \alpha)}{\pi} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha}}{(n - 1)!}, \quad n \in \mathbb{N} \setminus \{1\}.$$

Received March 16, 2018; Accepted July 24, 2018.

2010 *Mathematics Subject Classification*. Primary 30C45, 30C80.

Key words and phrases. coefficient estimate, starlike function, vertical strip domain.

This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIP; Ministry of Science, ICT & Future Planning) (No. NRF-2017R1C1B5076778).

Moreover, they obtained the sharp results for the second and third coefficients for functions in $\mathcal{S}(\alpha, \beta)$ as follows (See [4, 5]):

$$(3) \quad |a_2| \leq \frac{2(\beta - \alpha)}{\pi} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha}$$

and

$$(4) \quad |a_3| \leq \frac{(\beta - \alpha)}{\pi} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha} \left[\cos \frac{\pi(1 - \alpha)}{\beta - \alpha} + \frac{2(\beta - \alpha)}{\pi} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha} \right].$$

Given $\gamma \in [\pi/2, \pi)$, we consider a class denoted by $\mathcal{M}(\gamma)$ of functions $f \in \mathcal{A}$ such that

$$1 + \frac{\gamma - \pi}{2 \sin \gamma} < \Re \left\{ \frac{zf'(z)}{f(z)} \right\} < 1 + \frac{\gamma}{2 \sin \gamma}, \quad z \in \mathbb{D}.$$

The class $\mathcal{M}(\gamma)$ was introduced and investigated by Kargar *et al.* [3]. And this class can be reduced by the class $\mathcal{S}(\alpha, \beta)$ by putting

$$\alpha = 1 + \frac{\gamma - \pi}{2 \sin \gamma} \quad \text{and} \quad \beta = 1 + \frac{\gamma}{2 \sin \gamma}, \quad \gamma \in [\pi/2, \pi).$$

Therefore, for functions $f \in \mathcal{M}(\gamma)$ with the form given by (1), we obtain the inequalities $|a_n| \leq 1$ for $n \in \mathbb{N} \setminus \{1\}$ by (2). Furthermore, from (3) and (4), we get the sharp inequalities $|a_2| \leq 1$ and $|a_3| \leq (1 - \cos \gamma)/2$.

The purpose of this paper is to obtain the sharp bound for the fourth and fifth coefficients of function in $\mathcal{M}(\gamma)$. For this, the following notions and results on them are required.

For analytic functions f and g , we say that f is subordinate to g , denoted by $f \prec g$, if there is an analytic function $w : \mathbb{D} \rightarrow \mathbb{D}$ with $|w(z)| \leq |z|$ such that $f(z) = g(w(z))$. Further, if g is univalent, then the definition of subordination $f \prec g$ simplifies to the conditions $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$.

By using the notion of subordination, we can obtain the following equivalent condition for $f \in \mathcal{M}(\gamma)$:

$$(5) \quad \frac{zf'(z)}{f(z)} \prec B_\gamma(z) := 1 + \frac{1}{2i \sin \gamma} \log \left(\frac{1 + ze^{i\gamma}}{1 + ze^{-i\gamma}} \right), \quad z \in \mathbb{D},$$

where $\pi/2 \leq \gamma < \pi$. We note that the function B_γ defined by (5) can be represented by

$$B_\gamma(z) = 1 + \sum_{n=1}^{\infty} A_n z^n, \quad z \in \mathbb{D},$$

where

$$A_n = \frac{(-1)^{n-1} \sin n\gamma}{n \sin \gamma}, \quad n \in \mathbb{N}.$$

Throughout this paper, for given $\gamma \in [\pi/2, \pi)$, we will denote

$$\tau = \tau(\gamma) = \cos \gamma \in (-1, 0],$$

for the sake of our convenient notation. We note that, using the notation τ , several initial coefficients of B_γ can be represented by

$$A_1 = 1, \quad A_2 = -\tau, \quad A_3 = \frac{1}{3}(4\tau^2 - 1) \quad \text{and} \quad A_4 = \tau - 2\tau^3.$$

Let \mathcal{P} be the class of Carathéodory functions $p \in \mathcal{H}$ of the form

$$(6) \quad p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbb{D},$$

having a positive real part in \mathbb{D} . For given $p \in \mathcal{P}$ define a function $w : \mathbb{D} \rightarrow \mathbb{C}$ by $w(z) = (p(z) - 1)/(p(z) + 1)$. Then the property $w(\mathbb{D}) \subset \mathbb{D}$ holds and therefore $|w(z)| \leq |z|$ for all $z \in \mathbb{D}$.

Now, let us recall several results for the class \mathcal{P} will be used in further considerations.

