Commun. Korean Math. Soc. **0** (0), No. 0, pp. 1–0
https://doi.org/10.4134/CKMS.c180102
pISSN: 1225-1763 / eISSN: 2234-3024

OPTIMIZATIONS ON TOTALLY REAL SUBMANIFOLDS OF LCS-MANIFOLDS USING CASORATI CURVATURES

Mohammad Hasan Shahid and Aliya Naaz Siddiqui

Abstract. In the present paper, we derive two optimal inequalities for totally real submanifolds and C-totally real submanifolds of LCS-manifolds with respect to Levi-Civita connection and quarter symmetric metric connection by using T. Oprea's optimization method.

1. Introduction

The concept of Lorentzian concircular structure manifold (LCS-manifolds) is studied by A. A. Shaikh as a generalization of LP-Sasakian manifolds in [18]. These manifolds are of great interest in the general theory of relativity and cosmology [19, 20]. Many researchers have studied LCS-manifolds (for example [1, 8–11, 21]).

The notion of semi-symmetric linear connection on smooth manifolds is initiated by Friedmann and Schouten in [4]. Later on, Golab has introduced the idea of quarter symmetric linear connection on such smooth manifolds as a generalization of semi-symmetric connection in [6].

In 1890, F. Casorati [2] has defined Casorati curvature and used it at the place of traditional Gauss curvature. The geometrical importance of the Casorati curvatures has been discussed by many researchers [3, 7, 12, 25]. Due to its vast geometric significance it drew attention of researchers to construct optimal inequalities for Casorati curvatures for different set ups [5, 13, 14, 22–24, 26, 27].

The outline of the present paper is as follows: Section 2 is preliminary in nature. Section 3 deals with the study of Casorati curvatures. Section 4 derives the optimal inequalities for totally real submanifolds and C-totally real submanifolds of LCS-manifolds with respect to Levi-Civita connection. Section 5 gives the proof of the geometric inequalities for totally real submanifolds and C-totally real submanifolds of LCS-manifolds with respect to quarter symmetric metric connection.

Received March 15, 2018; Revised May 31, 2018; Accepted July 27, 2018.
2010 *Mathematics Subject Classification.* 53C15, 53C25.
Key words and phrases. LCS-manifolds, quarter symmetric metric connection, Casorati curvatures, totally real submanifolds.

2. LCS-manifolds and their submanifolds

Definition ([16, 18])**.** A Lorentzian manifold $\overline{M}$ together with the unit time-like concircular vector field ξ, its associated 1-form η and an $(1,1)$ tensor field φ is said to be a Lorentzian concircular structure manifold (or LCS-manifold).

In an n-dimensional $(\mathrm{LCS})_n$-manifold $\overline{M}$, $n > 2$, the following relations hold [18]:

$$\eta(\xi) = -1, \quad \varphi(\xi) = 0, \quad \eta \circ \varphi = 0, \tag{1}$$

$$g(\varphi X, \varphi Y) = g(X,Y) + \eta(X)\eta(Y), \quad \varphi^2 X = X + \eta(X)\xi, \tag{2}$$

$$\overline{R}(X,Y)Z = \varphi\overline{R}(X,Y)Z + (\alpha^2 - \rho)\Big[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\Big]\xi \tag{3}$$

for any $X, Y, Z \in \Gamma(T\overline{M})$.

We consider $\overline{\nabla}$ is the operator of covariant differentiation with respect to the Lorentzian metric g and α is a non-zero scalar function satisfying the following:

$$\overline{\nabla}_X \alpha = (X\alpha) = d\alpha(X) = \rho\eta(X) \tag{4}$$

for any $X \in \Gamma(T\overline{M})$, where ρ is a certain scalar function given by

$$\rho = -(\xi\alpha). \tag{5}$$

Also,

$$\begin{aligned} \overline{R}(X,Y,Z,W) = \overline{R}(X,Y,Z,\varphi W) + (\alpha^2 - \rho)\Big[&g(Y,Z)\eta(X) \\ &- g(X,Z)\eta(Y)\Big]\eta(W) \end{aligned} \tag{6}$$

for any $X, Y, Z, W \in \Gamma(T\overline{M})$.

Remark 2.1. If we assume that $\alpha = 1$, then Lorentzian concircular structure becomes LP-Sasakian structure [15].

Let M be an m-dimensional submanifold of an n-dimensional manifold $\overline{M}$ with induced metric g. The Gauss equation is given by [29]

$$\begin{aligned} \overline{R}(X,Y,Z,W) = R(X,Y,Z,W) &+ g(\zeta(X,Z), \zeta(Y,W)) \\ &- g(\zeta(X,W), \zeta(Y,Z)) \end{aligned} \tag{7}$$

for any $X, Y, Z, W \in \Gamma(TM)$. Here ζ is the second fundamental form of M in $\overline{M}$.

