Commun. Korean Math. Soc. **0** (0), No. 0, pp. 1–0
https://doi.org/10.4134/CKMS.c180046
pISSN: 1225-1763 / eISSN: 2234-3024

A VANISHING THEOREM FOR REDUCIBLE SPACE CURVES AND THE CONSTRUCTION OF SMOOTH SPACE CURVES IN THE RANGE C

Edoardo Ballico

Abstract. Let $Y \subset \mathbb{P}^3$ be a degree d reduced curve with only planar singularities. We prove that $h^i(\mathcal{I}_Y(t)) = 0$, $i = 1, 2$, for all $t \geq d-2$. We use this result and linkage to construct some triples (d, g, s), $d > s^2$, with very large g for which there is a smooth and connected curve of degree d and genus g, $h^0(\mathcal{I}_C(s)) = 1$ and describe the Hartshorne-Rao module of C.

1. Introduction

To construct smooth space curves using liaison we needed the following weak version of a Castelnuovo-type theorem for curves which are not irreducible (see [4, 6] for better results for integral curve; [16] in the part concerning space curves requires that the curve is integral).

Theorem 1. *Let $Y \subset \mathbb{P}^3$ be a reduced curve with only planar singularities defined over an algebraically closed field of characteristic* 0. *Set $d := \deg(Y)$. Then $h^1(\mathcal{I}_Y(t)) = h^2(\mathcal{I}_Y(t)) = 0$ for all $t \geq d-2$.*

For several classical result on the classification of space curves, see [7, 8, 10]. We use Theorem 1 to construct (for certain d, g, s) smooth and connected curves C such that $\deg(C) = d$, $p_a(C) = g$, $h^0(\mathcal{I}_C(s-1)) = 0$ and $h^0(\mathcal{I}_C(s)) \neq 0$. For all integers d, s such that $s > 1$ and $d > s^2$ set

$$G(d, s) := 1 + [d(d + s^2 - 4s) - r(s-r)(s-1)]/2s,$$

where r is the only integer such that $0 \leq r \leq s-1$ and $d + r \equiv 0 \pmod{s}$. We work in the so-called Range C, i.e., we take $d > s^2$. In the Range C, L. Gruson and Ch. Peskine proved that if $X \subset \mathbb{P}^3$ is a smooth connected curve of degree d and genus g with $h^0(\mathcal{I}_X(s-1)) = 0$, then $g \leq G(d, s)$ and equality holds if and only if X is linked to a plane curve of degree r by the complete

Received February 3, 2018; Revised May 21, 2018; Accepted May 24, 2018.
2010 *Mathematics Subject Classification.* 14H50.
Key words and phrases. space curve, Hilbert function, Halphen's gaps, linkage, Hartshorne-Rao module.
The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

intersection of a surface of degree s and a surface of degree $\lceil d/s\rceil$ ([7, Théorème 3.1], [8, Théorème A]).

See [1], [17], [13, 3.1 and 3.3] for the construction of a huge number of triples (d,g,s) such that there is a smooth and connected curve $T\subset \mathbb{P}^3$ with $\deg(T)=d$, $p_a(T)=g$, $h^0(\mathcal{I}_T(s-1))=0$ and $h^0(\mathcal{I}_T(s))\neq 0$.

Fix an integer q such that $G(d,s+1)<q<G(d,s)$. The triple (d,q,s) is called an Halphen's gap if there is no smooth and connected curve $T\subset\mathbb{P}^3$ with $\deg(T)=d$, $p_a(T)=q$ and $h^0(\mathcal{I}_T(s-1))=0$. It is known that Halphen's gaps exist ([2,3,5,12,13]). In particular $(d,G(d,s)-1,s)$ is an Halphen's gap if either $r=0$ and $s\geq 4$ ([2, Proposition 3.10] or $s\geq 5$ and $r\notin\{2,3,s-3,s-2\}$ ([3, Th. 3.3]). Roughly speaking, with a few exceptions the first genus less than the maximal one, i.e., $G(d,s)$, gives an Halphen's gap. Let $G^1(d,s)$ denote the largest integer $<G(d,s)$ obtained as the genus of a projectively normal curve of degree d not contained in a surface of degree $s-1$ (see [5, Definition VI.1] for its value; we only need that $G^1(d,s)=G(d,s)-s+2$ if $r=0$ and $s\geq 3$ and $G^1(d,s)=G(d,s)-r+2$ if either $3\leq r<s/2$ and $s\geq 6$ or $r=s-2,s-1$ and in the other cases it is at least $G(d,s)-r+2$). Ph. Ellia proved (with the weaker assumption $d>s(s-1)$) that for all integers g such that $\min\{G^1(d,s),G(d,s+1)\}<g<G(d,s)$ the triple (d,g,s) is an Halphen's gap, unless either $r=2$ or $r=s-2$ (see [5, Théorème at page 42]). Here we prove the following result which shows that in many cases $(d,G^1(d,s)-1,s)$ is not an Halphen's gap.

