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NON-REAL GROUPS WITH EXACTLY TWO CONJUGACY CLASSES OF SAME SIZE

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Abstract. In this paper, we show that A_4 is the only finite group with exactly two conjugacy classes of the same size having some non-real linear characters.

1. Introduction

There are many papers on the S_3-conjecture which state that S_3 is the only non-abelian finite group with conjugacy classes of distinct sizes. This open conjecture was solved for solvable groups in [14, 16]. In [5], Boner and Ward investigated a problem with the weakened hypothesis on conjugacy class sizes, that is, the classification of finite groups having exactly two conjugacy classes of the same size. Groups satisfying the weakened hypothesis are semi-rational groups, defined in [6]. Moreover, their conjugacy classes with unique size are rational and the two other conjugacy classes may be rational, real, or non-real. If a finite group G has two non-real conjugacy classes, then G contains two non-real irreducible characters, both of them may be linear or non-linear. Darafsheh in [7] conjectured that the only non-real groups with exactly two conjugacy classes of the same size are A_4, $PSL(2, 7)$. In this paper, we prove that A_4 is the only finite group with exactly two conjugacy classes of the same size having some non-real linear characters.

On the other hand, the authors in [1, 2] classify all finite groups with exactly two non-linear irreducible characters of equal degrees. In Proposition 2.6, we show that S_4, D_{10}, A_5, and $PSL(2, 7)$ are the only finite groups with exactly two non-central conjugacy classes of the same size and two non-linear irreducible characters of the same degree.

For the sake of convenience, we introduce some notations. Let $cl_G(a)$ denote the conjugacy class of a in G and $Lin(G)$ denote the set of all the linear characters of G. We define $Irr_1(G)$ to be the set of all the non-linear irreducible characters of G and following this notation, define $cd_1(G) = \{\chi(1) \mid \chi \in Irr_1(G)\}$.

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Moreover, let R denote the set of all real elements of G and $o(x)$ denote the order of x. Further notations and definitions are standard.

2. Main results

An element $g \in G$ is a Camina element of G if $\chi(g) = 0$ for all $\chi \in Irr_1(G)$. By Lemma 2.1 of [15], g is a Camina element if and only if $cl(g) = gG'$. We say that G is a Camina group if every element in $G \setminus G'$ is a Camina element of G. In [8], Camina groups were described as follows:

Lemma 2.1 ([8]). *Let G be a finite Camina group. Then G is a non-abelian p-group, a Frobenius group with Frobenius kernel G', or a Frobenius group with Frobenius complement isomorphic to Q_8 and G' is a subgroup of index* 4.

It is clear that the Frobenius complement of a Camina group is solvable.

Proposition 2.2. *Let G be a Camina group with exactly two non-central conjugacy classes of the same size. Then $G \cong A_4$ or D_{10}.*

Proof. By Lemma 2.1, a Camina group G is either a non-abelian p-group or a Frobenius group. Assume that G is a Frobenius group with a Frobenius kernel K and a solvable complement H. Since G has exactly two conjugacy classes of the same size and $|cl_G(x)| = |cl_H(x)||K|$ for all $x \in H \setminus \{1\}$, then H has at most two nontrivial conjugacy classes of the same size.

First, suppose that the non-trivial conjugacy classes of H are of distinct sizes, then by Theorem 16 of [16], H is either C_2 or S_3. Since H is of even order, then by Theorem 13.3 of [9], we have that K is abelian and by Theorem 13.8 of [9], $(|K| - 1)/|H| = 2$ and $|K| = 5$ or 13. Therefore since H is a subgroup of $\mathrm{Aut}(K)$, then $H \cong C_2$ and this follows that $G \cong C_5 \rtimes C_2 \cong D_{10}$.

Suppose, now, that H has exactly two non-trivial conjugacy classes of the same size, then by Theorem 6 of [3, p. 278] $|Z(H)| \neq 1$ and by Corollary 3 of [11] $H \cong C_3$. Consequently, since K is nilpotent and $|cl_G(x)| = |cl_K(x)||H|$ for all $x \in K \setminus \{1\}$, then $|cl_G(z)| = 3$ for all $z \in Z(K) \setminus \{1\}$ and so $|Z(K)| = 4$. Hence K is a 2-group and by Theorem 3 of [17], $K' \subseteq Z(K)$. Thus $|cl_K(x)| = 1,\ 2,$ or 4 and $|cl_G(x)| = 3,\ 6,$ or 12 for each $x \in K \setminus \{1\}$ and by the hypothesis we have that $2^n = |K| \leq 1 + 3 + 6 + 12 = 22$, thus n is equal to 2, 3, or 4 and $|G| = 12, 24,$ or 48. Using GAP [10] we can obtain G is isomorphic to A_4. Thus the only Frobenius groups satisfying the hypothesis are A_4 and D_{10}.