Definition ([6])**.** A linear connection $\hat{\overline{\nabla}}$ in an n-dimensional smooth manifold $\overline{M}$ is said to be a quarter symmetric connection if its torsion tensor T is of the form

$$T(X,Y) = \hat{\overline{\nabla}}_X Y - \hat{\overline{\nabla}}_Y X - [X,Y] = \eta(Y)\varphi X - \eta(X)\varphi Y,$$

where η is an 1-form and φ is a tensor of type $(1,1)$.

Remark 2.2. If we assume that $\varphi X = X$, then the quarter symmetric connection reduces to semi-symmetric connection.

Definition ([6])**.** The quarter symmetric connection $\hat{\overline{\nabla}}$ is said to be a quarter symmetric metric connection if $\hat{\overline{\nabla}}$ satisfies the following condition:

$$(\hat{\overline{\nabla}}_X g)(Y, Z) = 0$$

for any $X, Y, Z, W \in \Gamma(T\overline{M})$.

The relation between quarter symmetric metric connection $\hat{\overline{\nabla}}$ and Riemannian connection $\overline{\nabla}$ on a $(\mathrm{LCS})_n$-manifold $\overline{M}$ is given by [11]

$$\hat{\overline{\nabla}}_X Y = \overline{\nabla}_X Y + \eta(Y)\varphi X - g(\varphi X, Y)\xi.$$

If $\hat{\overline{R}}$ and $\overline{R}$ are the curvature tensors of a $(\mathrm{LCS})_n$-manifold $\overline{M}$ with respect to quarter symmetric metric connection $\hat{\overline{\nabla}}$ and Riemannian connection $\overline{\nabla}$, then [10]

$$\begin{aligned} \hat{\overline{R}}(X, Y, Z, W) = \overline{R}(X, Y, Z, W) &+ (2\alpha - 1)\Big[g(\varphi X, Z)g(\varphi Y, W) \\ &- g(\varphi Y, Z)g(\varphi X, W)\Big] + \alpha\Big[\eta(Y)g(X, W) \\ &- \eta(X)g(Y, W)\Big]\eta(Z) + \alpha\Big[g(Y, Z)\eta(X) \\ &- g(X, Z)\eta(Y)\Big]\eta(W) \end{aligned} \tag{8}$$

for any $X, Y, Z, W \in \Gamma(TM)$.

Let M be an m-dimensional submanifold of an n-dimensional $(\mathrm{LCS})_n$-manifold $\overline{M}$ with respect to quarter symmetric metric connection $\hat{\overline{\nabla}}$ and $\hat{\nabla}$ be the induced connection of M associated to the quarter symmetric metric connection. Also let $\hat{\zeta}$ be the second fundamental form of M with respect to $\hat{\nabla}$. Then the relation (7) becomes

$$\begin{aligned} \hat{\overline{R}}(X, Y, Z, W) = \hat{R}(X, Y, Z, W) &+ g(\hat{\zeta}(X, Z), \hat{\zeta}(Y, W)) \\ &- g(\hat{\zeta}(X, W), \hat{\zeta}(Y, Z)) \end{aligned} \tag{9}$$

for any $X, Y, Z, W \in \Gamma(TM)$. Here $\hat{R}$ is the curvature tensor of M with respect to the induced connection associated to the quarter symmetric metric connection.

Definition ([28, 29])**.** (i) A submanifold M of a contact metric manifold $\overline{M}$ is said to be anti-invariant if for any X tangent to M, φX is normal to M, i.e., $\varphi(TM) \subset T^{\perp}M$ at every point of M, where $T^{\perp}M$ denotes the normal bundle of M.

(ii) A submanifold M in a contact metric manifold $\overline{M}$ is called a C-totally real submanifold in $\overline{M}$ if every tangent vector of M belongs to the contact distribution.

Remark 2.3. We note that if a submanifold M of a contact metric manifold $\overline{M}$ is normal to the structure vector field ξ, then it is anti-invariant. Also, a submanifold M in a contact metric manifold $\overline{M}$ is a C-totally real submanifold if the structure vector field ξ is normal to M. Therefore it is clear that C-totally real submanifolds in a contact metric manifold are anti-invariant, as they are normal to ξ.