Lemma 1.1 ([2, p. 41]). *If $p \in \mathcal{P}$ is of the form (6), then*

$$(7) \quad |c_n| \leq 2, \quad n \in \mathbb{N}.$$

The inequality (7) is sharp and the equality holds for the function L defined by

$$L(z) = \frac{1+z}{1-z} = 1 + 2 \sum_{n=1}^{\infty} z^n, \quad z \in \mathbb{D}.$$

Lemma 1.2 ([8, 9, Libera and Zlotkiewicz]). *If $p \in \mathcal{P}$ is of the form (6) with $c_1 \geq 0$, then*

$$(8) \quad 2c_2 = c_1^2 + \zeta(4 - c_1^2)$$

and

$$(9) \quad 4c_3 = c_1^3 + 2c_1(4 - c_1^2)\zeta - c_1(4 - c_1^2)\zeta^2 + 2(4 - c_1^2)(1 - |\zeta|^2)\eta$$

for some ζ and η such that $|\zeta| \leq 1$ and $|\eta| \leq 1$.

Lemma 1.3 ([6, Kwon, Lecko and Sim]). *Let $p \in \mathcal{P}$ be of the form given by (6) and the formula (8) with $c_1 \in [0, 2)$ and $\zeta \in \mathbb{T}$ holds. Then p must be of the form*

$$p(z) = \frac{1 + \mu(1 + \zeta)z + \zeta z^2}{1 - \mu(1 - \zeta)z - \zeta z^2}, \quad z \in \mathbb{D},$$

where $\mu \in [0, 1)$. Here, $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$.

We remark here that a special case Lemma 1.3 with $\zeta = -1 \in \mathbb{T}$ implies the result in [7, Lemma 2.3].

The next lemma is a special case of more general results due to Choi, Kim and Sugawa [1] (see also [10]). Define

$$Y(a, b, c) = \max_{z \in \mathbb{D}} (|a + bz + cz^2| + 1 - |z|^2), \quad a, b, c \in \mathbb{R},$$

where $\mathbb{D} := \{z \in \mathbb{C} : |z| \leq 1\}$.

Lemma 1.4 ([1, Choi, Kim and Sugawa]). *If $ac \geq 0$, then*

$$Y(a, b, c) = \begin{cases} |a| + |b| + |c|, & |b| \geq 2(1 - |c|), \\ 1 + |a| + \frac{b^2}{4(1 - |c|)}, & |b| < 2(1 - |c|). \end{cases}$$

If $ac < 0$, then

$$Y(a, b, c) = \begin{cases} 1 - |a| + \frac{b^2}{4(1 - |c|)}, & -4ac(c^{-2} - 1) \leq b^2 \wedge |b| < 2(1 - |c|), \\ 1 + |a| + \frac{b^2}{4(1 + |c|)}, & b^2 < \min \{4(1 + |c|)^2, -4ac(c^{-2} - 1)\}, \\ R(a, b, c), & \text{otherwise,} \end{cases}$$

where

$$R(a, b, c) = \begin{cases} |a| + |b| - |c|, & |c|(|b| + 4|a|) \leq |ab|, \\ -|a| + |b| + |c|, & |ab| \leq |c|(|b| - 4|a|), \\ (|c| + |a|)\sqrt{1 - \frac{b^2}{4ac}}, & \text{otherwise.} \end{cases}$$

2. On the fourth coefficients of functions in $\mathcal{M}(\gamma)$

Let $x_1 \approx -0.468$, $x_2 \approx -0.454$, $x_3 \approx -0.141$ and $x_4 \approx -0.072$ be the unique zeros in $[-1, 0]$ of the polynomials q_1 , q_2 , q_3 and q_4 defined by

$$(10) \quad q_1(x) = -11 + 83x + 72x^2 - 272x^3 + 128x^4,$$

$$(11) \quad q_2(x) = 1001 - 9658x + 10533x^2 + 29820x^3 - 75792x^4 + 66624x^5 - 26624x^6 + 4096x^7,$$

$$(12) \quad q_3(x) = 71 + 420x - 560x^2 + 192x^3$$

and

$$q_4(x) = 11 + 140x - 176x^2 + 64x^3,$$

respectively. And define the functions g , h , $k : [-1, 0] \rightarrow \mathbb{R}$ by

$$(13) \quad g(x) = \begin{cases} \frac{32(479 - 1092x + 784x^2 - 192x^3)}{3(7 - 4x)^3}, & x \in [-1, x_3]; \\ 16, & x \in [x_3, 0], \end{cases}$$

$$(14) \quad h(x) = \begin{cases} \frac{8\sqrt{6}(7 - 4x)^2(1 - 9x + 8x^2)}{3 - 4x} \left(\frac{3 - 4x}{29 - 127x + 124x^2 - 32x^3} \right)^{3/2}, & x \in [-1, x_2]; \\ \frac{16(5 - 4x)^{3/2}}{3\sqrt{16 - 21x + 8x^2}}, & x \in [x_2, x_4]; \\ \frac{32(479 - 1092x + 784x^2 - 192x^3)}{3(7 - 4x)^3}, & x \in [x_4, 0] \end{cases}$$

and

(15)

$$k(x) = \begin{cases} \frac{8}{3}(1 - 9x + 8x^2), & x \in [-1, x_1]; \\ \frac{4(-7+39x+176x^2-336x^3+128x^4)}{57-48x^2} \sqrt{k_1(x)} \sqrt{k_2(x)}, & x \in [x_1, x_2]; \\ \frac{8\sqrt{6}(7-4x)(-7+67x-92x^2+32x^3)}{-29+127x-124x^2+32x^3} \left(\frac{3-4x}{29-127x+124x^2-32x^3} \right)^{1/2}, & x \in [x_2, 0] \end{cases}$$