If G is a non-abelian p-group and p is odd, then G has at least two non-central conjugacy classes of each size. If G satisfies the hypothesis, then G has exactly two non-central conjugacy classes which are of the same size which this is impossible by Theorem 1 of [11]. Now, assume that G is a 2-group and $|Z(G)| = 2^m$, then the class equation and our assumptions yield

$$\begin{aligned} 2^n = |G| &\leq 2^m + (2 + \cdots + 2^{n-m-1} + 2^{n-m-1}) \\ &= \frac{2^{n-m} - 1}{2 - 1} + 2^m + 2^{n-m-1} < 2^{n-m+1} + 2^m, \end{aligned} \tag{2.1}$$

which implies that

$$2^{n-m} \leq 2^{n-2m+1} \tag{2.2}$$

which is impossible. Thus there does not exist a p-group satisfying the assumption. □

In the following results, let P_q be a Sylow q-subgroup of G, $F(G)$ be the Fitting subgroup of G, and $\pi(G)$ be the set of prime numbers dividing the order of G.

Lemma 2.3 (Theorem 1 of [12])**.** *Let G be a soluble group all of whose elements have prime power order. Then $F(G) = O_p(G)$ for some $p \in \pi(G)$ and either G is a p-group or $G = P_pP_q$ is a Frobenius group with kernel P_p or $G/F(G)$ is a group of order p^aq^b with cyclic Sylow subgroups, q being a prime of the form $kp^a + 1$.*

In Lemma 2.3, P_q is either cyclic or a generalized quaternion group whenever $G = P_pP_q$ is a Frobenius group.

Theorem 2.4. *Let G be a non-real group with exactly two conjugacy classes of the same size. Then every non-real irreducible character of G is linear if and only if G is isomorphic to A_4.*

Proof. A_4 is obviously a non-real group with exactly two conjugacy classes of the same size and every non-real irreducible character of it is linear.

Let us suppose conversely that G is a non-real group with exactly two conjugacy classes of the same size such that every non-real irreducible character of G is linear. We shall break the proof into three steps.

Step 1. G is either A_4 or $G'P_2$, where G' is of odd order and P_2 is cyclic of order 4.

Since G has exactly two non-real linear characters, then $G/G' \cong \mathbb{Z}_3$ or $\mathbb{Z}_4$. Furthermore, $x \in R$ if and only if $\theta(x) = \theta(x^{-1})$ for each linear character θ if and only if $x^2 \in G'$.

Let $\lambda, \overline{\lambda}$ be the two linear non-real irreducible characters of G. Since if $a \in G\backslash R$ then a, a^{-1} are in conjugacy classes of the same size, by assumption $G\backslash R = cl(a) \cup cl(a^{-1})$. Moreover, since every $\lambda\chi$ must be real for $\chi \in Irr_1(G)$, thus every nonlinear irreducible character vanishes on $G\backslash R$. Hence every element of $G\backslash R$ is a Camina element and $cl(a) = aG'$. The previous arguments yield that either

$$G = cl(a) \cup cl(a^{-1}) \cup G' = aG' \cup a^{-1}G' \cup G', \tag{2.3}$$

where $R = G'$, or

$$G = cl(a) \cup cl(a^{-1}) \cup a^2G' \cup G' = aG' \cup a^{-1}G' \cup a^2G' \cup G', \tag{2.4}$$

where $R = a^2G' \cup G'$ for some $a \in G - R$.

First, assume that (2.3) holds. Hence, G is a Camina group and by Proposition 2.2, $G \cong A_4$ or D_{10}. We know that D_{10} is real, then G is isomorphic to A_4.

