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REGULARITIES OF MULTIFRACTAL HEWITT-STROMBERG MEASURES

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Abstract. We construct new metric outer measures (multifractal analogues of the Hewitt-Stromberg measure) $H_\mu^{q,t}$ and $P_\mu^{q,t}$ lying between the multifractal Hausdorff measure $\mathcal{H}_\mu^{q,t}$ and the multifractal packing measure $\mathcal{P}_\mu^{q,t}$. We set up a necessary and sufficient condition for which multifractal Hausdorff and packing measures are equivalent to the new ones. Also, we focus our study on some regularities for these given measures. In particular, we try to formulate a new version of Olsen's density theorem when μ satisfies the doubling condition. As an application, we extend the density theorem given in [3].

1. Introduction

Hewitt-Stromberg measures were introduced by Hewitt and Stromberg in [11, Exercise (10.51)]. Since then, they have been investigated by several authors, highlighting their importance in the study of local properties of fractals and products of fractals. One can cite, for example [9, 10, 12, 29]. In particular, Edgar's textbook [6, pp. 32–36] provides an excellent and systematic introduction to these measures, which also appears explicitly, for example, in Pesin's monograph [22, 5.3] and implicitly in Mattila's text [16]. The purpose of this paper is to define and study a class of natural multifractal generalizations of the Hewitt-Stromberg measures.

Let X be a metric space, $E \subseteq X$ and $t > 0$. The Hausdorff measure is defined, for $\varepsilon > 0$, as follows

$$\mathcal{H}_\varepsilon^t(E) = \inf\left\{\sum_i \Big(\operatorname{diam}(E_i)\Big)^t \ \Big|\ \ E \subseteq \bigcup_i E_i,\ \ \operatorname{diam}(E_i) < \varepsilon\right\}.$$

This allows to define the t-dimensional Hausdorff measure $\mathcal{H}^t(E)$ of E by

$$\mathcal{H}^t(E) = \sup_{\varepsilon>0} \mathcal{H}_\varepsilon^t(E).$$

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The packing measure is defined, for $\varepsilon > 0$, as follows

$$\overline{\mathcal{P}}^t_\varepsilon(E) = \sup\left\{\sum_i \Big(2r_i\Big)^t\right\},$$

where the supremum is taken over all closed balls $(C(x_i, r_i))_i$ such that $r_i \le \varepsilon$ and with $x_i \in E$ and $d(x_i, x_j) \ge \frac{r_i + r_j}{2}$ for $i \neq j$. The t-dimensional packing pre-measure $\overline{\mathcal{P}}^t(E)$ of E is now defined by

$$\overline{\mathcal{P}}^t(E) = \sup_{\varepsilon > 0} \overline{\mathcal{P}}^t_\varepsilon(E).$$

This makes us able to define the t-dimensional packing measure $\mathcal{P}^t(E)$ of E as

$$\mathcal{P}^t(E) = \inf\left\{\sum_i \overline{\mathcal{P}}^t(E_i) \;\;|\;\; E \subseteq \bigcup_i E_i\right\}.$$

While Hausdorff and packing measures are defined using coverings and packings by families of sets with diameters less than a given positive number ε, say, the Hewitt-Stromberg measures are defined using packings of balls with the same diameter ε. For $t > 0$, the Hewitt-Stromberg pre-measures are defined as follows:

$$\overline{\mathcal{U}}^t(E) = \liminf_{r \to 0} M_r(E)\;(2r)^t$$

and

$$\overline{\mathcal{V}}^t(E) = \limsup_{r \to 0} M_r(E)\;(2r)^t,$$

where the packing number $M_r(E)$ of E is given by

$$M_r(E) = \sup\Big\{\sharp\{I\} \;\;|\;\; (C(x_i, r_i))_{i \in I} \text{ is a family of closed balls with } x_i \in E \text{ and } d(x_i, x_j) \ge r \text{ for } i \neq j\Big\}.$$

Now, we define the lower and upper t-dimensional Hewitt-Stromberg measures, which we denote respectively by $\mathcal{U}^t(E)$ and $\mathcal{V}^t(E)$, as follows:

$$\mathcal{U}^t(E) = \inf\left\{\sum_i \overline{\mathcal{U}}^t(E_i) \;\;|\;\; E \subseteq \bigcup_i E_i\right\}$$

and

$$\mathcal{V}^t(E) = \inf\left\{\sum_i \overline{\mathcal{V}}^t(E_i) \;\;|\;\; E \subseteq \bigcup_i E_i\right\}.$$

We recall the basic inequalities satisfied by the Hewitt-Stromberg, the Hausdorff and the packing measure (see [12, Proposition 2.1])

$$\overline{\mathcal{U}}^t(E) \le \overline{\mathcal{V}}^t(E) \le \overline{\mathcal{P}}^t(E)$$

and

$$\mathcal{H}^t(E) \le \mathcal{U}^t(E) \le \mathcal{V}^t(E) \le \mathcal{P}^t(E).$$