with

$$(16) \quad k_1(x) := \frac{64 - 39x - 224x^2 + 336x^3 - 128x^4}{95 + 171x - 232x^2 - 144x^3 + 128x^4}$$

and

$$(17) \quad k_2(x) := \frac{266 + 912x - 832x^2 - 768x^3 + 512x^4}{64 - 39x - 224x^2 + 336x^3 - 128x^4},$$

respectively. By comparing the functions g , h and k for fixed $x \in [-1, 0]$, we can obtain the following relation which will be used for the proof of our result (see figures below):

$$(18) \quad \max \{g(x), h(x), k(x)\} = \begin{cases} k(x), & \text{when } x \in [-1, x_2]; \\ h(x), & \text{when } x \in [x_2, x_3]; \\ g(x), & \text{when } x \in [x_3, 0]. \end{cases}$$

Now, we suggest the sharp bound for the fourth coefficient of $f \in \mathcal{M}(\gamma)$ for $\gamma \in [\pi/2, \pi)$.

Theorem 2.1. *Let $\gamma \in [\pi/2, \pi)$. If $f \in \mathcal{M}(\gamma)$ has the form given by (1), then*

$$(19) \quad |a_4| \leq \begin{cases} \frac{1}{3}, & \text{when } \pi/2 \leq \gamma \leq \gamma_3, \\ \frac{(5-4\tau)^{3/2}}{9\sqrt{16-21\tau+8\tau^2}}, & \text{when } \gamma_3 \leq \gamma \leq \gamma_2, \\ \frac{-7+39\tau+176\tau^2-336\tau^3+128\tau^4}{12(57-48\tau^2)} \sqrt{k_1(\tau)} \sqrt{k_2(\tau)}, & \text{when } \gamma_2 \leq \gamma \leq \gamma_1, \\ \frac{1}{18}(1-\tau)(1-8\tau), & \text{when } \gamma_1 \leq \gamma < \pi \end{cases}$$

with k_1 and k_2 given by (16) and (17), respectively. Here, $\gamma_1 \approx 2.058$, $\gamma_2 \approx 2.042$ and $\gamma_3 \approx 1.712$ is the unique root in $[\pi/2, \pi]$ of the equation $q_1(\cos \gamma) = 0$, $q_2(\cos \gamma) = 0$ and $q_3(\cos \gamma) = 0$, where q_1 , q_2 and q_3 is defined by (10), (11) and (12), respectively. This result is sharp.

Proof. For given $\gamma \in [\pi/2, \pi)$, let $f \in \mathcal{M}(\gamma)$ be of the form (1). Then there exists $p \in \mathcal{P}$ with the form given by (6) such that

$$(20) \quad \frac{zf'(z)}{f(z)} = B_\gamma \left(\frac{p(z) - 1}{p(z) + 1} \right).$$

Putting the series (1) and (6) into (20) by equating the coefficient we get

$$(21) \quad 48a_4 = 9c_3 - 2(4\tau + 1)c_1c_2 + \frac{1}{3}(8\tau^2 + 3\tau - 2)c_1^3.$$

Since the class $\mathcal{M}(\gamma)$ is invariant under the rotations, by Lemma 1.1, we may assume that $c_1 := t \in [0, 2]$. By using (8) and (9) in Lemma 1.2 we have

$$(22) \quad 48|a_4| = \left| \frac{1}{3}(8\tau^2 - 9\tau + 1)t^3 + (3 - 4\tau)t(4 - t^2)\zeta - 2t(4 - t^2)\zeta^2 + 4(4 - t^2)(1 - |\zeta|^2)\eta \right|,$$

where $\zeta \in \overline{\mathbb{D}}$ and $\eta \in \overline{\mathbb{D}}$.

Assume first that $t = 2$. Then

$$(23) \quad |a_4| = \frac{1}{18}(1 - \tau)(1 - 8\tau).$$

On the other hand, for $t = 0$, we have

$$(24) \quad |a_4| = \frac{1}{3}(1 - |\zeta|^2)|\eta| \leq \frac{1}{3}, \quad (\zeta, \eta) \in \overline{\mathbb{D}} \times \overline{\mathbb{D}}.$$

Now let $t \in (0, 2)$. Applying the triangle inequality to (22) we have

$$(25) \quad 48|a_4| \leq 4(4 - t^2)\Gamma(A, B, C),$$

where Γ is defined by

$$\Gamma(A, B, C) = |A + B\zeta + C\zeta^2| + 1 - |\zeta|^2, \quad \zeta \in \overline{\mathbb{D}}$$

with

$$(26) \quad A = \frac{(8\tau^2 - 9\tau + 1)t^3}{12(4 - t^2)}, \quad B = \frac{1}{4}(3 - 4\tau)t, \quad \text{and} \quad C = -\frac{1}{2}t.$$