Proposition 1. *Take $d>s^2$ with $s\geq 3$. Let r be the only integer such that $0\leq r<s$ and $d+r\equiv 0 \pmod{s}$.*

(a) *If $r=0$, then $G(d,s)-s+1$ is not an Halphen's gap.*

(b) *If $r>0$, then $G(d,s)-r+1$ is not an Halphen's gap.*

If $r=1$, then Proposition 1 is trivial and also the cases $r=2,3$ are well-known with as a curve a projectively normal curve ([3, Th. 3.3]). If $r=3$ we also prove that $(d,G(d,s)-2,s)$ is not an Halphen's gap (see Remark 3). We use linkage to cover other triples (d,g,s) as being not an Halphen's gap, but the main point is to get examples for the same (d,g,s), but with very different cohomology groups $h^1(\mathcal{I}_C(t))$, $t\in\mathbb{Z}$, (see Proposition 2).

As in [7,8] we work over an algebraically closed field $\mathbb{K}$ of characteristic zero.

2. The proofs

Proof of Theorem 1. For any $t\in\mathbb{Z}$ we have $h^2(\mathcal{I}_Y(t))=h^1(\mathcal{O}_Y(t))$. To prove that $h^1(\mathcal{O}_Y(d-2))=0$ it is sufficient to do it when Y is connected, i.e., (since Y is reduced) when $h^0(\mathcal{O}_Y)=1$. In this case we have $h^1(\mathcal{O}_Y(d-2))=0$, because $\deg(\omega_Y)\leq d(d-3)$, which is true by Riemann-Roch, duality and the inequality $\chi(\mathcal{O}_Y)\geq 1-(d-2)(d-3)/2$ true by [11, Theorem 3.1]. Now we prove that $h^1(\mathcal{I}_Y(d-2))=0$. Fix a general $q\in\mathbb{P}^3$. Let $\ell_q:\mathbb{P}^3\setminus\{q\}\longrightarrow\mathbb{P}^2$ denote the linear projection from q. Since Y is reduced and with only planar singularities and q is general, q is not contained in the union of the Zariski tangent spaces of Y. Since we are in characteristic zero and q is general, no line L with $\deg(L \cap Y) \geq 3$ contains q and only finitely many secant lines of Y pass though q. Thus $\ell_q(Y)$ is a plane curve of degree d with only nodal singularities plus for each $a \in \mathrm{Sing}(Y)$ the curve $\ell_q(Y)$ has a singularity at $\ell_q(a)$ formally equivalent to the one of Y at a. Call S the union of the singular points of Y which are not images of a singular point of Y. Choose homogeneous coordinates x_0, x_1, x_2, x_3 on $\mathbb{P}^2$ such that $q = (1:0:0:0)$ and use x_1, x_2, x_3 as homogeneous coordinates of $\mathbb{P}^2$. So $\ell_q(x_0 : x_1 : x_2 : x_3) = (x_1 : x_2 : x_3)$. For each $\lambda \in \mathbb{K} \setminus \{0\}$ let $h_\lambda : \mathbb{P}^3 \longrightarrow \mathbb{P}^3$ be the automorphism defined by the formula $h_\lambda(x_0 : x_1 : x_2 : x_3) = (\lambda x_0 : x_1 : x_2 : x_3)$. For each $o \in \mathbb{P}^3$ let $\chi(o)$ denote the first infinitesimal neighborhood of o in $\mathbb{P}^3$, i.e. the closed subscheme of $\mathbb{P}^3$ with $(\mathcal{I}_o)^2$ as its ideal sheaf. For each $\lambda \in \mathbb{K} \setminus \{0\}$, we have $h^1(\mathcal{I}_{h_\lambda(Y)}(t)) = h^1(\mathcal{I}_Y(t))$, because h_λ is an automorphism. The flat family

