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ON 3-DIMENSIONAL LORENTZIAN CONCIRCULAR STRUCTURE MANIFOLDS

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Abstract. The aim of the present paper is to study the Eisenhart problems of finding the properties of second order parallel tensors (symmetric and skew-symmetric) on a 3-dimensional LCS-manifold. We also investigate the properties of Ricci solitons, Ricci semisymmetric, locally ϕ-symmetric, η-parallel Ricci tensor and a non-null concircular vector field on $(LCS)_3$-manifolds.

1. Introduction

Lorentzian manifold is one of the most important sub-class of pseudo Riemannian manifolds. It plays a crucial role in mathematical physics (specially in the development of the theory of relativity and cosmology). Matsumoto et al. ([23, 24]) gave the idea of Lorentzian para-Sasakian manifolds (briefly LP-Sasakian manifolds). In 2003, Shaikh [28] extended the notion of LP-Sasakian manifolds by considering the fact of concircular vector field and called them the Lorentzian concircular structure manifolds (briefly $(LCS)_n$-manifolds). Since then the properties of such manifolds have been studied by many geometers (for instance, see [12–19], [29–38], [46–48]).

Eisenhart [9] studied the properties of second order parallel symmetric tensor in 1923. He proved that if a positive definite Riemannian manifold confesses a second order parallel symmetric covariant tensor other than a constant multiple of the metric tensor, then it is reducible. Levy [22] in 1925, demonstrated that a second order parallel symmetric non-degenerated tensor of type $(0, 2)$ in a space form is proportional to the metric tensor. Eisenhart and Levy studied the properties of second order parallel tensor locally while Sharma [39] studied the properties of same tensor globally based on Ricci identities on complex space forms. Since then, many authors examined the Eisenhart problems of finding the properties of symmetric and skew-symmetric parallel tensors on various spaces and obtained many geometrical results. As illustrations, the

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Eisenhart problems on almost contact metric manifolds were considered by Sharma ([40–42]), on f-Kenmotsu manifold by Calin et al. [4], on $N(k)$-quasi Einstein manifold by Crasmareanu [7], on space with vanishing quasi constant curvature and Vaisman manifolds by Bejan et al. ([1, 2]), on almost Kenmotsu manifold by Wang et al. [45], on α-Sasakian manifold by Das [8], on space of quasi constant curvature by Jia [20], on $(LCS)_n$-manifold by second author et al. [6] and many others.

Hamilton [10] introduced the notion of Ricci flow to obtain a canonical metric on a differentiable manifold in the beginning of $80's$. After that it became a powerful tool to study Riemannian manifolds of positive curvature. To prove Poincaré conjecture, Perelman ([26, 27]) used Ricci flow and its surgery. Also Brendle and Schoen [3] proved the differentiable sphere theorem by using Ricci flow. The evolution equation for metrics on a Riemannian manifold, called Ricci flow and defined as

$$\frac{\partial}{\partial t} g_{ij}(t) = -2S_{ij},$$

where S_{ij} denote the components of Ricci tensors. The solutions of Ricci flow are called Ricci solitons if they are governed by a one parameter family of diffeomorphisms and scalings. A triplet (g, V, λ) on a Riemannian manifold (M, g) is called Ricci soliton [11], natural generalization of Einstein metric, and satisfies

$$\frac{1}{2} L_V g + S = \lambda g, \tag{1}$$

where S is the Ricci tensor, $L_V g$ denotes the Lie derivative of the metric g along the vector field V on M and λ is a real constant [11]. A Ricci soliton is said to be steady, expanding and shrinking if $\lambda = 0, < 0$ and > 0 respectively. The properties of Ricci solitons on $(LCS)_n$-manifolds have been studied by Hui and his co-authors ([6, 16–19]) and others.

Another important sub area of differential geometry is symmetric spaces. The perception of local symmetry on different spaces has been weakened by number of geometers in distinct extent. As a feeble version of local symmetry, Takahashi [44] introduced and studied the concept of local ϕ-symmetry on a Sasakian manifold. The properties of local ϕ-symmetry on $(LCS)_n$-manifolds have been noticed in [33]. From the above study, we come to the conclusion that the study of Ricci solitons, Ricci semisymmetric, locally ϕ-symmetric, η-parallelism, second order parallel symmetric and skew-symmetric tensors and non-null concircular vector fields are lacking in 3-dimensional LCS-manifolds. To fill these gaps, we are going to study all these properties on LCS-manifolds of dimension three. We will organize the present paper as follows:

Section 1 is about the brief introduction of $(LCS)_n$-manifolds, Ricci solitons, symmetric spaces and Eisenhart problem of finding symmetric and skew-symmetric properties of second order parallel tensors while in second section we give basic results of $(LCS)_3$-manifolds, Ricci soliton and η-parallel Ricci tensor. In Section 3, we prove that a $(LCS)_3$-manifold is space form if and only if it is Ricci semisymmetric. The properties of locally ϕ-symmetric and η-parallel Ricci tensor on a 3-dimensional LCS-manifold are studied in Section 4 and 5 respectively. Sections 6 and 7 deal with study of Eisenhart problem of finding the properties of symmetric and skew-symmetric second order parallel tensors. In the last section, we evaluate the geometrical properties of a non-null concircular vector field.