We will find $\max_{\zeta \in \overline{\mathbb{D}}} \Gamma(A, B, C)$ by using Lemma 1.4. For this, we note that $AC < 0$ since $\tau \in (-1, 0]$ and $t \in (0, 2)$. We also note that

$$-4AC(C^{-2} - 1) \leq B^2, \quad t \in (0, 2),$$

since

$$4AC(C^{-2} - 1) + B^2 = \frac{1}{48}c^2(19 - 16\tau^2) \geq 0, \quad t \in (0, 2).$$

Moreover, we note that

$$(27) \quad |C|(|B| + 4|A|) \geq |AB|$$

for all $t \in (0, 2)$ and $\tau \in (-1, 0]$. Indeed the inequality (27) is equivalent to $t^2 \leq \lambda_\tau$, where

$$\lambda_\tau = \frac{24(3 - 4\tau)}{-32\tau^3 - 4\tau^2 + 17\tau + 13}$$

and we can easily check that $\lambda_\tau > 4$ for given $\tau \in (-1, 0]$.

Now, let

$$\tau^* = \sqrt{\frac{24(3 - 4\tau)}{29 - 127\tau + 124\tau^2 - 32\tau^3}}.$$

We note that $0 < 8/(7 - 4\tau) < \tau^* < 2$ for given $\tau \in (-1, 0]$. And we consider subintervals I_1 , I_2 and I_3 of $(0, 2)$ defined by $I_1 = (0, 8/(7 - 4\tau)]$, $I_2 = [8/(7 - 4\tau), \tau^*]$ and $I_3 = [\tau^*, 2)$, respectively.

Let us consider the case $t \in I_1$. Then we have $|B| < 2(1 - |C|)$, and by Lemma 1.4 we get

$$4(4 - t^2)\Gamma(A, B, C) \leq 4(4 - t^2) \left[1 + |A| + \frac{B^2}{4(1 - |C|)} \right] = \Psi(t),$$

where $\Psi : I_1 \cup \{0\} \rightarrow \mathbb{R}$ is defined by

$$\Psi(x) = \frac{1}{24}(19 - 16\tau^2)x^3 - \frac{1}{4}(7 - 4\tau)(1 + 4\tau)x^2 + 16.$$

When $\tau \in (-1, -1/4]$, since

$$\Psi''(x) = \frac{1}{4}(19 - 16\tau^2)x - \frac{1}{2}(7 - 4\tau)(1 + 4\tau) \geq -\frac{1}{2}(1 + 4\tau)(7 - 4\tau) \geq 0$$

for $x \in (0, 2)$, Ψ is convex on the interval I_1 . Therefore, we have

$$(28) \quad \Psi(x) \leq \max \left\{ \Psi(0), \Psi \left(\frac{8}{7 - 4\tau} \right) \right\}, \quad x \in I_1.$$

When $\tau \in (-1/4, 0]$, we have $\Psi'(x) = 0$ occurs at $x = 0$ or $x = \hat{\tau}$, where

$$\hat{\tau} := \frac{4(7 - 4\tau)(1 + 4\tau)}{19 - 16\tau^2} > 0.$$

Since the leading coefficient of Ψ is positive for given $\tau \in (-1/4, 0]$, $\Psi(\hat{\tau})$ is a local minimum on I_1 and therefore we have the inequality (28) again for the case $\tau \in (-1/4, 0]$. Furthermore, we can obtain $\Psi(8/(7 - 4\tau)) \geq 16$ when $\tau \in [-1, x_3]$ and $\Psi(8/(7 - 4\tau)) \leq 16$ when $\tau \in [x_3, 0]$. Therefore, we can obtain

$$(29) \quad \Psi(x) \leq \begin{cases} \Psi \left(\frac{8}{7 - 4\tau} \right), & \text{when } \tau \in (-1, x_3], \\ 16, & \text{when } \tau \in [x_3, 0], \end{cases} \quad x \in I_1.$$

And, it follows from (24), (25) and (29) that

$$48|a_4| \leq g(\tau), \quad \text{when } t \in I_1 \cup \{0\} = [0, 8/(7 - 4\tau)],$$

where g is defined by (13).

Next, we consider the case $t \in I_2$. In this case we have $|AB| \leq |C|(|B| - 4|A|)$. Therefore, by Lemma 1.4,

$$(30) \quad 4(4 - t^2)\Gamma(A, B, C) \leq 4(4 - t^2)(-|A| + |B| + |C|) = \Lambda(t),$$

where $\Lambda : I_2 \rightarrow \mathbb{R}$ is defined by

$$\Lambda(x) = -\frac{1}{3}(8\tau^2 - 21\tau + 16)x^3 + (20 - 16\tau)x.$$

Moreover, $\Lambda'(x) = 0$ at $x = \tilde{\tau}$, where

$$(31) \quad \tilde{\tau} := \sqrt{\frac{20 - 16\tau}{16 - 21\tau + 8\tau^2}}.$$

Comparing the values $8/(7-4\tau)$, τ^* and $\tilde{\tau}$, we have

$$\begin{cases} \frac{8}{7-4\tau} \leq \tau^* \leq \tilde{\tau}, & \text{when } \tau \in (-1, x_2]; \\ \frac{8}{7-4\tau} \leq \tilde{\tau} \leq \tau^*, & \text{when } \tau \in [x_2, x_4]; \\ \tilde{\tau} \leq \frac{8}{7-4\tau} \leq \tau^*, & \text{when } \tau \in [x_4, 0]. \end{cases}$$