Regular sets are defined by density with respect to the Hausdorff measure [5, 7, 8, 17–19], to packing measure [23, 27, 28] or to Hewitt-Stromberg measure [4, 13–15]. Tricot et al. [23, 28] managed to show that a subset of $\mathbb{R}^n$ has an integer Hausdorff and packing dimension if it is strongly regular. Then, the results of [23] were improved to a generalized ϕ-Hausdorff measure in a Polish space by Mattila and Mauldin in [18]. Later, Baek [3] used the multifractal density theorems [20,21] to prove the decomposition theorem for the regularities of a generalized centered Hausdorff measure $\mathcal{H}_\mu^{q,t}$ and a generalized packing measure $\mathcal{P}_\mu^{q,t}$ in an Euclidean space which enables him to split a set into regular and irregular parts. In addition, he extended the Olsen's density theorem to any measurable set. Moreover, sets' regularities were also studied with respect to these measures, see for example [1, 2, 24–26].

In this paper, we set up a multifractal analogues of the Hewitt-Stromberg measure $H_\mu^{q,t}$ and $P_\mu^{q,t}$ lying between the multifractal Hausdorff measure $\mathcal{H}_\mu^{q,t}$ and the multifractal packing measure $\mathcal{P}_\mu^{q,t}$. We give a necessary and sufficient condition for which multifractal Hausdorff and packing measures are equivalent to the new ones. We also study some regularities with respect to $H_\mu^{q,t}$ and $P_\mu^{q,t}$. In addition, some density results are given. In particular, when μ satisfies the doubling condition, we formulate a new version of Olsen's density theorem (Theorem 2.3). As an application, we extend the density theorem stated in [3, Theorem 3.5].

2. Statements of results

2.1. Multifractal Hausdorff measure and packing measure

We start by introducing the generalized centered Hausdorff measure $\mathcal{H}_\mu^{q,t}$ and the generalized packing measure $\mathcal{P}_\mu^{q,t}$. We fix an integer $n \geq 1$ and we denote by $\mathcal{P}(\mathbb{R}^n)$ the family of compactly supported Borel probability measures on $\mathbb{R}^n$. Let $\mu \in \mathcal{P}(\mathbb{R}^n)$, $q, t \in \mathbb{R}$, $E \subseteq \mathbb{R}^n$ and $\delta > 0$. We define the generalized packing pre-measure,

$$\overline{\mathcal{P}}_\mu^{q,t}(E) = \inf_{\delta>0} \sup \Big\{ \sum_i \mu\big(B(x_i, r_i)\big)^q (2r_i)^t;\ \big(B(x_i, r_i)\big)_i \text{ is a centered } \delta\text{-packing of } E \Big\}.$$

In a similar way, we define the generalized Hausdorff pre-measure,

$$\overline{\mathcal{H}}_\mu^{q,t}(E) = \sup_{\delta>0} \inf \Big\{ \sum_i \mu\big(B(x_i, r_i)\big)^q (2r_i)^t;\ \big(B(x_i, r_i)\big)_i \text{ is a centered } \delta\text{-covering of } E \Big\},$$

with the conventions $0^q = \infty$ for $q \leq 0$ and $0^q = 0$ for $q > 0$.

The function $\overline{\mathcal{H}}_\mu^{q,t}$ is σ-subadditive but not increasing and the function $\overline{\mathcal{P}}_\mu^{q,t}$ is increasing but not σ-subadditive. That is the reason for which Olsen introduced the following modifications of the generalized Hausdorff and packing measures $\mathcal{H}_\mu^{q,t}$ and $\mathcal{P}_\mu^{q,t}$:

$$\mathcal{H}_\mu^{q,t}(E) = \sup_{F\subseteq E} \overline{\mathcal{H}}_\mu^{q,t}(F) \quad \text{and} \quad \mathcal{P}_\mu^{q,t}(E) = \inf_{E\subseteq \bigcup_i E_i} \sum_i \overline{\mathcal{P}}_\mu^{q,t}(E_i).$$

The functions $\mathcal{H}_\mu^{q,t}$ and $\mathcal{P}_\mu^{q,t}$ are metric outer measures and thus measures on the Borel family of subsets of $\mathbb{R}^n$. In addition, there exists an integer $\xi \in \mathbb{N}$ such that $\mathcal{H}_\mu^{q,t} \le \xi \mathcal{P}_\mu^{q,t}$. The measure $\mathcal{H}_\mu^{q,t}$ is of course a multifractal generalization of the centered Hausdorff measure, whereas $\mathcal{P}_\mu^{q,t}$ is a multifractal generalization of the packing measure. In fact, it is easily seen that, for $t \ge 0$, one has $2^{-t}\mathcal{H}_\mu^{0,t} \le \mathcal{H}^t \le \mathcal{H}_\mu^{0,t}$ and $\mathcal{P}_\mu^{0,t} = \mathcal{P}^t$, where $\mathcal{H}^t$ and $\mathcal{P}^t$ denote respectively the t-dimensional Hausdorff and t-dimensional packing measures.