For a totally real submanifold and a C-totally real submanifold of a $(\mathrm{LCS})_n$-manifold $\overline{M}$, $\hat{\zeta}$ is given by [10]

$$\hat{\zeta}(X,Y) = \zeta(X,Y) + \eta(Y)\varphi X \tag{10}$$

and

$$\hat{\zeta}(X,Y) = \zeta(X,Y), \tag{11}$$

respectively, for any $X, Y \in \Gamma(TM)$.

3. Casorati curvatures

Let $\overline{M}$ be an n-dimensional $(\mathrm{LCS})_n$-manifold and M be an m-dimensional submanifold in $\overline{M}$. Let $\{\mathcal{E}_1, \ldots, \mathcal{E}_m\}$ be an orthonormal basis of $T_\wp M$ and $\{\mathcal{E}_{m+1}, \ldots, \mathcal{E}_n\}$ be an orthonormal basis of $T^\perp_\wp M$ at any $\wp \in M$. Then the scalar curvature $\sigma(\wp)$ at $\wp$ is given by

$$\sigma(\wp) = \sum_{1\leq \imath < \jmath \leq m} K(\mathcal{E}_\imath \wedge \mathcal{E}_\jmath) \tag{12}$$

and the normalized scalar curvature ϱ is given by

$$\varrho = \frac{2\sigma}{m(m-1)}, \tag{13}$$

where $K(\Lambda)$ denotes the sectional curvature of the plane section $\Lambda \subset T_\wp M$.

The mean curvature vector $\mathcal{H}$ is defined as

$$\mathcal{H} = \frac{1}{m} \sum_{\imath,\jmath=1}^{m} \zeta(\mathcal{E}_\imath, \mathcal{E}_\imath) \tag{14}$$

and the squared norm of mean curvature is given by

$$\|\mathcal{H}\|^2 = \frac{1}{m^2} \sum_{a=m+1}^{n} \Big(\sum_{\imath=1}^{m} \zeta^a_{\imath\imath}\Big)^2. \tag{15}$$

We also put

$$\zeta^a_{\imath\jmath} = g(\zeta(\mathcal{E}_\imath, \mathcal{E}_\jmath), \mathcal{E}_a),\ \imath, \jmath \in \{1, 2, \ldots, m\},\ a \in \{m+1, m+2, \ldots, n\}.$$

The Casorati curvature $\mathcal{C}$ of M is defined by

$$\mathcal{C} = \frac{1}{m} \sum_{a=m+1}^{n} \sum_{\imath,\jmath=1}^{m} \left(\zeta_{\imath\jmath}^{a}\right)^2. \tag{16}$$

Let us assume an r-dimensional subspace Ψ of TM, $r \geq 2$, whose orthonormal basis is $\{\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_r\}$. Then we have

$$\sigma(\Psi) = \sum_{1\leq\alpha<\beta\leq r} K(\mathcal{E}_\alpha \wedge \mathcal{E}_\beta) \tag{17}$$

and

$$\mathcal{C}(\Psi) = \frac{1}{r} \sum_{a=m+1}^{n} \sum_{\imath,\jmath=1}^{m} \left(\zeta_{\imath\jmath}^{a}\right)^2, \tag{18}$$

where $\sigma(\Psi)$ and $\mathcal{C}(\Psi)$ are the scalar curvature and Casorati curvature of Ψ, respectively.

The following δ-Casorati curvatures $\delta_{\mathcal{C}}(m-1)$ and $\widehat{\delta}_{\mathcal{C}}(m-1)$

$$[\delta_{\mathcal{C}}(m-1)]_\wp = \frac{1}{2}\mathcal{C}_\wp + \frac{m+1}{2m} \inf\{\mathcal{C}(\Psi)|\Psi : \text{a hyperplane of } T_\wp M\} \tag{19}$$

and

$$[\widehat{\delta}_{\mathcal{C}}(m-1)]_\wp = 2\mathcal{C}_\wp + \frac{2m-1}{2m} \sup\{\mathcal{C}(\Psi)|\Psi : \text{a hyperplane of } T_\wp M\} \tag{20}$$

are known as the normalized δ-Casorati curvatures.

Definition ([14]). A point $p \in M$ is said to be an invariantly quasi-umbilical point if there exist $n-m$ orthogonal unit normal vector $\{\mathcal{E}_{m+1}, \ldots, \mathcal{E}_n\}$ such that the shape operator with respect to all directions $\mathcal{E}_r$ have an eigenvalue of multiplicity $m-1$ and that for each $\mathcal{E}_r$ the distinguished eigendirection is the same. The submanifold M is said to be an invariantly quasi-umbilical submanifold if each of its point is an invariantly quasi-umbilical point.