Furthermore, Λ is increasing on I_2 when $\tau \in (-1, x_2]$, and Λ is decreasing on I_2 when $\tau \in [x_4, 0]$. On the other hand, when $\tau \in [x_2, x_4]$, Λ has its local maximum at $\tilde{\tau}$. Using these facts, we can obtain the maximum for Λ on I_2 as follows:

$$(32) \quad \Lambda(x) \leq \begin{cases} \Lambda(\tau^*), & \text{when } \tau \in (-1, x_2]; \\ \Lambda(\tilde{\tau}), & \text{when } \tau \in [x_2, x_4]; \\ \Lambda\left(\frac{8}{7-4\tau}\right), & \text{when } \tau \in [x_4, 0], \end{cases} \quad x \in I_2.$$

Combining (25), (30) and (32), we have

$$48|a_4| \leq h(\tau), \quad \text{when } t \in I_2,$$

where h is defined by (14).

Next, let us consider the case $t \in I_3$. In this case, from Lemma 1.4, we have

$$(33) \quad 4(4-t^2)\Gamma(A, B, C) \leq 4(4-t^2)(|C|+|A|)\sqrt{1-\frac{B^2}{4AC}} = \Phi(t),$$

where $\Phi : I_3 \cup \{2\} \rightarrow \mathbb{R}$ is defined by

$$\Phi(x) = \frac{1}{3}(24x + (8\tau^2 - 9\tau - 5)x^3)\sqrt{\phi(x)}$$

with

$$\phi(x) = 1 + \frac{3(3-4\tau)^2(4-x^2)}{8(8\tau^2-9\tau+1)x^2}.$$

Differentiating Φ , we have

$$(34) \quad \Phi'(x) = \frac{\kappa_1 + \kappa_2 x^2}{2(1-9\tau+8\tau^2)\sqrt{\phi(x)}},$$

where

$$\kappa_1 = 8(-64 + 39\tau + 224\tau^2 - 336\tau^3 + 128\tau^4)$$

and

$$\kappa_2 = (5 + 9\tau - 8\tau^2)(19 - 16\tau^2).$$

We note that $\kappa_2 \geq 0$ for $\tau \in [\frac{9-\sqrt{241}}{16}, 0]$ and $\kappa_2 \leq 0$ for $\tau \in (-1, \frac{9-\sqrt{241}}{16}]$.

When $\tau \in [\frac{9-\sqrt{241}}{16}, 0]$, we have

$$\kappa_1 + \kappa_2 x^2 \leq \kappa_1 + 4\kappa_2 = 12(-11 + 83\tau + 72\tau^2 - 272\tau^2 + 128\tau^4) < 0, \quad x \in I_3.$$

Therefore, by (34), Φ is decreasing on I_3 . Also, Φ is decreasing on I_3 for the case $\tau \in [x_2, \frac{9-\sqrt{241}}{16}]$, since

$$\kappa_1 + \kappa_2 x^2 \leq \kappa_1 + \kappa_2 (\tau^*)^2 = \frac{-8q_2(\tau)}{29 - 127\tau + 124\tau^2 - 32\tau^3} \leq 0, \quad x \in I_3.$$

For the case $\tau \in (-1, x_2]$, we note that $\kappa_1 \kappa_2 < 0$ and $\Phi'(x) = 0$ occurs at $x = \hat{\tau}$, where

$$(35) \quad \hat{\tau} := \sqrt{\frac{\kappa_1}{-\kappa_2}} = \sqrt{\frac{8(-64 + 39\tau + 224\tau^2 - 336\tau^3 + 128\tau^4)}{-(5 + 9\tau - 8\tau^2)(19 - 16\tau^2)}}.$$

Comparing the values 2, τ^* and $\hat{\tau}$, we have

$$\begin{cases} \tau^* \leq 2 \leq \hat{\tau}, & \text{when } \tau \in (-1, x_1]; \\ \tau^* \leq \hat{\tau} \leq 2, & \text{when } \tau \in [x_1, x_2]. \end{cases}$$

Furthermore, Φ is increasing on $[\tau^*, 2]$ when $\tau \in (-1, x_1]$, and Φ has the local maximum at $x = \hat{\tau}$ when $\tau \in [x_1, x_2]$. Using these facts, we can obtain the following maximum for Φ on I_3 for each cases:

$$(36) \quad \Phi(x) \leq \begin{cases} \Phi(2), & \text{when } \tau \in (-1, x_1]; \\ \Phi(\hat{\tau}), & \text{when } \tau \in [x_1, x_2]; \\ \Phi(\tau^*), & \text{when } \tau \in [x_2, 0], \end{cases} \quad x \in I_3.$$

Combining (23), (25), (33) and (36), we have

$$48|a_4| \leq k(\tau), \quad \text{when } t \in I_3 \cup \{2\} = [\tau^*, 2],$$

where k is defined by (15).