For $\mu \in \mathcal{P}(\mathbb{R}^n)$ and $a > 1$, we write

$$P_a(\mu) = \limsup_{r\searrow 0} \left(\sup_{x\in \operatorname{supp}\mu} \frac{\mu\big(B_x(ar)\big)}{\mu\big(B_x(r)\big)} \right).$$

We will now say that the measure μ satisfies the doubling condition if there exists $a > 1$ such that $P_a(\mu) < \infty$. It is easily seen that the exact value of the parameter a is unimportant: $P_a(\mu) < \infty$ for some $a > 1$ if and only if $P_a(\mu) < \infty$ for all $a > 1$. Also, we will write $\mathcal{P}_D(\mathbb{R}^n)$ for the family of Borel probability measures on $\mathbb{R}^n$ which satisfy the doubling condition. We can cite as classical examples of doubling measures, the self-similar measures and the self-conformal ones [20]. In particular, if $\mu \in \mathcal{P}_D(\mathbb{R}^n)$, then $\mathcal{H}_\mu^{q,t} \le \mathcal{P}_\mu^{q,t}$.

2.2. Multifractal Hewitt-Stromberg measures

In the following, we will set up, for $q, t \in \mathbb{R}$ and $\mu \in \mathcal{P}(\mathbb{R}^n)$, the multifractal Hewitt-Stromberg measures $H_\mu^{q,t}$ and $P_\mu^{q,t}$. For $E \subset \operatorname{supp}\mu$, the pre-measure of E is defined by

$$C_\mu^{q,t}(E) = \limsup_{r\to 0} M_{\mu,r}^q(E)(2r)^t,$$

where

$$M_{\mu,r}^q(E) = \sup\left\{ \sum_i \mu(B(x_i,r))^q;\;\; \big(B(x_i,r)\big)_i \text{ is a centered packing of } E \right\}.$$

It's clear that $C_\mu^{q,t}$ is increasing and $C_\mu^{q,t}(\emptyset) = 0$, however it's not σ-additive. For this, we introduce the $P_\mu^{q,t}$-measure defined by

$$P_\mu^{q,t}(E) = \inf\left\{ \sum_i C_\mu^{q,t}(E_i);\;\; E \subseteq \cup_i E_i \text{ and the } E_i\text{'s are bounded} \right\}.$$

In a similar way we define

$$L_\mu^{q,t}(E) = \liminf_{r\to 0} N_{\mu,r}^q(E)(2r)^t,$$

where

$$N^q_{\mu,r}(E) = \inf\left\{\sum_i \mu(B(x_i,r))^q;\ \ \Big(B(x_i,r)\Big)_i \text{ is a centered covering of } E\right\}.$$

Since $L^{q,t}_\mu$ is not increasing and not countably subadditive, one needs a standard modification to get an outer measure. Hence we modify the definition to

$$\overline{H}^{q,t}_\mu(E) = \inf\left\{\sum_i L^{q,t}_\mu(E_i);\ \ E \subseteq \cup_i E_i \text{ and the } E_i\text{'s are bounded}\right\}$$

and

$$H^{q,t}_\mu(E) = \sup_{F\subset E} \overline{H}^{q,t}_\mu(F).$$

The measure $H^{q,t}_\mu$ is of course a multifractal generalization of the lower t-dimensional Hewitt-Stromberg measure $\mathcal{U}^t$, whereas $P^{q,t}_\mu$ is a multifractal generalization of the upper t-dimensional Hewitt-Stromberg measures $\mathcal{V}^t$. In fact, it is easily seen that, for $t > 0$, one has

$$H^{0,t}_\mu = \mathcal{U}^t \quad \text{and} \quad P^{0,t}_\mu = \mathcal{V}^t.$$

2.3. Main results

Our first main result describes some of the basic properties of the multifractal Hewitt-Stromberg measures including the fact that $H^{q,t}_\mu$ and $P^{q,t}_\mu$ are Borel metric outer measures and summarises the basic inequalities satisfied by the multifractal Hewitt-Stromberg measures, the generalized Hausdorff measure and the generalized packing measure.