2. Preliminaries

A Lorentzian manifold of dimension n is a doublet (M, g), where M is a smooth connected para-compact Hausdorff manifold of dimension n and g is a Lorentzian metric, that is, M admits a smooth symmetric tensor field g of type $(0, 2)$ such that for each point $p \in M$ the tensor $g_p : T_pM \times T_pM \to \Re$ is a non degenerate inner product of signature $(-, +, \dots, +)$, where T_pM denotes the tangent space of M at p and $\Re$ is the real number space. A non-zero vector field $V \in T_pM$ is said to be time like (resp., non-space like, null, and space like) if it satisfies $g_p(V, V) < 0$ (resp., $\leq 0, = 0, > 0$) [25].

Definition. In a Lorentzian manifold (M, g) a vector field ρ defined by

$$g(X, \rho) = B(X)$$

for any $X \in \chi(M)$ is said to be a concircular vector field if

$$(\nabla_X B)(Y) = \alpha\{g(X, Y) + \omega(X)\omega(Y)\},$$

where α is a non-zero scalar and ω is a closed 1-form [49].

If a Lorentzian manifold M admits a unit time like concircular vector field ξ, called the generator of the manifold, then we have

$$g(\xi, \xi) = -1. \tag{2}$$

Since ξ is the unit concircular vector field on M, there exists a non-zero 1-form η such that

$$g(X, \xi) = \eta(X), \tag{3}$$

which satisfies the following equation

$$(\nabla_X \eta)(Y) = \alpha\{g(X, Y) + \eta(X)\eta(Y)\} (\alpha \neq 0) \tag{4}$$

for all vector fields X and Y, where ∇ denotes the operator of covariant differentiation with respect to the Lorentzan metric g and α is a non-zero scalar function satisfies

$$(\nabla_X \alpha) = X\alpha = d\alpha(X) = \rho\eta(X), \tag{5}$$

where ρ being a certain scalar function given by $\rho = -(\xi\alpha)$. If we put

$$\phi X = \frac{1}{\alpha}\nabla_X \xi, \tag{6}$$

then with the help of (3), (4) and (6), we can find

$$\phi X = X + \eta(X)\xi, \tag{7}$$

which shows that ϕ is a tensor field of type $(1,1)$, called the structure tensor of the manifold M. Hence the Lorentzian manifold M of class C^∞ equipped with a unit timelike concircular vector field ξ, its associated 1-form η and $(1,1)$-tensor field ϕ is said to be a Lorentzian concircular structure manifold (briefly $(LCS)_n$-manifolds) [28]. Especially, if we take $\alpha = 1$, then we can obtain the LP-Sasakian structure of Matsumoto [23]. Thus we can say that the $(LCS)_n$-manifold is the generalization of LP-Sasakian manifold. It is noteworthy to mention that LCS-manifold is invariant under a conformal change whereas LP-Sasakian structure is not so [36]. In $(LCS)_3$-manifolds, the following relations hold [28]:

$$\eta(\xi) = -1, \quad \phi\xi = 0, \quad \phi^2 X = X + \eta(X)\xi, \quad \eta(\phi X) = 0$$
$$\text{and} \quad g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{8}$$

$$\eta(R(X,Y)Z) = (\alpha^2 - \rho)\{\eta(X)g(Y,Z) - \eta(Y)g(X,Z)\}, \tag{9}$$

$$R(X,Y)\xi = (\alpha^2 - \rho)\{\eta(Y)X - \eta(X)Y\}, \tag{10}$$

$$R(\xi, X)Y = (\alpha^2 - \rho)\{g(X,Y)\xi - \eta(Y)X\}, \tag{11}$$

$$(\nabla_X \phi)(Y) = \alpha\{g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\}, \tag{12}$$

$$S(X,\xi) = 2(\alpha^2 - \rho)\eta(X), \tag{13}$$

$$S(\phi X, \phi Y) = S(X,Y) + 2(\alpha^2 - \rho)\eta(X)\eta(Y), \tag{14}$$

$$X\rho = d\rho(X) = \beta\eta(X) \tag{15}$$

for any vector fields X, Y, Z, where R, S denote the curvature tensor and the Ricci tensor of the manifold M respectively. In consequence of equations (6), (7) and (8), we get

$$(\mathcal{L}_\xi g)(X,Y) = 2\alpha\{g(X,Y) + \eta(X)\eta(Y)\}. \tag{16}$$

The notion of η-parallelism on a Sasakian manifold was introduced by Kon [21]. A $(LCS)_3$-manifold is said to be η-parallel Ricci tensor if its non-vanishing Ricci tensor S satisfies the tensorial relation

$$(\nabla_X S)(\phi Y, \phi Z) = 0 \tag{17}$$

for all vector fields X, Y and Z on M.