Consequently, we obtain

$$(37) \quad 48|a_4| \leq \begin{cases} g(\tau), & \text{when } t \in [0, 8/(7-4\tau)]; \\ h(\tau), & \text{when } t \in [8/(7-4\tau), \tau^*]; \\ k(\tau), & \text{when } t \in [\tau^*, 2]. \end{cases}$$

We note that each conditions $\gamma \in [\pi/2, \gamma_3]$, $\gamma \in [\gamma_3, \gamma_2]$, $\gamma \in [\gamma_2, \gamma_1]$ and $\gamma \in [\gamma_1, \pi)$ are equivalent to $\tau \in [x_3, 0]$, $\tau \in [x_2, x_3]$, $\tau \in [x_1, x_2]$ and $\tau \in (-1, x_1]$, respectively. Thus, it follows from (37) and (18) that the inequality (19) holds.

From now, we will show that this result is sharp. The equality for the first case (i.e., $\gamma \in [\pi/2, \gamma_3]$) in (19) holds for the function $f_1 \in \mathcal{M}(\gamma)$ defined by

$$f_1(z) = z \exp \left[\int_0^z \frac{1}{\xi} (B_\gamma(\xi^3) - 1) d\xi \right] = z + \frac{1}{3}z^4 + \cdots, \quad z \in \mathbb{D}.$$

And the equality for the fourth case (i.e., $\gamma \in [\gamma_1, \pi)$) in (19) holds for the function $f_2 \in \mathcal{M}(\gamma)$ defined by

$$(38) \quad \begin{aligned} f_2(z) &= z \exp \left[\int_0^z \frac{1}{\xi} (B_\gamma(\xi) - 1) d\xi \right] \\ &= z + z^2 + \frac{1}{2}(1 - \tau)z^3 + \frac{1}{18}(1 - 9\tau + 8\tau^2)z^4 \\ &\quad + \frac{1}{72}(-5 + 41\tau^2 - 36\tau^3)z^5 + \dots, \quad z \in \mathbb{D}. \end{aligned}$$

To consider the sharpness for the second case in (19), let us fix $\gamma \in [\gamma_3, \gamma_2]$. We note that the equality in (19) holds for ζ which satisfies

$$4(4 - t^2)(|A + B\zeta + C\zeta^2| + 1 - |\zeta|^2) = 4(4 - t^2)(-|A| + |B| + |C|) = \Lambda(\tilde{\tau}),$$

where $\tilde{\tau}$ is given by (31) and A, B, C are given by (26) with $t = \tilde{\tau}$. And we can easily check that this relation is satisfied for $\zeta = -1$ and $t = c_1 = \tilde{\tau}$. Therefore, by Lemma 1.2, we have

$$c_2 = \tilde{\tau}^2 - 2 \quad \text{and} \quad c_3 = \tilde{\tau}^3 - 3\tilde{\tau}.$$

We consider the function $f_3 : \mathbb{D} \rightarrow \mathbb{C}$ defined by

$$(39) \quad f_3(z) = z \exp \left[\int_0^z \frac{1}{\xi} \left(B_\gamma \left(\frac{p(\xi) - 1}{p(\xi) + 1} \right) - 1 \right) d\xi \right], \quad z \in \mathbb{D},$$

where $p \in \mathcal{P}$ has the form given by

$$(40) \quad p(z) = 1 + \tilde{\tau}z + (\tilde{\tau}^2 - 2)z^2 + (\tilde{\tau}^3 - 3\tilde{\tau})z^3 + \dots.$$

From (21), we can easily check that equality in (19) holds for this function f_3 . Therefore, it is enough to construct a function $p \in \mathcal{P}$ with the form given by (40). Since $c_2 = c_1^2 - 2$, it follows from Lemma 1.3 with $\zeta = -1$ that the function p should be defined by

$$p(z) = \frac{1 - z^2}{1 - \tilde{\tau}z + z^2}, \quad z \in \mathbb{D}$$

and this guarantees the sharpness of the inequality (19) for the second case.

Finally, it is remained to show the sharpness of (19) for the third case. Fix $\gamma \in [\gamma_2, \gamma_1]$. We note that the equality in (19) holds for ζ which satisfies

$$4(4 - t^2)(|A + B\zeta + C\zeta^2| + 1 - |\zeta|^2) = 4(4 - t^2)(|C| + |A|)\sqrt{1 - \frac{B^2}{4AC}} = \Phi(\hat{\tau}),$$

where $\hat{\tau}$ is given by (35) and A, B, C are given by (26) with $t = \hat{\tau}$. Moreover, this relation holds for $\zeta = e^{i\theta}$, where $\theta \in [0, 2\pi)$ is defined so that

$$\cos \theta = -\frac{B(A + C)}{4AC} = \frac{(3 - 4\tau)(-163 - 105\tau + 368\tau^2 - 336\tau^3 + 128\tau^4)}{8(-64 + 39\tau + 224\tau^2 - 336\tau^3 + 128\tau^4)}.$$