Theorem 2.1. *Let $q,t \in \mathbb{R}$, $\mu \in \mathcal{P}(\mathbb{R}^n)$ and $E \subseteq \mathbb{R}^n$. Then*

(1) *the set functions $H^{q,t}_\mu$ and $P^{q,t}_\mu$ are metric outer measures and thus they are measures on the Borel algebra.*

(2) *There exists an integer $\xi \in \mathbb{N}$, such that*

$$\mathcal{H}^{q,t}_\mu(E) \le H^{q,t}_\mu(E) \le \xi P^{q,t}_\mu(E) \le \xi \mathcal{P}^{q,t}_\mu(E).$$

(3) *When $q \le 0$ or $q > 0$ and $\mu \in \mathcal{P}_D(\mathbb{R}^n)$, we have*

$$\mathcal{H}^{q,t}_\mu(E) \le H^{q,t}_\mu(E) \le P^{q,t}_\mu(E) \le \mathcal{P}^{q,t}_\mu(E).$$

Given two locally finite Borel measures μ and ν on $\mathbb{R}^n$, $q,t \in \mathbb{R}$ and $x \in \operatorname{supp}\mu$, we define the upper and lower (q,t)-densities of ν at x with respect to μ by

$$\overline{d}^{q,t}_\mu(x,\nu) = \limsup_{r\to 0} \frac{\nu\big(B(x,r)\big)}{\mu\big(B(x,r)\big)^q (2r)^t} \text{ and } \underline{d}^{q,t}_\mu(x,\nu) = \liminf_{r\to 0} \frac{\nu\big(B(x,r)\big)}{\mu\big(B(x,r)\big)^q (2r)^t}.$$

We consider a Borel set E of $\mathbb{R}^n$ and we denote by $\mathcal{H}^{q,t}_{\mu}{}_{\llcorner E}$ (resp. $\mathcal{P}^{q,t}_{\mu}{}_{\llcorner E}$) the s-dimensional centered Hausdorff measure $\mathcal{H}^{q,s}_\mu$ (resp. packing measure $\mathcal{P}^{q,t}_\mu$)

restricted to E. We define

$$\begin{cases}\overline{\Lambda}_{\mu}^{q,t}(x,E)=\overline{d}_{\mu}^{q,t}\left(x,\mathcal{H}_{\mu}^{q,t}{}_{\llcorner E}\right)\\ \\ \underline{\Lambda}_{\mu}^{q,t}(x,E)=\underline{d}_{\mu}^{q,t}\left(x,\mathcal{H}_{\mu}^{q,t}{}_{\llcorner E}\right)\end{cases}\quad\text{and}\quad\begin{cases}\overline{\Delta}_{\mu}^{q,t}(x,E)=\overline{d}_{\mu}^{q,t}\left(x,\mathcal{P}_{\mu}^{q,t}{}_{\llcorner E}\right)\\ \\ \underline{\Delta}_{\mu}^{q,t}(x,E)=\underline{d}_{\mu}^{q,t}\left(x,\mathcal{P}_{\mu}^{q,t}{}_{\llcorner E}\right).\end{cases}$$

If $\overline{\Lambda}_{\mu}^{q,t}(x,E)=\underline{\Lambda}_{\mu}^{q,t}(x,E)$ $\big($resp. $\overline{\Delta}_{\mu}^{q,t}(x,E)=\underline{\Delta}_{\mu}^{q,t}(x,E)\big)$, we write $\Lambda_{\mu}^{q,t}(x,E)$ $\big($resp. $\Delta_{\mu}^{q,t}(x,E)\big)$ for the common value. Similarly, we define

$$\begin{cases}\overline{d}_{\mu}^{q,t}(x,E)=\overline{d}_{\mu}^{q,t}\left(x,H_{\mu}^{q,t}{}_{\llcorner E}\right)\\ \\ \underline{d}_{\mu}^{q,t}(x,E)=\underline{d}_{\mu}^{q,t}\left(x,H_{\mu}^{q,t}{}_{\llcorner E}\right)\end{cases}\quad\text{and}\quad\begin{cases}\overline{D}_{\mu}^{q,t}(x,E)=\overline{d}_{\mu}^{q,t}\left(x,P_{\mu}^{q,t}{}_{\llcorner E}\right)\\ \\ \underline{D}_{\mu}^{q,t}(x,E)=\underline{d}_{\mu}^{q,t}\left(x,P_{\mu}^{q,t}{}_{\llcorner E}\right).\end{cases}$$

If $\overline{d}_{\mu}^{q,t}(x,E)=\underline{d}_{\mu}^{q,t}(x,E)$ $\big($resp. $\overline{D}_{\mu}^{q,t}(x,E)=\underline{D}_{\mu}^{q,t}(x,E)\big)$, we write $d_{\mu}^{q,t}(x,E)$ $\big($resp. $D_{\mu}^{q,t}(x,E)\big)$ for the common value.

Definition. Let ν be a measure on $\mathcal{P}(\mathbb{R}^n)$ and E be a Borel subset of $\operatorname{supp}\mu$ such that $0<\nu(E)<\infty$. A point $x\in E$ is called a ν-regular point of E if $\overline{d}_{\mu}^{q,t}(x,\nu)=\underline{d}_{\mu}^{q,t}(x,\nu)=1$, otherwise we say that x is an irregular point of E. Then E is said to be ν-regular if ν-a.a. of its points are ν-regular and ν-irregular if ν-a.a. of its points are ν-irregular.