On a 3-dimensional semi-Riemannian manifold, we have

$$R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X$$
$$- S(X,Z)Y - \frac{r}{2}\{g(Y,Z)X - g(X,Z)Y\}, \tag{18}$$

where Q denotes the endomorphism of the tangent space corresponding to Ricci tensor, *i.e.*, $S(X,Y) = g(QY,Z)$ and r is the scalar curvature of the manifold. Putting $Z = \xi$ in (18) and using (3), (10) and (13), we get

$$\eta(Y)QX - \eta(X)QY = \{\frac{r}{2} - (\alpha^2 - \rho)\}[\eta(Y)X - \eta(X)Y]. \tag{19}$$

Again putting $Y = \xi$ in (19) and using (8) and (13), we obtain

$$QX = \{\frac{r}{2} - (\alpha^2 - \rho)\}X + \{\frac{r}{2} - 3(\alpha^2 - \rho)\}\eta(X)\xi, \tag{20}$$

which is equivalent to

$$S(X,Y) = \{\frac{r}{2} - (\alpha^2 - \rho)\}g(X,Y) + \{\frac{r}{2} - 3(\alpha^2 - \rho)\}\eta(X)\eta(Y). \tag{21}$$

A Lorentzian concircular structure manifold is said to be a space form if it is a space of constant curvature. In consequence of (20) and (21), (18) reduces to the form

$$\begin{aligned} R(X,Y)Z = (\frac{r - 6(\alpha^2 - \rho)}{2})\{\eta(X)g(Y,Z)\xi - \eta(Y)g(X,Z)\xi + \eta(Y)\eta(Z)X \\ - \eta(X)\eta(Z)Y\} + (\frac{r - 4(\alpha^2 - \rho)}{2})\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned} \tag{22}$$

From (22), it is obvious that the manifold is of constant curvature $(\alpha^2 - \rho)$ if and only if the scalar curvature of the manifold is $6(\alpha^2 - \rho)$. Thus we have:

Corollary 2.1. *A LCS-manifold of dimension 3 is a space form if and only the scalar curvature of the manifold is $6(\alpha^2 - \rho)$.*

Remark 2.2. If we suppose that $\alpha = 1$, then the $(LCS)_3$-manifold reduces to LP-Sasakian manifold and therefore $r = 6$. This result has been proved by Shaikh and De in [36] and therefore the Corollary 2.1 is the generalization of the Lemma 1.1 (see [36], p. 361).

3. Ricci semisymmetric $(LCS)_3$-manifolds

In this section, we study the properties of 3-dimensional Ricci semisymmetric Lorentzian concircular structure manifolds. It is obvious that

$$(R(X,Y).S)(Z,U) = -S(R(X,Y)Z,U) - S(Z,R(X,Y)U). \tag{23}$$

Let us consider $R(X,Y).S = 0$ and therefore equation (23) gives

$$S(R(X,Y)Z,U) + S(Z,R(X,Y)U) = 0$$

for all X, Y, Z, $U \in \chi(M)$. Replacing Y by ξ in the last expression and using (11), we obtain

$$(\alpha^2 - \rho)\{\eta(Z)S(X,U) - g(X,Z)S(\xi,U) + S(Z,X)g(U,\xi) - g(X,U)S(Z,\xi) = 0,$$

which shows that

$$\eta(Z)S(X,U) - g(X,Z)S(\xi,U) + S(Z,X)g(U,\xi) - g(X,U)S(Z,\xi) = 0$$

because $\alpha^2-\rho \neq 0$ (in general). In view (13), last expression assumes the form

(24) $\eta(Z)S(X,U)+S(Z,X)\eta(U)-2(\alpha^2-\rho)\{g(X,Z)\eta(U)+g(X,U)\eta(Z)\}=0.$

Let $\{e_i, i=1,2,3\}$ be an orthonormal basis of the tangent space at any point of the manifold M_3. Setting $X=U=e_i$ in (24) and taking summation over i, $1 \leq i \leq 3$, we get

$$r = 6(\alpha^2-\rho). \tag{25}$$

Thus with the help of Corollary 2.1 and equation (25), we can easily see that the Ricci semisymmetric $(LCS)_3$-manifold is a space form. Conversely, let us suppose that the $(LCS)_3$-manifold is a space form with $r=6(\alpha^2-\rho)$, *i.e.*,

$$R(X,Y)Z = (\alpha^2-\rho)\{g(Y,Z)X - g(X,Z)Y\}. \tag{26}$$

In consequence of (23) and (26), we find that $R(X,Y).S=0$. Hence we can state the following:

Theorem 3.1. *A $(LCS)_3$-manifold is Ricci semisymmetric if and only if it is a space form.*

Also putting $Z=\xi$ in (24) and then using (8) and (13) we obtain

$$S(X,U) = 2(\alpha^2-\rho)g(X,U), \tag{27}$$

provided $(\alpha^2-\rho) \neq 0$. From (27), it is obvious that the manifold M is an Einstein manifold with scalar curvature $r=6(\alpha^2-\rho)$. In view of above discussions and Theorem 3.1, we can state:

Corollary 3.2. *On a 3-dimensional LCS-manifold, the following conditions are equivalent*:
(i) *M is semisymmetric ($R.R=0$),*
(ii) *M is locally symmetric,*
(iii) *M is of constant curvature,*
(iv) *M is Einstein.*

In the light of (16) and (27), (1) turns into the form

$$\{2\alpha + 2(\alpha^2-\rho) - \lambda\}g(X,Y) + 2\alpha\eta(X)\eta(Y) = 0. \tag{28}$$

Setting $X=Y=e_i$ in (28) for $1 \leq i \leq 3$, where e_i denotes the orthonornal basis of the tangent space at each point of the manifold M_3, and taking summation over i, we get

$$\frac{\lambda}{2} = (\alpha+\frac{1}{3})^2 - (\rho+\frac{1}{9}).$$

If $\frac{1}{2}L_\xi g + S$ is parallel on a Ricci semisymmetric $(LCS)_3$-manifold, then the Ricci soliton (g,ξ,λ) will be shrinking, expanding and steady accordingly $(\alpha+\frac{1}{3})^2 >, <$ and $= \rho+\frac{1}{9}$. Thus we have