For the main results, we need following lemma:

Lemma 3.1 ([17]). *Let M be a Riemannian submanifold of Riemannian manifold $(\overline{M}, \overline{g})$, where g is the induced metric on M from $\overline{g}$ and $\iota : M \to \mathbb{R}$ is a differentiable function. If $y \in M$ is the solution of the constrained extremum problem $\min_{x\in M} \iota(x)$, then*

(i) *$(grad\ \iota)(y) \in T_y^{\perp} M$;*
(ii) *the bilinear form $Ł : T_yM \times T_yM \to \mathbb{R}$;*

$$Ł(X, Y) = \overline{g}(\varsigma(X, Y), (grad\ \iota)(y)) + \mathcal{H}ess_\iota(X, Y)$$

is positive semi-definite, where ς is the second fundamental form of M in $\overline{M}$.

4. Main result 1

Theorem 4.1. *Let M be an m-dimensional totally real submanifold in an n-dimensional $(LCS)_n$-manifold $\overline{M}$. Then*

(i) *The normalized δ-Casorati curvature $\delta_{\mathcal{C}}(m-1)$ satisfies*

$$\varrho \leq \delta_{\mathcal{C}}(m-1) - \frac{\alpha^2-\rho}{m}.$$

Furthermore, the equality sign holds if and only if M is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{M}(c)$, such that with respect to orthonormal frames of $T_\wp M$ and $T_\wp^\perp M$, respectively, the shape operators $\mathcal{S}_a \equiv \mathcal{S}_{\mathcal{E}_a}$, $a \in \{m+1,\dots,n\}$, are of the following form

$$\mathcal{S}_{m+1} = \begin{pmatrix} \beta & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \beta & 0 \\ 0 & \dots & 0 & 2\beta \end{pmatrix}, \ \mathcal{S}_{m+2} = \dots = \mathcal{S}_n = 0. \tag{21}$$

(ii) *The normalized δ-Casorati curvature $\widehat{\delta}_{\mathcal{C}}(m-1)$ satisfies*

$$\varrho \leq \widehat{\delta}_{\mathcal{C}}(m-1) - \frac{\alpha^2-\rho}{m}.$$

Furthermore, the equality sign holds if and only if M is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{M}(c)$, such that with respect to orthonormal frames of $T_\wp M$ and $T_\wp^\perp M$, respectively, the shape operators $\mathcal{S}_a \equiv \mathcal{S}_{\mathcal{E}_a}$, $a \in \{m+1,\dots,n\}$, are of the following form

$$\mathcal{S}_{m+1} = \begin{pmatrix} 2\beta & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 2\beta & 0 \\ 0 & \dots & 0 & \beta \end{pmatrix}, \ \mathcal{S}_{m+2} = \dots = \mathcal{S}_n = 0. \tag{22}$$

Proof. Let $\{\mathcal{E}_1,\dots,\mathcal{E}_m\}$ be an orthonormal frame of $T_\wp M$ and $\{\mathcal{E}_{m+1},\dots,\mathcal{E}_n\}$ be an orthonormal frame of $T_\wp^\perp M$, $\wp \in M$. From [10], we get

$$2\sigma = -(m-1)(\alpha^2-\rho) + m^2||\mathcal{H}||^2 - m\mathcal{C}.$$

Let us take a quadratic polynomial $\mathbb{K}$ in the components of the second fundamental form

$$\mathbb{K} = \frac{m(m-1)}{2}\mathcal{C} + \frac{m^2-1}{2}\mathcal{C}(\Psi) + 2\sigma - (m-1)(\alpha^2-\rho). \tag{23}$$

Without loss of generality, we assume that Ψ is spanned by $\mathcal{E}_1, \ldots, \mathcal{E}_{m-1}$ and together (23), we find that

$$\mathbb{K} = \frac{m+1}{2} \sum_{a=m+1}^{n} \Big[\sum_{\imath,\jmath=1}^{m} (\zeta_{\imath\jmath}^{a})^2 + \sum_{\imath,\jmath=1}^{m-1} (\zeta_{\imath\jmath}^{a})^2 \Big] - \sum_{a=m+1}^{n} \Big[\sum_{\imath=1}^{m} \zeta_{\imath\imath}^{a} \Big]^2 . \tag{24}$$