Consider the sets of $H_{\mu}^{q,t}$-regular points and $P_{\mu}^{q,t}$-regular points respectively of a set $E\subset\mathbb{R}^n$

$$F=\Big\{x\in E;\ \ \underline{d}_{\mu}^{q,t}(x,E)=1=\overline{d}_{\mu}^{q,t}(x,E)\Big\},$$
$$G=\Big\{x\in E;\ \ \underline{D}_{\mu}^{q,t}(x,E)=1=\overline{D}_{\mu}^{q,t}(x,E)\Big\}.$$

The next theorem is a multifractal analogues of the Hewitt-Stromberg measure version of one of the fundamental facts in geometric measure theory. This theorem says that the set of regular points with respect to $H_{\mu}^{q,t}$(resp. $P_{\mu}^{q,t}$) is regular and the set of irregular points with respect to $H_{\mu}^{q,t}$(resp. $P_{\mu}^{q,t}$) is irregular.

Theorem 2.2. *Suppose that $\mathcal{P}_{\mu}^{q,t}(E)<\infty$. Then*

(1) $\overline{d}_{\mu}^{q,t}(x,F)=1=\underline{d}_{\mu}^{q,t}(x,F)$ *for $H_{\mu}^{q,t}$-a.a. on F.*
(2) $H_{\mu}^{q,t}\left(\Big\{x\in E\setminus F;\ \ \underline{d}_{\mu}^{q,t}(x,E\setminus F)=1=\overline{d}_{\mu}^{q,t}(x,E\setminus F)\Big\}\right)=0.$
(3) $\overline{D}_{\mu}^{q,t}(x,G)=1=\underline{D}_{\mu}^{q,t}(x,G)$ *for $P_{\mu}^{q,t}$-a.a. on G.*
(4) $P_{\mu}^{q,t}\left(\Big\{x\in E\setminus G;\ \ \underline{D}_{\mu}^{q,t}(x,E\setminus G)=1=\overline{D}_{\mu}^{q,t}(x,E\setminus G)\Big\}\right)=0.$

Let E be a Borel subset of the support of μ, we say that E is strongly regular if $\mathcal{H}_{\mu}^{q,t}(E)=\mathcal{P}_{\mu}^{q,t}(E)\in(0,\infty)$. Now we study the strong regularity of E when μ satisfies the doubling condition. In particular, we try to formulate a new version of Olsen's density result [20, Corollary 2.16].

Theorem 2.3. *Let $\mu \in \mathcal{P}_D(\mathbb{R}^n)$ and E be a Borel subset of* $\operatorname{supp}\mu$ *such that $\mathcal{P}_\mu^{q,t}(E) < +\infty$. Then the following assertions are equivalent*:

(1) $\mathcal{H}_\mu^{q,t}(E) = \mathcal{P}_\mu^{q,t}(E)$.

(2) $\underline{d}_\mu^{q,t}(x, E) = 1 = \overline{d}_\mu^{q,t}(x, E)$ *for $\mathcal{P}_\mu^{q,t}$-a.a. on E.*

(3) $\underline{D}_\mu^{q,t}(x, E) = 1 = \overline{D}_\mu^{q,t}(x, E)$ *for $\mathcal{P}_\mu^{q,t}$-a.a. on E.*

(4) $\underline{\Lambda}_\mu^{q,t}(x, E) = 1 = \overline{\Lambda}_\mu^{q,t}(x, E)$ *for $\mathcal{P}_\mu^{q,t}$-a.a. on E.*

(5) $\underline{\Delta}_\mu^{q,t}(x, E) = 1 = \overline{\Delta}_\mu^{q,t}(x, E)$ *for $\mathcal{P}_\mu^{q,t}$-a.a. on E.*

3. Proof of the main results

3.1. Preliminary results

In this section we prove some multifractal density results which we will need in order to prove the main results, however, we believe that the density results below also have some interest in their own right.

Two Borel measures μ and ν are equivalent, and we write $\mu \sim \nu$, if for any Borel set E we have

$$\mu(E) = 0 \iff \nu(E) = 0.$$

It's clear that $\mathcal{H}_\mu^{q,t} \ll H_\mu^{q,t} \ll P_\mu^{q,t} \ll \mathcal{P}_\mu^{q,t}$. In particular, if $\mathcal{P}_\mu^{q,t}(E) = 0 \Rightarrow \mathcal{H}_\mu^{q,t}(E) = H_\mu^{q,t}(E) = P_\mu^{q,t}(E) = 0$. Then, it is worth investigating the different cases of equivalence between these measures. The density result was also proven with respect to multifractal Hausdorff measure and packing measure in [20,21]. More precisely, we have the following proposition,

Proposition 3.1. *Let $\mu \in \mathcal{P}(\mathbb{R}^n)$ and E be a Borel subset of* $\operatorname{supp}\mu$.