Therefore, by Lemma 1.2, we have

$$c_2 = \frac{1}{2}[\hat{\tau}^2 + \zeta(4 - \hat{\tau}^2)] \quad \text{and} \quad c_3 = \frac{1}{4}\hat{\tau}[\hat{\tau}^2 + \zeta(2 - \zeta)(4 - \hat{\tau}^2)].$$

Since the equality in (19) holds for the function f_3 defined by (39) with $p \in \mathcal{P}$, where

$$p(z) = 1 + \hat{\tau}z + \frac{1}{2}[\hat{\tau}^2 + \zeta(4 - \hat{\tau}^2)]z^2 + \frac{1}{4}\hat{\tau}[\hat{\tau}^2 + \zeta(2 - \zeta)(4 - \hat{\tau}^2)]z^3 + \dots,$$

it is enough to construct a function $p \in \mathcal{P}$ with this representation. Similar methods with the second case and Lemma 1.3 lead us to get the desired function $p \in \mathcal{P}$ defined by

$$p(z) = \frac{2 + \hat{\tau}(1 + \zeta)z + 2\zeta z^2}{2 - \hat{\tau}(1 - \zeta)z - 2\zeta z^2}$$

and this completes the proof of Theorem 2.1. \square

We give here 5 figures which present the graphs of g , h and k defined by (13), (14) and (15), respectively, to justify the equality (18).

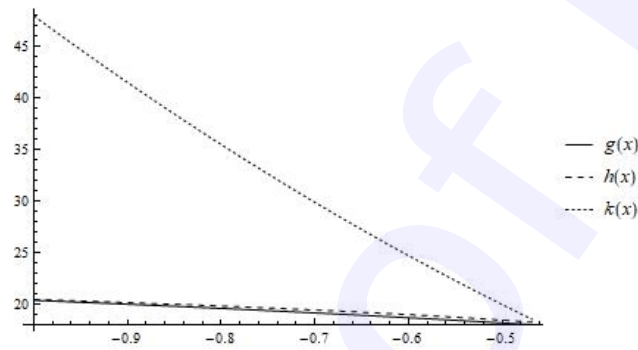


FIGURE 1. Graphs of g , h , k on the interval $[-1, x_1]$

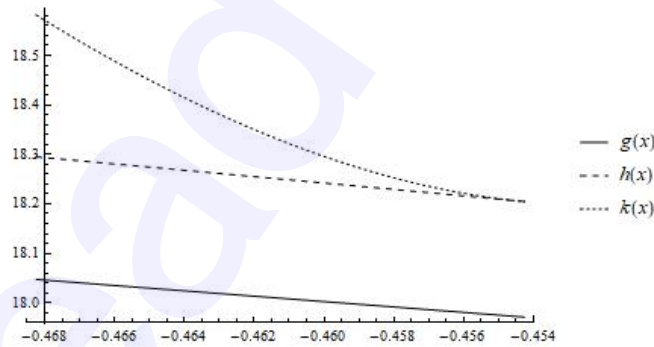
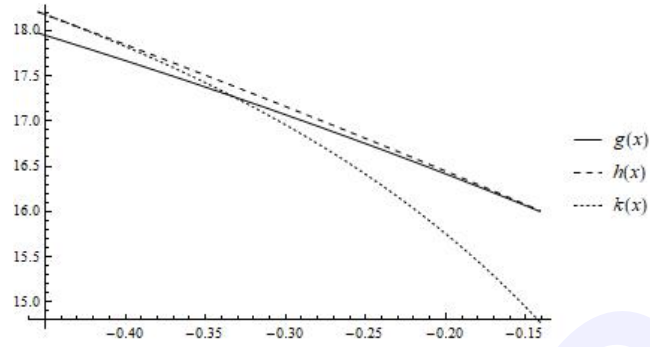
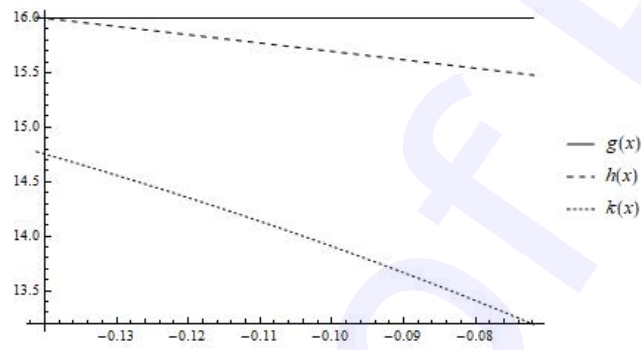
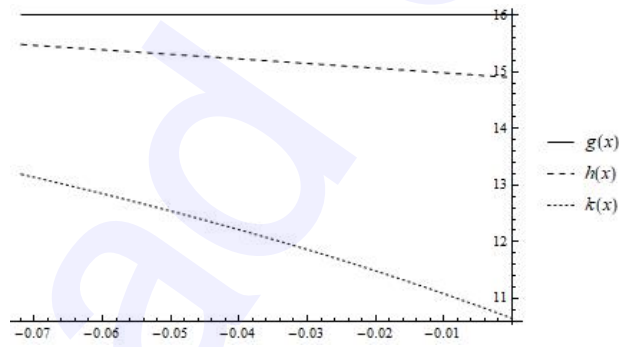