Also,

$$\begin{aligned}
\mathbb{K} = & \sum_{a=m+1}^{n} \sum_{\imath=1}^{m-1} \Big[m(\zeta_{\imath\imath}^{a})^2 + (m+1)(\zeta_{\imath m}^{a})^2 \Big] \\
& + \sum_{a=m+1}^{n} \Big[2(m+1) \sum_{1\leq \imath<\jmath\leq m-1} (\zeta_{\imath\jmath}^{a})^2 - 2 \sum_{1\leq \imath<\jmath\leq m} \zeta_{\imath\imath}^{a}\zeta_{\jmath\jmath}^{a} + \frac{m-1}{2}(\zeta_{mm}^{a})^2 \Big] \\
\geq & \sum_{a=m+1}^{n} \sum_{\imath=1}^{m-1} m(\zeta_{\imath\imath}^{a})^2 + \sum_{a=m+1}^{n} \Big[-2 \sum_{1\leq \imath<\jmath\leq m} \zeta_{\imath\imath}^{a}\zeta_{\jmath\jmath}^{a} + \frac{m-1}{2}(\zeta_{mm}^{a})^2 \Big].
\end{aligned} \tag{25}$$

For $a = m+1, \ldots, n$, we suppose the following quadratic form $f_a : \mathbb{R}^m \to \mathbb{R}$,

$$f_a\big(\zeta_{11}^{a}, \ldots, \zeta_{mm}^{a}\big) = \sum_{\imath=1}^{m-1} m(\zeta_{\imath\imath}^{a})^2 - 2 \sum_{1\leq \imath<\jmath\leq m} \zeta_{\imath\imath}^{a}\zeta_{\jmath\jmath}^{a} + \frac{m-1}{2}(\zeta_{mm}^{a})^2 \tag{26}$$

and the constrained extremum problem $\min f_a$ subject to the component of trace $\mathcal{H}$,

$$\varphi : \zeta_{11}^{a} + \cdots + \zeta_{mm}^{a} = \gamma^{a},$$

where γ^a is a real constant.

The function f_a has the following partial derivatives:

$$\begin{aligned}
\tfrac{\partial f_a}{\partial \zeta_{11}^{a}} &= 2m\zeta_{11}^{a} - 2\textstyle\sum_{\imath=2}^{m} \zeta_{\imath\imath}^{a}, \\
\tfrac{\partial f_a}{\partial \zeta_{22}^{a}} &= 2m\zeta_{22}^{a} - 2\zeta_{11}^{a} - 2\textstyle\sum_{\imath=3}^{m} \zeta_{\imath\imath}^{a}, \\
&\ \ \vdots \\
\tfrac{\partial f_a}{\partial \zeta_{m-1\ m-1}^{a}} &= 2m\zeta_{m-1\ m-1}^{a} - 2\textstyle\sum_{\imath=1}^{m-2} \zeta_{\imath\imath}^{a} - 2\zeta_{mm}^{a}, \\
\tfrac{\partial f_a}{\partial \zeta_{mm}^{a}} &= -2\textstyle\sum_{\imath=1}^{m-1} \zeta_{\imath\imath}^{a} + (m-1)\zeta_{mm}^{a}.
\end{aligned} \tag{27}$$

For an optimal solution $\big(\zeta_{11}^{a}, \ldots, \zeta_{mm}^{a}\big)$ of the problem in question, the vector *grad* f_a is normal at φ. From (27), we have a following critical point of the considered problem:

$$\zeta_{11}^{a} = \zeta_{22}^{a} = \cdots = \zeta_{m-1\ m-1}^{a} = \frac{\gamma^a}{m+1},\ \zeta_{mm}^{a} = \frac{2\gamma^a}{m+1}. \tag{28}$$

Now, we use Lemma 3.1 and for this, we fix an arbitrary point $y \in \varphi$. The bilinear form

$$\text{Ł} : T_y\varphi \times T_y\varphi \to \mathbb{R}$$

is defined by

$$Ł(X,Y) = \langle \hbar(X,Y), (grad\ f_a)(y)\rangle + \mathcal{H}ess_{f_a}(X,Y),$$

where $\hbar$ denotes the second fundamental form of φ in $\mathbb{R}^m$ and $\langle , \rangle$ denotes the standard inner product on $\mathbb{R}^m$. So, we have the following:

$$\begin{aligned} Ł(Z,Z) = &\ -2(Z_1,\dots,Z_{m-1},Z_m) \\ &\begin{pmatrix} -m & 1 & 1 & \dots & 1 & 1 \\ 1 & -m & 1 & \dots & 1 & 1 \\ 1 & 1 & -m & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & -m & 1 \\ 1 & 1 & 1 & \dots & 1 & \frac{1-m}{2} \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \\ \vdots \\ \vdots \\ Z_{m-1} \\ Z_m \end{pmatrix} \\ &= 2(m+1)\sum_{\imath=1}^{m-1} Z_\imath^2 + (m+1)Z_m^2 - 2(Z_1+\dots+Z_m)^2 \\ &= 2(m+1)\sum_{\imath=1}^{m-1} Z_\imath^2 + (m+1)Z_m^2 \\ &\geq 0, \end{aligned}$$