(1) *Assume that $\mathcal{H}_\mu^{q,t}(E) < \infty$. We have*

$$\frac{1}{\xi}\mathcal{H}_\mu^{q,t}(E) \inf_{x\in E} \overline{d}_\mu^{q,t}(x,\nu) \le \nu(E) \le \mathcal{H}_\mu^{q,t}(E) \sup_{x\in E} \overline{d}_\mu^{q,t}(x,\nu). \tag{3.1}$$

(2) *If $\mathcal{H}_\mu^{q,t}(E) < \infty$ and $\mu \in \mathcal{P}_D(\mathbb{R}^n)$, then*

$$\mathcal{H}_\mu^{q,t}(E) \inf_{x\in E} \overline{d}_\mu^{q,t}(x,\nu) \le \nu(E) \le \mathcal{H}_\mu^{q,t}(E) \sup_{x\in E} \overline{d}_\mu^{q,t}(x,\nu). \tag{3.2}$$

(3) *If $\mathcal{P}_\mu^{q,t}(E) < \infty$, then*

$$\mathcal{P}_\mu^{q,t}(E) \inf_{x\in E} \underline{d}_\mu^{q,t}(x,\nu) \le \nu(E) \le \mathcal{P}_\mu^{q,t}(E) \sup_{x\in E} \underline{d}_\mu^{q,t}(x,\nu). \tag{3.3}$$

As a consequence we have, the following corollary.

Corollary 3.2. *Let $\mu \in \mathcal{P}(\mathbb{R}^n)$ and E be a Borel subset of* $\operatorname{supp}\mu$. *If $\mathcal{P}_\mu^{q,t}(E) < \infty$, then*

$$\frac{1}{\xi} H_\mu^{q,t}(E) \inf_{x\in E} \underline{d}_\mu^{q,t}(x,\nu) \le \nu(E) \le H_\mu^{q,t}(E) \sup_{x\in E} \overline{d}_\mu^{q,t}(x,\nu), \tag{3.4}$$

$$P_\mu^{q,t}(E) \inf_{x\in E} \underline{d}_\mu^{q,t}(x,\nu) \le \nu(E) \le \xi P_\mu^{q,t}(E) \sup_{x\in E} \overline{d}_\mu^{q,t}(x,\nu). \tag{3.5}$$

In addition, if $\mu \in \mathcal{P}_D(\mathbb{R}^n)$*, we have*

$$H_\mu^{q,t}(E) \inf_{x\in E} \underline{d}_\mu^{q,t}(x,\nu) \le \nu(E) \le H_\mu^{q,t}(E) \sup_{x\in E} \overline{d}_\mu^{q,t}(x,\nu),$$

$$P_\mu^{q,t}(E) \inf_{x\in E} \underline{d}_\mu^{q,t}(x,\nu) \le \nu(E) \le P_\mu^{q,t}(E) \sup_{x\in E} \overline{d}_\mu^{q,t}(x,\nu).$$

For $\nu \in \mathcal{P}(\mathbb{R}^n)$ let

$$\mathcal{E}(\nu) = \Big\{ E \text{ be a Borel subset of } \mathbb{R}^n, \ \ \overline{d}_\mu^{q,t}(x,\nu) < \infty \text{ for all } x \in E \Big\}.$$

As a consequence we have, the following result.

Proposition 3.3. *Let* $F \subset E \in \mathcal{E}(\nu)$ *such that* $\mathcal{H}_\mu^{q,t}(E) < \infty$*. Then*

$$\mathcal{H}_\mu^{q,t}(F) = 0 \Rightarrow \nu(F) = 0.$$

In particular $\mathcal{H}_\mu^{q,t}(F) = 0 \Rightarrow \mathcal{P}_\mu^{q,t}(F) = 0$ *for all* $F \in \mathcal{E}\left(\mathcal{P}_\mu^{q,t}\right)$*.*

Remark 3.4. This proposition gives a necessary and sufficient condition for which

$$\mathcal{H}_\mu^{q,t} \sim H_\mu^{q,t} \text{ on } \mathcal{E}\left(H_\mu^{q,t}\right), \ \mathcal{H}_\mu^{q,t} \sim P_\mu^{q,t} \text{ on } \mathcal{E}\left(P_\mu^{q,t}\right) \text{ and } \mathcal{H}_\mu^{q,t} \sim \mathcal{P}_\mu^{q,t} \text{ on } \mathcal{E}\left(\mathcal{P}_\mu^{q,t}\right).$$

Remark 3.5. We can relax the assumption of Proposition 3.3 by taking

$$\mathcal{E}(\nu) = \Big\{ E \text{ be a Borel subset of } \mathbb{R}^n, \ \ \overline{d}_\mu^{q,t}(x,\nu) < \infty, \ \nu \ \text{ a.a. on } E \Big\}.$$

In this case, we get the equivalence mentioned in Remark 3.4 only in this new set.