Corollary 3.3. *A Ricci soliton (g,ξ,λ) on a Ricci semisymmetric $(LCS)_3$-manifold together with parallel tensor $\frac{1}{2}L_\xi g + S$ is expanding, shrinking and steady if $(\alpha+\frac{1}{3})^2 <, >$ and $= (\rho+\frac{1}{9})$ respectively.*

4. Locally ϕ-symmetric $(LCS)_3$-manifolds

This section deals with the study of locally ϕ-symmetric $(LCS)_3$-manifolds. The covariant derivative of (22) with respect to the Levi-Civita connection ∇ along the vector field W gives

$$
\begin{aligned}
&(\nabla_W R)(X,Y)Z \\
&= \frac{1}{2}\{dr(W) - 4(2\alpha d\alpha(W) - d\rho(W))\}\{g(Y,Z)X - g(X,Z)Y\} \\
&+ \frac{1}{2}\{dr(W) - 6(2\alpha d\alpha(W) - d\rho(W))\}\{g(Y,Z)\eta(X)\xi \\
&- g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\} \\
&+ \frac{\alpha}{2}\{r - 6(\alpha^2 - \rho)\}[g(Y,Z)g(\phi W,\phi X)\xi + \eta(X)g(Y,Z)W \\
&+ \eta(X)\eta(W)g(Y,Z)\xi - g(X,Z)g(\phi W,\phi Y)\xi - g(X,Z)\eta(Y)W \\
&- g(X,Z)\eta(Y)\eta(W)\xi + g(\phi W,\phi Y)\eta(Z)X + \eta(Y)g(\phi W,\phi Z)X \\
&- g(\phi W,\phi X)\eta(Z)Y - g(\phi W,\phi Z)\eta(X)Y].
\end{aligned} \tag{29}
$$

Operating ϕ^2 on both sides of (29) and consider that the vector fields W, X, Y and Z are orthogonal to ξ, we have

$$
\begin{aligned}
&\phi^2(\nabla_W R)(X,Y)Z \\
&= \frac{1}{2}\{dr(W) - 4(2\alpha d\alpha(W) - d\rho(W))\}\{g(Y,Z)X - g(X,Z)Y\}.
\end{aligned} \tag{30}
$$

By considering the fact $\nabla_X \alpha = d\alpha(X) = \rho\eta(X)$ and $d\rho(X) = \beta\eta(X)$, equation (30) reduces to the form

$$
\begin{aligned}
&\phi^2(\nabla_W R)(X,Y)Z \\
&= \frac{1}{2}\{dr(W) - 4(2\alpha\rho - \beta))\eta(W)\}\{g(Y,Z)X - g(X,Z)Y\}.
\end{aligned} \tag{31}
$$

If we suppose that α is a non zero constant, then with the help of (5) and (15), (31) becomes

$$\phi^2(\nabla_W R)(X,Y)Z = \frac{1}{2}dr(W)\{g(Y,Z)X - g(X,Z)Y\}. \tag{32}$$

This leads to the following:

Theorem 4.1. *A 3-dimensional LCS-manifold is locally ϕ-symmetric if and only if the scalar curvature of the manifold is constant.*

From Theorems 3.1 and 4.1, we conclude the following:

Corollary 4.2. *Every Ricci semisymmetric $(LCS)_3$-manifold is locally symmetric.*

5. η-parallel Ricci tensor on $(LCS)_3$-manifolds

From (8) and (21), it is obvious that

$$S(\phi X,\phi Y)=\frac{1}{2}\left(r-2(\alpha^2-\rho)\right)\{g(X,Y)+\eta(X)\eta(Y)\}. \tag{33}$$

The covariant differentiation of (33) along the vector field Z gives

$$\begin{aligned}(\nabla_Z S)(\phi X,\phi Y)=&\frac{\alpha}{2}[r-2(\alpha^2-\rho)]\{\eta(X)g(\phi Y,Z)+\eta(Y)g(\phi X,Z)\\&-\eta(X)S(\phi Y,Z)-\eta(Y)S(\phi X,Z)\}\\&+\frac{1}{2}\{dr(Z)-2(2\alpha d\alpha(Z)-d\rho(Z))\}g(\phi X,\phi Y).\end{aligned} \tag{34}$$

From (17) and (34), we get

$$\begin{aligned}&\alpha[r-2(\alpha^2-\rho)]\{\eta(X)g(\phi Y,Z)+\eta(Y)g(\phi X,Z)-\eta(X)S(\phi Y,Z)\\&-\eta(Y)S(\phi X,Z)\}+\{dr(Z)-2(2\alpha d\alpha(Z)-d\rho(Z))\}g(\phi X,\phi Y)=0.\end{aligned} \tag{35}$$

Setting $Y=X=e_i$ in (35) and taking summation over i, $1\leq i\leq 3$, where $\{e_i, i=1,2,3\}$ be the orthonormal basis of the tangent space at each point of the manifold M_3, we find that

$$dr(Z)=2(2\alpha\rho-\beta)\eta(Z). \tag{36}$$

If we suppose that α is a non-zero constant, then equation (36) becomes

$$dr(Z)=0\Rightarrow r=constant. \tag{37}$$

Thus with the above discussion, we state:

Theorem 5.1. *If the Ricci tensor on a $(LCS)_3$-manifold is η-parallel, then the scalar curvature is constant, provided α is a non-zero constant.*

In consequence of Theorems 4.1 and 5.1, we conclude:

Corollary 5.2. *A 3-dimensional LCS-manifold with η-parallel Ricci tensor is locally ϕ-symmetric, provided α is constant.*

Differentiating (21) covariantly along the vector field Z, we immediately obtain

$$\begin{aligned}(\nabla_Z S)(X,Y)=&\frac{\alpha}{2}[r-6(\alpha^2-\rho)]\{\eta(Y)g(\phi Z,\phi X)+\eta(X)g(\phi Z,\phi Y)\}\\&+\frac{1}{2}\{dr(Z)-2(2\alpha d\alpha(Z)-d\rho(Z))\}g(X,Y)\\&+\frac{1}{2}\{dr(Z)-6(2\alpha d\alpha(Z)-d\rho(Z))\}\eta(X)\eta(Y).\end{aligned} \tag{38}$$

If we presume that α is a non-zero constant, then by the equations (37) and (38), we can find

$$\begin{aligned}&(\nabla_Z S)(X,Y)\\&=\frac{\alpha}{2}[r-6(\alpha^2-\rho)]\{\eta(Y)g(Z,X)+\eta(X)g(Z,Y)+2\eta(X)\eta(Y)\eta(Z)\}.\end{aligned} \tag{39}$$

The cyclic sum of (39) gives

$$
\begin{aligned}
(40)\qquad & (\nabla_Z S)(X,Y) + (\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) \\
&= \alpha[r - 6(\alpha^2 - \rho)]\{\eta(Y)g(Z,X) + \eta(X)g(Z,Y) + \eta(Z)g(X,Y) \\
&\quad + 3\eta(X)\eta(Y)\eta(Z)\}.
\end{aligned}
$$

We assume that the manifold M_3 is cyclic Ricci parallel, *i.e.* $(\nabla_Z S)(X,Y) + (\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) = 0$, then (40) becomes

$$
\begin{aligned}
&\alpha[r - 6(\alpha^2 - \rho)]\{\eta(Y)g(Z,X) + \eta(X)g(Z,Y) \\
&\quad + \eta(Z)g(X,Y) + 3\eta(X)\eta(Y)\eta(Z)\} = 0.
\end{aligned}
$$

Let $\{e_i, i = 1,2,3\}$ denotes the set of orthonormal vector fields at each point of the manifold M_3. Setting $X = Y = e_i$ in above equation and taking summation over i, $1 \le i \le 3$ gives

$$\alpha\{r - 6(\alpha^2 - \rho)\}\eta(Z) = 0,$$

which reflects that

$$r = 6(\alpha^2 - \rho).$$

Thus Corollary 2.1 and last expression state the following statement as:

Theorem 5.3. *If the Ricci tensor on a $(LCS)_3$-manifold is η-parallel as well as cyclic parallel, then it is a space form.*

Also we consider that the Ricci tensor on M_3 is of Codazzi type, *i.e.*,

$$(\nabla_Z S)(X,Y) = (\nabla_X S)(Z,Y),$$

then from (39), we have

$$\frac{\alpha}{2}[r - 6(\alpha^2 - \rho)]\{\eta(X)g(Z,Y) - \eta(Z)g(X,Y)\} = 0, \tag{41}$$

which shows that $r = 6(\alpha^2 - \rho), (\alpha \neq 0)$. Thus we state:

Corollary 5.4. *If the Ricci tensor of a 3-dimensional LCS-manifold is of Codazzi type and η-parallel, then the manifold is a space form.*

6. Second order parallel symmetric tensor

A symmetric tensor δ of type $(0,2)$ is called parallel with respect to the Levi-Civita connection ∇ if $\nabla\delta = 0$. By considering $\nabla\delta = 0$ and the Ricci identity $\nabla^2_{X,Y}\delta(Z,U) - \nabla^2_{X,Y}\delta(U,Z) = 0$, we get

$$\delta(R(X,Y)Z,U) + \delta(Z,R(X,Y)U) = 0$$

for arbitrary vector fields X, Y, Z and W on (M_3, g). Setting $U = X = \xi$ in above equation and using (3), (8) and (11), we find that

$$-\eta(Z)\delta(Y,\xi) + g(Y,Z)\delta(\xi,\xi) + \delta(Y,Z) + \eta(Y)\delta(\xi,Z) = 0,$$

provided $\alpha^2 - \rho \neq 0$. Also replacing Z by ξ in above equation and then using equations (3) and (8), we obtain

$$\delta(Y, \xi) = -g(Y, \xi)\delta(\xi, \xi). \tag{42}$$

Covariant differentiation of (42) along the vector field X reveals that

$$\delta(Y, \nabla_X \xi) = -g(Y, \nabla_X \xi)\delta(\xi, \xi) - 2g(Y, \xi)\delta(\nabla_X \xi, \xi). \tag{43}$$

Replacing the vector field Y with $\nabla_X Y$ in (42), we get

$$g(\nabla_X Y, \xi)\delta(\xi, \xi) + \delta(\nabla_X Y, \xi) = 0. \tag{44}$$

In consequence of equations (43) and (44), we find that

$$\delta(Y, \nabla_X \xi) = \{2g(Y, \xi)g(\nabla_X \xi, \xi) - g(Y, \nabla_X \xi)\}\delta(\xi, \xi). \tag{45}$$