Suppose, now, that (2.4) holds. Hence, $|C_G(a)| = |G/G'| = 4$ and $\langle a \rangle \subseteq C_G(a)$. Since a is not a real element, we can deduce $o(a) \neq 2$ and $\langle a \rangle = C_G(a)$. Furthermore, we have that

$$\left| \frac{N_G(\langle a \rangle)}{C_G(a)} \right| = \phi(o(a))/2 = 1$$

and so $\langle a \rangle = C_G(a) = N_G(\langle a \rangle)$. On the other hand, let P_2 be a Sylow 2-group of G containing $\langle a \rangle$. We can write that

$$\langle a \rangle \subseteq N_{P_2}(\langle a \rangle) = N_G(\langle a \rangle) \cap P_2 \subseteq N_G(\langle a \rangle) = \langle a \rangle$$

which follows that $\langle a \rangle = N_{P_2}(\langle a \rangle)$ and hence $P_2 = \langle a \rangle$. Therefore $G = G'P_2$ and $|G| = 4|G'|$, where $|G'|$ is odd. This implies that G is solvable.

Step 2. If $G \ncong A_4$, then $G' = P_pP_q$ in which $p, q \in \{3, 5, 7\}$ and $F(G') = O_p(G)$. Moreover, the order of P_q divides q.

Since G is an inverse semi-rational group, by Theorem 2 of [6], $\pi(G') \subseteq \{3, 5, 7, 13\}$. Moreover, we observe that

$$\frac{N_G(\langle x \rangle)}{C_G(x)} \cong \mathrm{Aut}(\langle x \rangle) \tag{2.5}$$

for all $x \in G'$ because every generator of $\langle x \rangle$ has the same conjugacy class size, so it must be conjugate. However, G has no element of order $4n$, where $n > 1$ and so a factor of a subgroup cannot have. Thus, $\mathrm{Aut}(\langle x \rangle)$ cannot have elements of order $4n$, $n > 1$. This implies that G has no elements of order $13, 25, 10, 15, 35, 42, 49$. Similarly, as $\mathrm{Aut}(C_{21})$ has $C_2 \times C_2$ as a factor, G cannot have an element of order 21. Therefore each element of G' has prime power order and, by Lemma 2.3, $G' = P_pP_q$ in which $p, q \in \{3, 5, 7\}$, $F(G') = O_p(G)$, and P_q is cyclic. Moreover, since $P_q \subseteq C_G(P_q)$, then by (2.5) the order of P_q divides q.

Step 3. If $G \ncong A_4$, then G' is a p-group.

By Step 2, we know that $G' \cong P_pP_q$ for $p, q \in \{3, 5, 7\}$ and $F(G') = O_p(G)$. Assume that P_q is of order q. Since $|\mathrm{Aut}(\langle z \rangle)| = \phi(7) = 6$ whenever $z \in Z(P_7)$ of order 7, then by (2.5) $\pi(G') \neq \{5, 7\}$. Now, we distinguish three cases for p.

`Case (1):` $p = 5$.

Lemma 2.3 implies that $G' = P_5P_3$ is a Frobenius group with kernel P_5. Since $\phi(o(z))$ divides $|cl_G(z)|$, the only possibility is $o(z) = 5$ and $|cl_G(z)| = 12$ for every $z \in Z(P_5) \setminus \{1\}$. As $Z(P_5)$ is a union of conjugacy classes, there must be at least two nontrivial among them, their both sizes 12 contradicting the hypothesis.

`Case (2):` $p = 3$.

As $q = 5$ is not of the form $kp^a + 1$, so Lemma 2.3 implies that $G' = P_3P_5$ a Frobenius group with kernel P_3. Following the previous case, we obtain that $o(z) = 3$ and $|cl_G(z)| = 10$ or 20 for every $z \in Z(P_3) \setminus \{1\}$. As $10 + 1$, $20 + 1$

and $20+10+1$ are not powers of 3, $Z(P_3)$ is a union of at least two conjugacy classes of the same size, contradicting the hypothesis.

So consider $q = 7$ and $G' = P_3P_7$. Apply (2.5) for the generator x of P_7 to see that $P_7 \leq C_{G'}(x) < N_{G'}(P_7)$ so G' is not a Frobenius group. By Lemma 2.3, $G'/O_3(G')$ is of order 21. As G' does not contain elements of order 21, $C_{G'}(x) = P_7$ so $Z(O_3(G'))P_7$ is a Frobenius group with complement P_7. As above, $o(z) = 3$ and $|cl_G(z)| = 14, 28, 42$ or 84 for every $z \in Z(O_3(G')) \setminus \{1\}$, all divisible by 14. So $|Z(O_3(G'))|$ is a power of 3 congruent to 1 modulo 7 so at least 3^6. But $1 + 14 + 28 + 42 + 84 = 169 < 3^6$ so there must be two conjugacy classes of the same size, contradicting the hypothesis.