where we have used the relation $\sum_{\imath=1}^{m} Z_\imath^2 = 0$ (because a vector Z is tangent to φ at $y \in \varphi$ and φ is totally geodesic in $\mathbb{R}^m$). Thus, the point $(\zeta_{11}^a, \dots, \zeta_{mm}^a)$ (see (28)) is a global minimum point. From relations (25) and (28), we get $\mathbb{K} \geq 0$ and hence we have

$$2\sigma \leq m(m-1)\mathcal{C} + \frac{m^2-1}{2}\mathcal{C}(\Psi) - (m-1)(\alpha^2-\rho).$$

Further, we find that

$$\varrho \leq \mathcal{C} + \frac{(m+1)}{2m}\mathcal{C}(\Psi) - \frac{\alpha^2-\rho}{m}.$$

This is the required inequality in (i). The equality in (i) holds if and only if

$$\zeta_{\imath\jmath}^a = 0,\ \forall\ \imath,\jmath \in \{1,\dots,m\},\ \imath \neq \jmath\ ,\ a \in \{m+1,\dots,n\} \tag{29}$$

and

$$\zeta_{mm}^a = 2\zeta_{11}^a = \dots = 2\zeta_{m-1\ m-1}^a\ \forall a \in \{m+1,\dots,n\}. \tag{30}$$

With the help of (29) and (30), we find that the submanifold is invariantly quasi-umbilical and the shape operators are given by (21).

Similarly, one can easily prove the geometric inequality (ii). □

Corollary 4.2. *Let M be an m-dimensional C-totally real submanifold in an n-dimensional $(LCS)_n$-manifold $\overline{M}$. Then:*

(i) *The normalized δ-Casorati curvature $\delta_{\mathcal{C}}(m-1)$ satisfies*

$$\varrho \leq \delta_{\mathcal{C}}(m-1).$$

Furthermore, the equality sign holds if and only if M is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{M}(c)$, such that with respect to orthonormal frames of $T_{\wp}M$ and $T_{\wp}^{\perp}M$, respectively, the shape operators $\mathcal{S}_a \equiv \mathcal{S}_{\mathcal{E}_a}$, $a \in \{m+1, \ldots, n\}$, is given by (21)*.*

(ii) *The normalized δ-Casorati curvature $\widehat{\delta}_{\mathcal{C}}(m-1)$ satisfies*

$$\varrho \leq \widehat{\delta}_{\mathcal{C}}(m-1).$$

Furthermore, the equality sign holds if and only if M is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{M}(c)$, such that with respect to orthonormal frames of $T_{\wp}M$ and $T_{\wp}^{\perp}M$, respectively, the shape operators $\mathcal{S}_a \equiv \mathcal{S}_{\mathcal{E}_a}$, $a \in \{m+1, \ldots, n\}$, is given by (22)*.*

5. Main result 2

Let $\{\mathcal{E}_1, \ldots, \mathcal{E}_n\}$ be an orthonormal basis of the tangent space $\overline{M}$ and $\mathcal{N}$ be a unit tangent vector at $\wp \in \overline{M}^n$ such that $\mathcal{E}_1 = \mathcal{N}$ refracting to M^m, $\{\mathcal{E}_1, \ldots, \mathcal{E}_m\}$ is the orthonormal basis to the tangent space $T_{\wp}M$ with respect to induced quarter symmetric metric connection. Let us denote the scalar curvature and normalized scalar curvature of M with respect to induced connection associated to the quarter symmetric metric connection by $\hat{\sigma}(\wp)$ at $\wp$ and $\hat{\varrho}$, respectively. Then we prove the following:

Theorem 5.1. *Let M be an m-dimensional totally real submanifold in an n-dimensional $(LCS)_n$-manifold $\overline{M}$ with respect to quarter symmetric metric connection. Then*

(i) *The normalized δ-Casorati curvature $\delta_{\mathcal{C}}(m-1)$ satisfies*

$$\hat{\varrho} \leq \delta_{\mathcal{C}}(m-1) - \frac{(2m-1)\alpha}{m(m-1)} - \frac{\alpha\eta^2(\mathcal{N})}{m-1}.$$

Furthermore, the equality sign holds if and only if M is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{M}(c)$, such that with respect to orthonormal frames of $T_{\wp}M$ and $T_{\wp}^{\perp}M$, respectively, the shape operators $\mathcal{S}_a \equiv \mathcal{S}_{\mathcal{E}_a}$, $a \in \{m+1, \ldots, n\}$, is given by (21)*.*