$$\{h_\lambda(Y)\}_{\lambda \in \mathbb{K} \setminus \{0\}}$$

has as a flat limit the one-dimensional scheme $E := \ell_q(Y) \cup \bigcup_{o \in S} \chi(o)$ ([9, III.9.8.4 and figure 11 at page 260]). By the semicontinuity theorem for cohomology it is sufficient to prove that $h^1(\mathcal{I}_E(t)) = 0$ for all $t \geq d-2$. Let H denote the plane $\{x_0 = 0\}$. See $\ell_q(Y)$ as a subscheme of H. For any scheme $W \subset \mathbb{P}^3$ let $\mathrm{Res}_H(W)$ denote the residual scheme of W with respect to H, i.e., the closed subscheme of $\mathbb{P}^3$ with $\mathcal{I}_W : \mathcal{I}_H$ as its ideal sheaf. Since $\mathrm{Res}_H(\chi(o)) = \{o\}$ for each $o \in H$ and $\ell_q(Y) \subset H$, we have a residual exact sequence

$$0 \longrightarrow \mathcal{I}_S(t-1) \longrightarrow \mathcal{I}_E(t) \longrightarrow \mathcal{I}_{\ell_q(Y),H}(t) \longrightarrow 0. \quad (1)$$

Since $\ell_q(Y)$ is a plane curve, we have $h^1(H, \mathcal{I}_{\ell_q(Y),H}(t)) = 0$. Since S is a subset of the set of all singular points of the reduced degree d plane curve, adjunction theory gives $h^1(H, \mathcal{I}_{S,H}(d-3)) = 0$. Thus $h^1(H, \mathcal{I}_{S,H}(x)) = 0$ for all $x \geq d-3$. Thus $h^1(\mathcal{I}_S(t-1)) = 0$ for all $t \geq d-2$. Use the long cohomology exact sequence of (1). □

The following remark gives the relations between the numerical and cohomological invariants of two linked space curves.

Remark 1. Let $A, B \subset \mathbb{P}^3$ be locally Cohen-Macaulay schemes with pure dimension 1. Assume that A and B are linked by a complete intersection X of a curve of degree s and a curve of degree m. Then for each $t \in \mathbb{Z}$ we have ([14, Proposition III.1.2]):

$$h^1(\mathcal{I}_A(t)) = h^1(\mathcal{I}_B(s+m-4-t)); \quad (2)$$

$$h^0(\mathcal{I}_A(t)) = h^0(\mathcal{I}_X(t)) = h^1(\mathcal{O}_B(s+m-4-t)); \quad (3)$$

$$h^0(\mathcal{I}_B(t)) = h^0(\mathcal{I}_X(t)) = h^1(\mathcal{O}_A(s+m-4-t)); \quad (4)$$

$$\chi(\mathcal{O}_B) - \chi(\mathcal{O}_A) = (s+m-4)(\deg(A) - \deg(B))/2. \quad (5)$$

We obviously have $\deg(A) + \deg(B) = sm$.

Lemma 1. *Let $Y \subset \mathbb{P}^3$ be a reduced curve with only planar singularities. Fix an integer $b > 0$ and assume $h^1(\mathcal{I}_Y(b-1)) = h^0(\mathcal{O}_Y(b-2)) = 0$. We have $|\mathcal{I}_Y(b)| \neq \emptyset$, $\mathcal{I}_Y(b)$ is globally generated and a general $G \in |\mathcal{I}_Y(b)|$ is smooth.*