With the help of equations (2), (6), (7), (42) and (45), we can find that

$$\delta(X, Y) = -g(X, Y)\delta(\xi, \xi). \tag{46}$$

The covariant differentiation of (46) with respect to the Levi-Civita connection ∇ along any arbitrary vector field on (M_3, g) together with (3), (6) and (8) reveals that $\delta(\xi, \xi)$ is constant. Thus the equation (46) implies that the second order symmetric parallel tensor with respect to the connection ∇ in a regular 3-dimensional LCS-manifold (M_3, g) is a constant multiple of metric tensor g. Thus we have the following:

Theorem 6.1. *A Lorentzian metric on a 3-dimensional regular LCS-manifold M_3 is irreducible. In other words, the tangent bundle of M_3 does not admit a decomposition $TM = E_1 \times E_2$ parallel with respect to the connection ∇ of g.*

Let us suppose that the $(LCS)_3$-manifold M_3 is Ricci-symmetric, *i.e.*, $\nabla S = 0$ and therefore from equation (46), we get

$$S(X, Y) = -S(\xi, \xi)g(X, Y), \tag{47}$$

where

$$S(\xi, \xi) = -2(\alpha^2 - \rho).$$

Thus it is clear from equation (47) that the Ricci tensor is a constant multiple of the metric tensor g and hence the manifold is Einstein. Let $\{e_i, i = 1, 2, 3\}$ be an orthonormal basis of the tangent space at each point of the manifold M_3. Setting $X = Y = e_i$ in (47) and then summing over i, $1 \leq i \leq 3$, we find that

$$r = 6(\alpha^2 - \rho). \tag{48}$$

Thus with the help of (39), (47), (48), Corollary 2.1 and Theorem 6.1, we state the following corollaries:

Corollary 6.2. *Every Ricci symmetric $(LCS)_3$-manifold is an Einstein manifold.*

Corollary 6.3. *A $(LCS)_3$-manifold is Ricci symmetric if and only if it is space form.*

Let us consider that $L_V g$ is parallel and a regular $(LCS)_3$-manifold is Ricci symmetric, where $L_V g$ denotes the Lie derivative of g along the vector field V. Here we have two situations regarding the vector field V: the first is that $V \in Span(\xi)$ and second $V \perp \xi$. From the analysis point of view, second situation becomes complex and therefore we are going to consider the first case, *i.e.*, $V = \xi$. We consider $\delta_1(X,Y) = (\frac{1}{2}L_\xi g + S)(X,Y)$, then from equations (2), (8), (13), (16) and (46), we find

$$\delta_1(X,Y) = 2(\alpha^2 - \rho)g(X,Y). \tag{49}$$

From (1) and (49), we observe that $\lambda = 2(\alpha^2 - \rho)(\neq 0)$. Thus the Ricci soliton (g, ξ, λ) on a regular Ricci symmetric $(LCS)_3$-manifold with parallel tensor $\frac{1}{2}L_\xi g$ is expanding and shrinking accordingly $\alpha^2 - \rho <$ and > 0 respectively. Hence we can state the following:

Corollary 6.4. *If on a regular Ricci symmetric $(LCS)_3$-manifold $\frac{1}{2}L_\xi g$ is parallel, then the Ricci soliton (g, ξ, λ) on M_3 is expanding and shrinking if α^2 $<$ and $> \rho$ respectively.*

Definition. A vector field $X \in \chi(M_3)$ on a semi-Riemannian manifold is said to be affine Killing vector field if $\nabla L_X g = 0$.

With the help of equation (46) and above definition, we have

$$(L_X g)(Y,Z) = cg(Y,Z),$$

where $c = 2g(L_X\xi, \xi)$. With the help of equation (2), (5), (13) and (15), we can easily calculate that $(L_X Q)(\xi) = 0$ and hence $(L_X S)(\xi, \xi) = 0$, provided α is a non-zero constant. Also, $(L_X S)(\xi,\xi) = -2S(L_X\xi,\xi) = -4(\alpha^2 - \rho)g(L_X\xi,\xi) = 0$ and thus $g(L_X\xi,\xi) = 0$. It is obvious that $(L_X g)(\xi,\xi) = -2g(L_X\xi,\xi) = 0$ and therefore $(L_X g)(Y,Z) = 0$. This shows that the vector field X is a Killing vector field. Thus we have the following result:

Corollary 6.5. *An affine Killing vector field on a regular $(LCS)_3$-manifold is Killing.*

In [5], Chaki defined and studied pseudo Ricci-symmetric manifold $(PRS)_n$. A Riemannian manifold (M_n, g) is said to be pseudo Ricci-symmetric manifold if $S \neq 0$ and satisfies

$$(\nabla_X S)(Y,Z) = 2A(X)S(Y,Z) + A(Y)S(X,Z) + A(Z)S(X,Y),$$

for arbitrary vector fields X, Y and Z, where A is a non-zero 1-form defined as $A(\cdot) = g(\cdot, P)$ for associated vector field P [5]. We define:

Definition. A $(LCS)_3$-manifold is called pseudo Ricci-symmetric manifold (briefly $(PRS)_3$-manifold) if $S \neq 0$ and satisfies

$$(\nabla_X S)(Y,Z) = 2A(X)S(Y,Z) + A(Y)S(X,Z) + A(Z)S(X,Y) \tag{50}$$

for arbitrary vector fields X, Y and Z.