(ii) *The normalized δ-Casorati curvature $\widehat{\delta}_{\mathcal{C}}(m-1)$ satisfies*

$$\hat{\varrho} \leq \widehat{\delta}_{\mathcal{C}}(m-1) - \frac{(2m-1)\alpha}{m(m-1)} - \frac{\alpha\eta^2(\mathcal{N})}{m-1}.$$

Furthermore, the equality sign holds if and only if M is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{M}(c)$, *such that with respect to orthonormal frames of* $T_\wp M$ *and* $T_\wp^\perp M$*, respectively, the shape operators* $\mathcal{S}_a \equiv \mathcal{S}_{\mathcal{E}_a}$*,* $a \in \{m+1, \ldots, n\}$*, is given by* (22).

Proof. Following [10], we have

$$2\hat{\sigma} = -(2m-1)\alpha - m\alpha\eta^2(\mathcal{N}) + m^2||\mathcal{H}||^2 - ||\zeta||^2. \tag{31}$$

Again by using T. Oprea's optimization technique, one can prove the theorem. □

Note that the scalar curvature and hence the normalized scalar curvature of C-totally real submanifold of a $(\mathrm{LCS})_n$-manifold with respect to induced Levi-Civita connection and induced quarter symmetric metric connection are identical (see [10]). Thus, we have the following:

Corollary 5.2. *Let* M *be an* m*-dimensional* C*-totally real submanifold in an* n*-dimensional* $(LCS)_n$*-manifold* $\overline{M}$ *with respect to quarter symmetric metric connection. Then*

(i) *The normalized* δ*-Casorati curvature* $\delta_{\mathcal{C}}(m-1)$ *satisfies*

$$\hat{\varrho} \leq \delta_{\mathcal{C}}(m-1).$$

Furthermore, the equality sign holds if and only if M *is an invariantly quasi-umbilical submanifold with trivial normal connection in* $\overline{M}(c)$*, such that with respect to orthonormal frames of* $T_\wp M$ *and* $T_\wp^\perp M$*, respectively, the shape operators* $\mathcal{S}_a \equiv \mathcal{S}_{\mathcal{E}_a}$*,* $a \in \{m+1, \ldots, n\}$*, is given by* (21).

(ii) *The normalized* δ*-Casorati curvature* $\widehat{\delta}_{\mathcal{C}}(m-1)$ *satisfies*

$$\hat{\varrho} \leq \widehat{\delta}_{\mathcal{C}}(m-1).$$

Furthermore, the equality sign holds if and only if M *is an invariantly quasi-umbilical submanifold with trivial normal connection in* $\overline{M}(c)$*, such that with respect to orthonormal frames of* $T_\wp M$ *and* $T_\wp^\perp M$*, respectively, the shape operators* $\mathcal{S}_a \equiv \mathcal{S}_{\mathcal{E}_a}$*,* $a \in \{m+1, \ldots, n\}$*, is given by* (22).

Acknowledgment. The authors thank the reviewer for his/her valuable suggestions to improve the presentation of this paper.

References

[1] M. Ateceken and S. K. Hui, *Slant and pseudo-slant submanifolds in LCS-manifolds*, Czechoslovak Math. J. **63(138)** (2013), no. 1, 177–190.

[2] F. Casorati, *Mesure de la courbure des surfaces suivant l'idée commune*, Acta Math. **14** (1890), no. 1, 95–110.

[3] S. Decu, S. Haesen, and L. Verstraelen, *Optimal inequalities involving Casorati curvatures*, Bull. Transilv. Univ. Braşov Ser. B (N.S.) **14(49)** (2007), suppl., 85–93.

[4] A. Friedmann and J. A. Schouten, *Über die Geometrie der halbsymmetrischen Übertragungen*, Math. Z. **21** (1924), no. 1, 211–223.