Proof. Since $h^2(\mathcal{I}_Y(t)) = h^1(\mathcal{O}_Y(t))$ for all $t \in \mathbb{Z}$, the Castelnuovo-Mumford's lemma gives that $\mathcal{I}_Y(b)$ is globally generated and in particular $|\mathcal{I}_Y(b)| \neq \emptyset$. By Bertini's theorem a general $G \in |\mathcal{I}_Y(b)|$ is smooth outside b. Since Y has only planar singularities, the conormal sheaf $\mathcal{A} := \mathcal{I}_Y/{\mathcal{I}_Y}^2$ is a rank 2 vector bundle on Y. Since $\mathcal{I}_Y(b)$ is globally generated, the image of the map $H^0(\mathcal{I}_Y(b)) \longrightarrow H^0(Y, \mathcal{A}(b))$ spans the vector bundle $\mathcal{A}(b)$. Since $\mathcal{A}(b)$ is a vector bundle whose rank is $> \dim(Y)$, there is $s \in H^0(\mathcal{I}_Y(b))$ whose image in $H^0(Y, \mathcal{A}(b))$ has no zero in Y. The element $\{s = 0\} \in |\mathcal{I}_Y(b)|$ is smooth at all smooth points of Y. Since Y is reduced, it has only finitely many singular points. Since $H^0(\mathcal{I}_Y(b))$ (as any vector space) is irreducible, to conclude the proof of the lemma it is sufficient to prove that for each $q \in \mathrm{Sing}(Y)$ the set of all $G \in |\mathcal{I}_Y(b)|$ singular at q is a proper linear subspace of $|\mathcal{I}_Y(b)|$. Let $v \subset \mathbb{P}^3$ be a connected zero-dimensional scheme with $\deg(v) = 2$, $v_{\mathrm{red}} = \{q\}$ and v not contained in the Zariski tangent space to Y at q. Since $\mathcal{I}_Y(b)$ is globally generated, $|\mathcal{I}_{Y \cup v}(b)|$ is a hyperplane of $|\mathcal{I}_Y(b)|$. The projective space $|\mathcal{I}_{Y \cup v}(b)|$ is the set of all $G \in |\mathcal{I}_Y(b)|$ singular at q. □

Lemma 2. *Let $Y \subset \mathbb{P}^3$ be a reduced curve with only planar singularities. Fix integers $k \ge b > 0$ and assume $h^1(\mathcal{I}_Y(b-1)) = h^2(\mathcal{I}_Y(b-2)) = 0$. Let C be a general curve linked to Y by a complete intersection of a surface of degree b by a surface of degree k. Then C is smooth. If $k \ge 3$, then C is connected.*

Proof. The linked curve C exists because $\mathcal{I}_Y(b)$ and $\mathcal{I}_Y(k)$ are globally generated by the Castelnuovo-Mumford's lemma. Fix a general $G \in |\mathcal{I}_Y(k)|$. By Lemma 1 G is smooth. Thus Y is a Cartier divisor of G. Since $\mathcal{I}_Y(k)$ is spanned, the line bundle $\mathcal{L} := \mathcal{O}_G(k)(-Y)$ is spanned. Apply Bertini's theorem to $\mathcal{L}$ and get the smoothness part. By (2) we have $h^1(\mathcal{I}_C) = h^1(\mathcal{I}_Y(b+k-4))$. Since $k \ge 3$ and $h^1(\mathcal{I}_Y(b-1)) = h^2(\mathcal{I}_Y(b-2)) = 0$, the Castelnuovo-Mumford's lemma gives $h^1(\mathcal{I}_Y(b+k-4)) = 0$ and so $h^1(\mathcal{I}_C) = 0$. Since $h^1(\mathcal{I}_C) = 0$, C is connected. □

Lemma 3. *Fix an integer $s \ge 3$. Let $Y \subset \mathbb{P}^3$ be the union of a smooth plane curve A and a line L with $\deg(A) = s-1$ and $A \cap L = \emptyset$. Then $h^1(\mathcal{I}_Y(t)) = 0$ if either $t \ge s-1$, or $t < 0$ and $h^1(\mathcal{I}_Y(t)) = 1$ if $0 \le t \le s-2$.*

Proof. Since $s - 1 \ge 2$, A spans a plane, M. Set $q := M \cap L$. Since $q \notin A$, we have the following exact sequence of coherent sheaves on M:

$$0 \longrightarrow \mathcal{I}_{q,M}(t-s+1) \longrightarrow \mathcal{I}_{A\cup\{q\},M}(t) \longrightarrow \mathcal{O}_A(t) \longrightarrow 0.$$