`Case (3):` $p = 7$.

Hence $G' = P_7P_3$ is a Frobenius group where $P_3 = \langle x\rangle$ and we conclude from the class equation and the hypothesis

$$\begin{aligned} 3.7^n = |G'| &= |cl_G(x)| + \sum_{b\in A} |cl_G(b)| + \sum_{b\in B} |cl_G(b)| \\ &< 2.7^n + 2.3 \cdot (1 + \cdots + 7^{n-2}) + 4.3 \cdot (1 + \cdots + 7^{n-2}) \\ &= 2.7^n + 2.3 \cdot \frac{7^{n-1}-1}{7-1} + 4.3 \cdot \frac{7^{n-1}-1}{7-1} \\ &= 2.7^n + 3.7^{n-1} - 3 = 17.7^{n-1} - 3 < 3.7^n - 3 \end{aligned} \tag{2.6}$$

in which $A = \{b \in P_7 \mid b$ commutes with an involution$\}$ and $B = P_7 \setminus A$. This is impossible.

By these cases, we deduce that P_q is trivial and G is a p-group.

Step 4. The alternating group A_4 is the unique group satisfying the hypothesis.

If G' is a p-group, by the class equation we have

$$\begin{aligned} p^n = |G'| &< 4(1 + \cdots + p^{n-2}) + 2(1 + \cdots + p^{n-2}) \\ &= 6 \cdot \frac{p^{n-1}-1}{p-1} \end{aligned} \tag{2.7}$$

which is impossible for $p = 3, 5, 7$, as desired. □

A group G is said to be a D_1-group if $|cd_1(G)| = |Irr_1(G)| - 1$, in other words, a D_1-group G is a finite group with exactly two non-linear irreducible characters of the same degree.

In the following theorem, let (A, B) be a Frobenius group with kernel B and complement A and $ccl(G)$ denote the set of numbers which occur as the lengths of conjugacy classes of G.

Theorem 2.5 (Theorem 7 of [1])**.** *Suppose that G is a solvable D_1-group. Then one of the following assertions is true*:

(a) *G is an extraspecial group of order p^{1+2m}*,
(b) *G is a 2-group with $|G'| = 2$ and $|Z(G)| = 4$*,
(c) *G is a 2-group with $|G'| = 4$ and $|Z(G)| = 2$*,

(d) *$G = (C_{(p^m-1)/2}, E(p^m))$, where $E(p^m)$ is the elementary abelian group of order p^m,*
(e) *$G = (C_{p^m-1}, E(p^m))) \times C(2)$,*
(f) *$G = (C_{2^m-1}, G') \cong M(2^m)$, the normalizer of a Sylow 2-subgroup of the Suzuki simple group $Sz(2^m)$, $m > 1$,*
(g) *$G = (C_{2^m-1}, G')$, where $G/Z(G') \cong M(2^m)$, $m > 1$,*
(h) *G is a group with $cd_1(G) = \{p^n - 1\}$ which $G/Z(G)$ is a Frobenius group,*
(i) *$G = S_4$.*

Proposition 2.6. *Let G be a D_1-group with exactly two non-central conjugacy classes of the same size. Then G is isomorphic to S_4, A_5, D_{10}, or $PSL(2, 7)$.*

Proof. Consider that G is a solvable group then G satisfies Theorem 2.5. By Proposition 2.2, since D_{10} and A_4 are the only Frobenius groups with solvable complement having two non-central conjugacy classes of the same size, then G is one of Cases (d), (e), (f), and (g) of Theorem 2.5 satisfying our hypotheses if and only if $G \cong D_{10}$.

If G is one of Cases (a), (b), and (c) of Theorem 2.5, then G is a p-group and so by the proof of Proposition 2.2, such groups do not satisfy the hypotheses.

Assume that G is a group with $cd_1(G) = \{p^n - 1\}$ for which $G/Z(G)$ is a Frobenius group. Then, by Theorem 1 and Corollary 2 of [4], $ccl(G) = \{1, p^n - 1, p^n\}$ with corresponding frequencies $\{2, 2, 2(p^n - 2)\}$ for some odd prime number p and $n \in \mathbb{N}$ which contradicts our hypotheses.

Additionally, we can easily check that S_4 satisfies the hypotheses.

Now, consider that G is a non-solvable group, by Main Theorem of [2], G is either A_5 or $PSL(2, 7)$. By the character tables in Appendix of [13], these groups satisfy the our hypotheses. □

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