Example 3.6. Let E be a Moran set satisfying strong separation condition and μ be a measure on a complete metric space denoted by X. Attia et al. have proved in [2] that ${\mathcal{H}_\mu^{q,t}}_{\llcorner E} \sim {\mathcal{P}_\mu^{q,t}}_{\llcorner E}$, then in this case we have

$${\mathcal{H}_\mu^{q,t}}_{\llcorner E} \sim {H_\mu^{q,t}}_{\llcorner E} \sim {P_\mu^{q,t}}_{\llcorner E} \sim {\mathcal{P}_\mu^{q,t}}_{\llcorner E}.$$

Proposition 3.7. *Let* $\theta \in \mathcal{P}(\mathbb{R}^n)$ *and* $E \in \Big\{ F$ *be a Borel subset of* $\mathbb{R}^n$; $0 < \theta(F) < +\infty \Big\}$*. Then*

$$\mathcal{H}_\mu^{q,t}(E) = 0 \iff \overline{d}_\mu^{q,t}(x,\theta) = +\infty \ \ \textit{for } \ \theta\textit{-a.a. on } E.$$

Proof. For $n \in \mathbb{N}$, we consider the set

$$E_n = \Big\{ x \in E; \ \ \overline{d}_\mu^{q,t}(x,\theta) \le n \Big\}.$$

Let $\nu = \theta_{\llcorner E}$, it follows immediately from Proposition 3.1 that

$$\theta(E \cap E_n) \le n\mathcal{H}_\mu^{q,t}(E_n) = 0, \ \ \forall n \in \mathbb{N}.$$

We therefore conclude that $\overline{d}_\mu^{q,t}(x,\theta) = +\infty$ for θ-a.a. on E.

Now, write $F = \Big\{x \in E;\ \ \overline{d}_\mu^{q,t}(x,\theta) = +\infty\Big\}$. Let $\nu = \theta\llcorner_E$, we deduce from (3.1) that

$$\frac{1}{\xi}\mathcal{H}_\mu^{q,t}(F) \inf_{x\in F} \overline{d}_\mu^{q,t}(x,\theta) \le \theta(F) < +\infty.$$

This implies that $\mathcal{H}_\mu^{q,t}(F) = 0$ and $\mathcal{H}_\mu^{q,t}(E) = \mathcal{H}_\mu^{q,t}(F) + \mathcal{H}_\mu^{q,t}(E\setminus F) = 0$. $\square$

Because of the importance of the multifractal Hausdorff measures, the multifractal packing measures and the multifractal Hewitt-Stromberg measures, the following corollary of Proposition 3.7 seems worthwhile stating separately.

Corollary 3.8. *Let E be a Borel subset of $\mathbb{R}^n$ such that $H_\mu^{q,t}(E) > 0$.*

(1) *If $H_\mu^{q,t}(E) < +\infty$, then*

$$\mathcal{H}_\mu^{q,t}(E) = 0 \iff \overline{d}_\mu^{q,t}(x,E) = +\infty \ \text{ for } H_\mu^{q,t}\text{-a.a. on } E.$$

(2) *If $P_\mu^{q,t}(E) < +\infty$, then*

$$\mathcal{H}_\mu^{q,t}(E) = 0 \iff \overline{D}_\mu^{q,t}(x,E) = +\infty \ \text{ for } P_\mu^{q,t}\text{-a.a. on } E.$$

(3) *If $\mathcal{P}_\mu^{q,t}(E) < +\infty$, then*

$$\mathcal{H}_\mu^{q,t}(E) = 0 \iff \overline{\Delta}_\mu^{q,t}(x,E) = +\infty \ \text{ for } \mathcal{P}_\mu^{q,t}\text{-a.a. on } E.$$

The next proposition treats the special case when $d_\mu^{q,t}(x,\theta)$ exists.

Proposition 3.9. *Let $\theta \in \mathcal{P}(\mathbb{R}^n)$ and $E \in \Big\{F$ be a Borel subset of $\mathbb{R}^n$; $0 < \theta(F) < +\infty$ and for all $x \in F$, $\underline{d}_\mu^{q,t}(x,\theta) = \overline{d}_\mu^{q,t}(x,\theta)\Big\}$. Then*

$$H_\mu^{q,t}(E) = 0 \iff d_\mu^{q,t}(x,\theta) = +\infty \ \text{ for } \theta\text{-a.a. on } E.$$

Proof. The proof is similar to the proof of Proposition 3.7 when we use Corollary 3.2 instead of Proposition 3.1. $\square$

Corollary 3.10. *Let $\theta \in \mathcal{P}(\mathbb{R}^n)$ and $E \in \Big\{F$ be a Borel subset of $\mathbb{R}^n$; $0 < \theta(F) < +\infty$ and for all $x \in F$, $\underline{\Delta}_\mu^{q,t}(x,\theta) = \overline{\Delta}_\mu^{q,t}(x,\theta)\Big\}$. Then*

$$\mathcal{H}_\mu^{q,t}(E) = H_\mu^{q,t}(E) = P_\mu^{q,t}(E) = 0 \iff \Delta_\mu^{q,t}(x,E) = +\infty \ \text{ for } \mathcal{P}_\mu^{q,t}\text{-a.a. on } E.$$