Thus $h^0(M, \mathcal{I}_{A\cup\{q\},M}(t)) = 0$ for all $t \le s-1$, $h^1(M, \mathcal{I}_{A\cup\{q\},M}(t)) = 0$ for all $t \ge s-1$ and $h^1(M, \mathcal{I}_{A\cup\{q\},M}(t)) = 1$ for all $t \le s-2$. We have the residual exact sequence of M in $\mathbb{P}^3$:

$$0 \longrightarrow \mathcal{I}_L(t-1) \longrightarrow \mathcal{I}_Y(t) \longrightarrow \mathcal{I}_{A\cup\{q\},M}(t) \longrightarrow 0. \tag{6}$$

Since L is arithmetically normal, (6) gives $h^1(\mathcal{I}_Y(t)) = 0$ for all $t \ge s-1$. Since $h^2(\mathcal{I}_L(t-1)) = h^1(\mathcal{O}_L(t-1)) = 0$ for all $t \ge 2$, (6) also gives $h^1(\mathcal{I}_Y(t)) = 1$ if $2 \le t \le s-2$. Since $h^0(\mathcal{O}_Y(1)) = 5$, $h^0(\mathcal{O}_Y) = 2$ and $h^0(\mathcal{O}_Y(t)) = 0$ for all $t < 0$, we get $h^1(\mathcal{I}_Y(1)) = h^1(\mathcal{I}_Y) = 1$ and $h^1(\mathcal{I}_Y(t)) = 0$ for all $t < 0$. □

Remark 2. Fix integers $k > s \ge 3$. Recall that $G(ks, s) = 1 + ks(k+s-4)/2$. Take Y as in Lemma 3 and let C be the curve linked to Y by the complete intersection of a general surface of degree s and a general surface of degree $k+1$ containing Y (C exists by Lemma 1 and it is smooth and connected by Lemma 2). We apply (5) with $B := Y$, $A := C$ and $m := k+1$. Let g be the genus of C. Since C is smooth and connected, we have $\chi(\mathcal{O}_C) = 1-g$. Since Y is the disjoint union of a line and a plane curve of degree $s-1$, we have $\chi(\mathcal{O}_Y) = 2-(s-2)(s-3)/2$. Thus (5) gives $g = (s-2)(s-3)/2 - 1 + (s+k-3)(sk-s) = (s-2)(s-3)/2 - 1 + sk(s+k-4)/2 + (-s^2+3s)/2 = 2 - s + ks(k+s-4)/2 = G(ks, s) - s + 1$. By (2) and Lemma 3 we have $h^1(\mathcal{I}_C(t)) = 0$ if either $t > s+k-3$ or $t \le k-2$ and $h^1(\mathcal{I}_C(t)) = 1$ if $k-1 \le t \le s+k-3$. For the case $2 \le r < s$ we may apply Lemma 3 with the integer r instead of the integer s; call Y' this curve of degree r. Call X the curve linked to Y' by a smooth surface G of degree g and a curve of degree k. It has degree $sk - r$. By (5) it has genus $G(sk-r, s) - r + 1$.

Proof of Proposition 1. First assume $r = 0$. Let $Y \subset \mathbb{P}^3$ be the union of a smooth plane curve A and a line L with $\deg(A) = s-1$ and $A \cap L = \emptyset$. By Lemma 3 (or Theorem 1) we have $h^1(\mathcal{I}_Y(s-1)) = 0$. Since $h^2(\mathcal{I}_Y(s-2)) = h^1(\mathcal{O}_Y(s-2)) = h^1(\mathcal{O}_A(s-2)) + h^1(\mathcal{O}_L(s-2)) = 0$. By the Castelnuovo-Mumford's lemma $\mathcal{I}_Y(s)$ is globally generated. By Lemmas 3 and 2 a general curve F linked to Y by a complete intersection of a surface of degree s and a surface of degrees $k+1$ is a smooth and connected curve and to take the linkage we may take a smooth surface G of degree s. Obviously F has degree d. By Remark 2 F has genus $G(d, s) - s + 1$. By construction $F \subset G$ with G an irreducible surface of degree s. Since $d > s(s-1)$, Bezout's theorem gives $h^0(\mathcal{I}_E(s-1)) = 0$. The curve F shows that $(d, G(d, s) - s + 1, s)$ is not an Halphen's gap.