It may be noted that the same notion was studied by Khan et al. [43] in the name of special Weakly Ricci-symmetric manifold. Replacing Z by ξ in (50) and using (8) and (13), we have

$$(\nabla_X S)(Y,\xi) = 2(\alpha^2-\rho)\{2A(X)\eta(Y)+A(Y)\eta(X)\}+A(\xi)S(X,Y). \tag{51}$$

It is obvious that

$$(\nabla_X S)(Y,\xi) = \nabla_X S(Y,\xi) - S(\nabla_X Y,\xi) - S(Y,\nabla_X \xi),$$

which becomes

$$(\nabla_X S)(Y,\xi) = 2[(2\alpha\rho-\beta)\eta(X)\eta(Y)+\alpha(\alpha^2-\rho)g(X,Y)]-\alpha S(X,Y), \tag{52}$$

by considering the equations (3), (4), (8) and (13). In view of (51) and (52), we get

$$\begin{aligned} S(X,Y) = {} & \frac{2}{\alpha+A(\xi)}\{(2\alpha\rho-\beta)\eta(X)\eta(Y)+\alpha(\alpha^2-\rho)g(X,Y) \\ & -(\alpha^2-\rho)[2A(X)\eta(Y)+A(Y)\eta(X)]\}, \end{aligned} \tag{53}$$

provided $\alpha + A(\xi) \neq 0$. Let us define

$$\delta_1(X,Y) = \frac{1}{2}(L_\xi g)(X,Y)+S(X,Y).$$

By virtue of (16), (53) and above definition, we acquire

$$\begin{aligned} \delta_1(X,Y) = {} & \frac{2}{\alpha+A(\xi)}\{(2\alpha\rho-\beta)\eta(X)\eta(Y)+\alpha(\alpha^2-\rho)g(X,Y) \\ & -(\alpha^2-\rho)[2A(X)\eta(Y)+A(Y)\eta(X)]\}+\alpha g(\phi X,\phi Y). \end{aligned} \tag{54}$$

Changing X and Y with ξ in (54), we procure

$$\delta_1(\xi,\xi) = \frac{2}{\alpha+A(\xi)}[-\alpha(\alpha^2-3\rho)+3(\alpha^2-\rho)A(\xi)-\beta]. \tag{55}$$

From equations (1) and (55), it is clear that

$$\lambda = -\delta_1(\xi,\xi),$$

and hence we can state:

Theorem 6.6. *If $\alpha^2-\rho \neq 0$ and $\frac{1}{2}L_\xi g + S$ is parallel on a pseudo Ricci symmetric $(LCS)_3$-manifold, then the Ricci soliton (g,ξ,λ) is shrinking, expanding and steady if $\frac{\beta+\alpha(\alpha^2-3\rho)-3(\alpha^2-\rho)A(\xi)}{\alpha+A(\xi)}$ $>$, $<$ and $=0$ respectively.*

In particular, if λ is a non-vanishing constant, then from (1), (5), (15), (55) and Theorem 6.6, we find that

$$\lambda = \frac{2}{\alpha+A(\xi)}\alpha^2(\alpha-3A(\xi)).$$

It is evident from the above expression that the Ricci soliton (g,ξ,λ) is shrinking, expanding and steady accordingly $\frac{4A(\xi)}{\alpha+A(\xi)}$ $<, >$ and $=1$ respectively. Thus we conclude the following:

Corollary 6.7. *If α is non-vanishing constant and $\frac{1}{2}L_\xi g + S$ is parallel on a pseudo Ricci symmetric $(LCS)_3$-manifold, then the Ricci soliton (g, ξ, λ) is expanding, shrinking and steady accordingly $\frac{4A(\xi)}{\alpha+A(\xi)} >, <$ and $= 1$ respectively.*

7. Second order parallel skew-symmetric tensor

In this section, we study the properties of second order parallel skew-symmetric tensor with respect to a Levi-Civita connection ∇ in a regular 3-dimensional LCS-manifold. Let us suppose that δ is a second order skew symmetric parallel tensor, *i.e.*, $\delta(X,Y) = -\delta(Y,X)$ and $\nabla\delta = 0$. By considering $\nabla\delta = 0$ and the Ricci identity $\nabla^2_{W,X}\delta(Y,Z) - \nabla^2_{W,X}\delta(Z,Y) = 0$, we get

$$\delta(R(W,X)Y,Z) + \delta(Y,R(W,X)Z) = 0$$

for arbitrary vector fields X, Y, Z and W on (M_3, g). Setting $W = Y = \xi$ in above equation and using (3), (8) and (11), we obtain

$$\delta(X,Z) = \eta(Z)\delta(\xi,X) - \eta(X)\delta(\xi,Z). \tag{56}$$

Let A_1 be an $(1,1)$ tensor field which is metrically equivalent to δ, *i.e.*,

$$\delta(X,Y) = g(A_1X,Y). \tag{57}$$

From equations (56) and (57), we conclude that

$$A_1X = -\eta(X)A_1\xi + g(A_1\xi,X)\xi. \tag{58}$$

Since δ is parallel and therefore A_1 is parallel and thus

$$\nabla_X(A_1\xi) = \alpha g(A_1\xi,X)\xi. \tag{59}$$

In view of (2) and (58), we have

$$g(A_1X,\xi) = -\eta(X)g(A_1\xi,\xi) - g(A_1\xi,X). \tag{60}$$

Putting $X = \xi$ in (60), we obtain

$$g(A_1\xi,\xi) = 0.$$

It is obvious from the above discussion that

$$g(\nabla_X(A_1\xi), A_1\xi) = 0,$$

which reflects that $||A_1\xi|| = constant$ on M_3. It is also reveals from above equations that