[5] V. Ghisoiu, *Inequalities for the Casorati curvatures of slant submanifolds in complex space forms*, in Riemannian geometry and applications—Proceedings RIGA 2011, 145–150, Ed. Univ. Bucureşti, Bucharest, 2011.
[6] S. Golab, *On semi-symmetric and quarter-symmetric linear connections*, Tensor (N.S.) **29** (1975), no. 3, 249–254.
[7] S. Haesen, D. Kowalczyk, and L. Verstraelen, *On the extrinsic principal directions of Riemannian submanifolds*, Note Mat. **29** (2009), no. 2, 41–53.
[8] S. K. Hui and M. Atceken, *Contact warped product semi-slant submanifolds of* $(\mathrm{LCS})_n$*-manifolds*, Acta Univ. Sapientiae Math. **3** (2011), no. 2, 212–224.
[9] S. K. Hui, M. Atceken, and T. Pal, *Warped product pseudo slant submanifolds of* $(LCS)_n$*-manifolds*, New Trends in Math. Sci. **5** (2017), 204–212.
[10] S. K. Hui and T. Pal, *Totally real submanifolds of* $(LCS)_n$*-manifolds*, arXiv:1710.04873v1 [math.DG] 13 Oct 2017.
[11] S. K. Hui, L. I. Piscoran, and T. Pal, *Invariant submanifolds of* $(LCS)_n$*-manifolds with respect to quarter symmetric metric connection*, arXiv:1706.09159 [Math. DG], 2017.
[12] D. Kowalczyk, *Casorati curvatures*, Bull. Transilv. Univ. Braşov Ser. III **1(50)** (2008), 209–213.
[13] C. W. Lee, J. W. Lee, G. E. Vilcu, and D. W. Yoon, *Optimal inequalities for the Casorati curvatures of submanifolds of generalized space forms endowed with semi-symmetric metric connections*, Bull. Korean Math. Soc. **52** (2015), no. 5, 1631–1647.
[14] J. Lee and G.-E. Vilcu, *Inequalities for generalized normalized* δ*-Casorati curvatures of slant submanifolds in quaternionic space forms*, Taiwanese J. Math. **19** (2015), no. 3, 691–702.
[15] K. Matsumoto, *On Lorentzian paracontact manifolds*, Bull. Yamagata Univ. Natur. Sci. **12** (1989), no. 2, 151–156.
[16] B. O'Neill, *Semi-Riemannian Geometry*, Pure and Applied Mathematics, **103**, Academic Press, Inc., New York, 1983.
[17] T. Oprea, *Optimization methods on Riemannian submanifolds*, An. Univ. Bucureşti Mat. **54** (2005), no. 1, 127–136.
[18] A. A. Shaikh, *On Lorentzian almost paracontact manifolds with a structure of the concircular type*, Kyungpook Math. J. **43** (2003), no. 2, 305–314.
[19] A. A. Shaikh and K. K. Baishya, *On concircular structure spacetimes*, J. Math. Stat. **1** (2005), no. 2, 129–132.
[20] ———, *On concircular structure spacetimes. II*, American J. Appl. Sci. **3** (2006), no. 4, 1790–1794.
[21] A. A. Shaikh, Y. Matsuyama, and S. K. Hui, On invariant submanifolds of $(LCS)_n$-manifolds, J. Egyptian Math. Soc. **24** (2016), no. 2, 263–269.
[22] A. N. Siddiqui, *Upper bound inequalities for* δ*-Casorati curvatures of submanifolds in generalized Sasakian space forms admitting a semi-symmetric metric connection*, Int. Electron. J. Geom. **11** (2018), no. 1, 57–67.
[23] A. N. Siddiqui and M. H. Shahid, *A lower bound of normalized scalar curvature for bi-slant submanifolds in generalized Sasakian space forms using Casorati curvatures*, Acta Math. Univ. Comenian. (N.S.) **87** (2018), no. 1, 127–140.
[24] V. Slesar, B. Sahin, and G.-E. Vilcu, *Inequalities for the Casorati curvatures of slant submanifolds in quaternionic space forms*, J. Inequal. Appl. **2014** (2014), 123, 10 pp.
[25] L. Verstraelen, *Geometry of submanifolds I. The first Casorati curvature indicatrices*, Kragujevac J. Math. **37** (2013), no. 1, 5–23.
[26] G.-E. Vilcu, *On Chen invariants and inequalities in quaternionic geometry*, J. Inequal. Appl. **2013** (2013), 66, 14 pp.
[27] A.-D. Vilcu and G.-E. Vilcu, *Statistical manifolds with almost quaternionic structures and quaternionic Kähler-like statistical submersions*, Entropy **17** (2015), no. 9, 6213–6228.

[28] S. Yamaguchi, M. Kon, and T. Ikawa, *C-totally real submanifolds*, J. Differential Geometry **11** (1976), no. 1, 59–64.

[29] K. Yano and M. Kon, *Structures on Manifolds*, Series in Pure Mathematics, **3**, World Scientific Publishing Co., Singapore, 1984.

Mohammad Hasan Shahid
Department of Mathematics
Faculty of Natural Sciences
Jamia Millia Islamia
New Delhi-110025, India
Email address: hasan_jmi@yahoo.com

Aliya Naaz Siddiqui
Department of Mathematics
Faculty of Natural Sciences
Jamia Millia Islamia
New Delhi-110025, India
Email address: aliyanaazsiddiqui9@gmail.com