Now assume $0 < r < s$. The case $r = 1$ is obvious, because $G(d, s) - r + 1 = G(d, s)$ in this case. Assume $r \ge 2$. Let $Y \subset \mathbb{P}^3$ be the union of a smooth plane curve A and a line L with $\deg(A) = e-1$ and $A \cap L = \emptyset$. By Lemma 3 (or Theorem 1) we have $h^1(\mathcal{I}_Y(r-1)) = 0$. Since $h^2(\mathcal{I}_Y(r-2)) = h^1(\mathcal{O}_Y(r-2)) = h^1(\mathcal{O}_A(r-2)) + h^1(\mathcal{O}_L(r-2)) = 0$. By the Castelnuovo-Mumford's lemma for all $x \ge r$ the sheaf $\mathcal{I}_Y(x)$ is globally generated. By Lemmas 3 and 2 a general curve F linked to Y by a complete intersection of a surface of degree s and a surface of degrees k is a smooth and connected curve and to take the linkage we may take a smooth surface G of degree s. Obviously F has degree

d. By Remark 2, F has genus $G(d,s)-r+1$. By construction $F \subset G$ with G an irreducible surface of degree s. Since $d > s(s-1)$, Bezout's theorem gives $h^0(\mathcal{I}_F(s-1)) = 0$. The curve F shows that $(d, G(d,s)-r+1, s)$ is not an Halphen's gap. □

For any positive integer d let $E(d)$ denote the set of all reduced degree d space curves with only planar singularities. For all positive integers d, s set $E'(d,s) := \{E \in E(d) \mid h^1(\mathcal{I}_E(s-1)) = h^2(\mathcal{I}_E(s-2)) = 0\}$. Fix any $E \in E'(d,s)$. By the Castelnuovo-Mumford's lemma for each integer $t \geq s$ we have $h^1(\mathcal{I}_E(t)) = h^2(\mathcal{I}_E(t-1)) = 0$ and the sheaf $\mathcal{I}_E(t)$ is globally generated. Thus we may use E to do a linkage with respect to two surfaces of degree at least s.

Proposition 2. *Fix integers d, s with $d > s^2$ and let r be the only integer such that $0 \leq r < s$. Set $k := \lceil d/s \rceil$. Fix an integer $x \geq k$ and take $Y \in E'(xs-d, s)$. Set $q := 1 - \chi(\mathcal{O}_Y)$. Let C be a curve obtained linking Y by a general complete intersection of a surface of degree s by a surface of degree x. Then C is smooth and connected, $\deg(C) = d$, $g := p_a(C) = q+(x+s-4)(2d-xs)$, $h^0(\mathcal{I}_C(s-1)) = 0$ and $h^0(\mathcal{I}_C(s)) \neq 0$. The Hartshorne-Rao module of C is, up to shift by $s+x-4$, the dual of the one of Y. The curve C shows that (d,g,s) is not an Halphen's gap.*

Proof. By Lemmas 1 and 2 the smooth curve C exists. Since $x \geq 3$, C is connected by Lemma 2. The genus g follows from (5). The statement about Hartshorne-Rao modules is a well-known property of linked curves ([15]). □

Remark 3. Take d, s and r as in Proposition 2 with $s \geq 3$. Assume $r = 2$. Let Y be the disjoint union of 3 lines. This curve is the curve of $E(3)$ with the larger $\chi(\mathcal{O}_Y)$. Taking a curve C linked to Y by a surface of degree s and a surface of degree k we get that $(d, G(d,s)-2, s)$ is not an Halphen's gap. Since $h^1(\mathcal{I}_Y) = h^1(\mathcal{I}_Y(1)) = 2$ and $h^1(\mathcal{I}_Y(t)) = 0$ is either $t < 0$ or $t \geq 2$, we also see that $h^1(\mathcal{I}_C(t)) = 0$ if either $t > s+k-4$ or $t \leq s+k+2$ and $h^1(\mathcal{I}_C(t+k-4)) = h^1(\mathcal{I}_C(t+k-3)) = 2$.

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Edoardo Ballico
Department of Mathematics
University of Trento
38123 Povo (TN), Italy
Email address: `ballico@science.unitn.it`