$$A_1^2\xi = -||A_1\xi||^2\xi. \tag{61}$$

Differentiating (61) covariantly with respect to the Levi-Civita connection ∇ along the vector field X and then using the equations (6), (7) and (8), we find that

$$A_1^2X = -||A_1\xi||^2X. \tag{62}$$

Now if $||A_1\xi||^2 \neq 0$, then $J = \frac{1}{||A_1\xi||}A_1$ is an almost complex structure on M. Indeed, (J,g) is a Kähler structure on M. Thus the fundamental 2-form is $g(JX,Y) = \lambda\delta(X,Y)$ with $\lambda = \frac{1}{||A_1\xi||}$ = constant. But δ satisfies the relation

(56) and thus it is degenerate, which is a contradiction. Therefore $||A_1\xi|| = 0$ and hence $\delta = 0$ on M_3. Thus we state:

Theorem 7.1. *There doesn't exist a second order skew-symmetric parallel tensor field δ on a regular $(LCS)_3$-manifold.*

8. Three dimensional LCS-manifolds admitting a non-null concircular vector field

A non vanishing vector field V on a 3-dimensional LCS-manifold M_3 is said to be a concircular vector field if

$$\nabla_X V = \sigma X, \quad \forall X \in \chi(M_3) \tag{63}$$

and σ is a scalar function. In particular, if $\sigma = 0$, then the vector field V is parallel. It is obvious from (63) that

$$\nabla_Y \nabla_X V = \sigma \nabla_Y X + (\nabla_Y \sigma)X,$$

which gives

$$'R(X, Y, V, Z) = d\sigma(X)g(Y, Z) - d\sigma(Y)g(X, Z), \tag{64}$$

where $'R(X, Y, V, Z) = g(R(X, Y)V, Z)$. Putting $Z = \xi$ in (64) and using (3) and (9), we have

$$(\alpha^2 - \rho)\{\eta(X)g(Y, V) - \eta(Y)g(X, V)\} = \eta(Y)d\sigma(X) - \eta(X)d\sigma(Y). \tag{65}$$

Setting $X = \phi X$ and $Y = \xi$ in (65), we find

$$d\sigma(X) + \eta(X)d\sigma(\xi) = -(\alpha^2 - \rho)\{g(X, V) + \eta(X)g(\xi, V)\}. \tag{66}$$

If we suppose that $g(X, V) = 0$, then $g(V, V) = 0$ and therefore $||V||^2 = 0$. This shows that V is a null vector field which contradicts our supposition. Thus $g(X, V) \neq 0$ and therefore equation (64) yields

$$d\sigma(X)g(Y, V) = d\sigma(Y)g(X, V). \tag{67}$$

Replacing the vector field Y with ξ in (67), we have

$$d\sigma(X)\eta(V) = d\sigma(\xi)g(X, V). \tag{68}$$

Since the vector field X is not orthogonal to the structure vector field ξ, (in general), therefore (68) takes the form

$$d\sigma(X)\eta(V)\eta(X) = d\sigma(\xi)g(X, V)\eta(X). \tag{69}$$

In consequence of (66), (67) and (69), we obtain

$$\{g(X, V) + \eta(X)\eta(V)\}\{d\sigma(X) + (\alpha^2 - \rho)g(X, V)\} = 0, \tag{70}$$

which reveals that either

$$g(X, V) + \eta(X)\eta(V) = 0, \tag{i}$$

or,

$$d\sigma(X) + (\alpha^2 - \rho)g(X, V) = 0. \tag{ii}$$

If we suppose that $g(X,V)+\eta(X)\eta(V)\neq 0$, then with the help of (64) and (ii), we conclude that

$$'R(X,Y,V,Z)=(\alpha^2-\rho)\{g(X,Z)g(Y,V)-g(Y,Z)g(X,V)\}. \tag{71}$$

By setting $X=Z=e_i$ in (71), where $\{e_i, i=1,2,3\}$ denotes the set of orthonormal vector field at each point of the manifold, and taking summation over i, $1\leq i\leq 3$, we conclude

$$S(Y,V)=2(\alpha^2-\rho)g(Y,V), \forall Y\in\chi(M_3). \tag{72}$$

With the help of equations (21) and (72), we observe that

$$\{r-6(\alpha^2-\rho)\}[g(Y,V)+\eta(Y)\eta(V)]=0,$$

which shows that $r=6(\alpha^2-\rho)$. On the other hand, if we suppose that $g(X,V)+\eta(X)\eta(V)=0$ and $d\sigma(X)+(\alpha^2-\rho)g(X,V)\neq 0$. Taking covariant derivative of (i) along the vector field Y, we get

$$(\nabla_Y\eta)(X)\eta(V)+(\nabla_Y\eta)(V)\eta(X)=0. \tag{73}$$

In view of (4), (73) gives us only one possibility that $V\perp\xi$. Thus we state:

Theorem 8.1. *A 3-dimensional LCS-manifold equipped with a non null concircular vector field which is not orthogonal to ξ is a space form.*

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