FIGURE 2. Graphs of g , h , k on the interval $[x_1, x_2]$

FIGURE 3. Graphs of g, h, k on the interval $[x_2, x_3]$ FIGURE 4. Graphs of g, h, k on the interval $[x_3, x_4]$ FIGURE 5. Graphs of g, h, k on the interval $[x_4, 0]$

3. On the fifth coefficients of functions in $\mathcal{M}(\gamma)$

In this section, we obtain the sharp bound for the fifth coefficient of $f \in \mathcal{M}(\gamma)$ for γ in a subset of $[\pi/2, \pi)$. For this, let $x_5 \approx -0.882$ be the zero in the

interval $(-1, -1/2)$ of the polynomial r_1 defined by

$$(41) \quad r_1(x) = 10 + 12x - 31x^2 - 36x^3.$$

Theorem 3.1. *Let $\gamma_5 \approx 2.651$ be a root of the equation $r_1(\cos \gamma) = 0$ and let $\gamma \in [\gamma_5, \pi)$. If $f \in \mathcal{M}(\gamma)$ has the form given by (1), then*

$$(42) \quad |a_5| \leq \frac{1}{72}(-5 + 41\tau^2 - 36\tau^3).$$

This result is sharp.

Proof. By the same methods in the proof of Theorem 2.1, we have

$$(43) \quad 64a_5 = \frac{1}{18}r_1(\tau)c_1^4 + \frac{2}{3}r_2(\tau)c_1^2c_2 + r_3(\tau)c_2^2 + \frac{8}{3}r_4(\tau)c_1c_3 + 8c_4,$$

where r_1 is defined by (41),

$$r_2(x) = -2 + 7x + 12x^2,$$

$$r_3(x) = -4x - 2$$

and

$$r_4(x) = -3x - 1.$$

And we can easily check that

$$r_i(x) \geq 0, \quad i = 1, 2, 3, 4$$

for $x \in [-1, x_5]$. Or, equivalently, $r_i(\tau) \geq 0$ ($i = 1, 2, 3, 4$) hold when $\gamma \in [\gamma_5, \pi)$. Therefore, by applying Lemma 1.1 to the inequality which obtained by applying the triangle inequality to (43), we obtain

$$64|a_5| \leq \frac{8}{9}r_1(\tau) + \frac{16}{3}r_2(\tau) + 4r_3(\tau) + \frac{32}{3}r_4(\tau) + 16 = -\frac{8}{9}(5 - 41\tau^2 + 36\tau^3),$$

which implies the inequality (42). The equality in (42) holds for f_2 defined by (38) and this completes the proof of our result. \square

References

- [1] J. H. Choi, Y. C. Kim, and T. Sugawa, *A general approach to the Fekete-Szegő problem*, J. Math. Soc. Japan **59** (2007), no. 3, 707–727.
- [2] P. L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, **259**, Springer-Verlag, New York, 1983.
- [3] R. Kargar, A. Ebadian, and J. J. Sokół, *Radius problems for some subclasses of analytic functions*, Complex Anal. Oper. Theory **11** (2017), no. 7, 1639–1649.
- [4] K. Kuroki and S. Owa, *Notes on new class for certain analytic functions*, RIMS Kokyuroku. Kyoto Univ. **1772** (2011), 21–25.
- [5] ———, *Notes on new class for certain analytic functions*, Adv. Math. Sci. J. **1** (2012), no. 2, 127–131.
- [6] O. S. Kwon, A. Lecko, and Y. J. Sim, *On the fourth coefficient of functions in the Carathéodory class*, Comput. Methods Funct. Theory **18** (2018), no. 2, 307–314.
- [7] M. Li and T. Sugawa, *A note on successive coefficients of convex functions*, Comput. Methods Funct. Theory **17** (2017), no. 2, 179–193.
- [8] R. J. Libera and E. J. Zlotkiewicz, *Early coefficients of the inverse of a regular convex function*, Proc. Amer. Math. Soc. **85** (1982), no. 2, 225–230.

- [9] ———, *Coefficient bounds for the inverse of a function with derivative in \mathcal{P}* , Proc. Amer. Math. Soc. **87** (1983), no. 2, 251–257.
- [10] R. Ohno and T. Sugawa, *Coefficient estimates of analytic endomorphisms of the unit disk fixing a point with applications to concave functions*, Kyoto J. Math. **58** (2018), no. 2, 227–241.

OH SANG KWON
DEPARTMENT OF MATHEMATICS
KYUNGSUNG UNIVERSITY
BUSAN 48434, KOREA
Email address: oskwon@ks.ac.kr

YOUNG JAE SIM
DEPARTMENT OF MATHEMATICS
KYUNGSUNG UNIVERSITY
BUSAN 48434, KOREA
Email address: yjsim@ks.ac.kr