Proposition 3.11. *Let E be a Borel subset of* $\operatorname{supp}\mu$ *such that $\mathcal{P}_\mu^{q,t}(E) < +\infty$.*

(1) *If $\underline{d}_\mu^{q,t}(x,E) = 1$ for $H_\mu^{q,t}$-a.a. on E, then $\mathcal{P}_\mu^{q,t}(E) = P_\mu^{q,t}(E) = H_\mu^{q,t}(E)$.*
(2) *If $\underline{D}_\mu^{q,t}(x,E) = 1$ for $P_\mu^{q,t}$-a.a. on E, then $\mathcal{P}_\mu^{q,t}(E) = P_\mu^{q,t}(E)$.*
(3) *If $\underline{\Delta}_\mu^{q,t}(x,E) = \xi$ for $\mathcal{H}_\mu^{q,t}$-a.a. on E, then $\mathcal{H}_\mu^{q,t}(E) = H_\mu^{q,t}(E)$.*

Proof. It is sufficient to take $H_\mu^{q,t}\llcorner_E$, $P_\mu^{q,t}\llcorner_E$ in (3.3) and $\mathcal{H}_\mu^{q,t}\llcorner_E$ in (3.4). $\square$

3.2. Proof of Theorem 2.1

(1) Let $E, F \subset \mathbb{R}^n$ such that $d(E,F) > 0$. Since $H_\mu^{q,t}$ is a outer measure, it suffices to prove that

$$H_\mu^{q,t}\Big(E\bigcup F\Big) \geq H_\mu^{q,t}(E) + H_\mu^{q,t}(F).$$

Let $0 < r < d(E,F)/2$ and $\big(B(x_i,r)\big)_i$ be a centered covering of $E'\bigcup F'$ where $E' \subset E$ and $F' \subset F$. Put $I = \Big\{i;\ B(x_i,r)\bigcap E' \neq \emptyset\Big\}$ and $J = \Big\{i;\ B(x_i,r)\bigcap F' \neq \emptyset\Big\}$. It now follows from the definitions that

$$\begin{aligned}\sum_i \mu(B(x_i,r))^q &= \sum_{i\in I} \mu(B(x_i,r))^q + \sum_{i\in J} \mu(B(x_i,r))^q \\ &\geq N_{\mu,r}^q(E') + N_{\mu,r}^q(F').\end{aligned}$$

This yields

$$L_\mu^{q,t}\Big(E'\bigcup F'\Big) \geq L_\mu^{q,t}(E') + L_\mu^{q,t}(F').$$

This clearly implies that

$$\begin{aligned}
& H_\mu^{q,t}\Big(E\bigcup F\Big) \geq \overline{H}_\mu^{q,t}\Big(E'\bigcup F'\Big) \\
=& \inf_{E'\cup F'\subseteq\bigcup_i E_i} \left\{\sum_i L_\mu^{q,t}(E_i);\ \ E_i \ \text{ are bounded}\right\} \\
\geq& \inf_{E'\cup F'\subseteq\bigcup_i E_i} \left\{\sum_i L_\mu^{q,t}(E_i\cap E') + \sum_i L_\mu^{q,t}(E_i\cap F');\ \ E_i \ \text{ are bounded}\right\} \\
\geq& \inf_{E'\cup F'\subseteq\bigcup_i E_i} \left\{\sum_i L_\mu^{q,t}(E_i\cap E');\ \ E_i \ \text{ are bounded}\right\} \\
& + \inf_{E'\cup F'\subseteq\bigcup_i E_i} \left\{\sum_i L_\mu^{q,t}(E_i\cap F');\ \ E_i \ \text{ are bounded}\right\} \\
\geq& \overline{H}_\mu^{q,t}(E') + \overline{H}_\mu^{q,t}(F').
\end{aligned}$$

Finally, we conclude that

$$H_\mu^{q,t}\Big(E\bigcup F\Big) \geq H_\mu^{q,t}(E) + H_\mu^{q,t}(F).$$

(2) Let $F \subseteq E$ and let $\big(B(x_i,r)\big)_i$ be a centered covering of F. Then,

$$\overline{\mathcal{H}}_{\mu,r}^{q,t}(F) \leq \sum_i \mu(B(x_i,r))^q (2r)^t \quad \text{and} \quad \overline{\mathcal{H}}_{\mu,r}^{q,t}(F) \leq N_{\mu,r}^q(F)(2r)^t.$$

Also observe that it follows from the definitions that $\overline{\mathcal{H}}_\mu^{q,t}(F) \leq L_\mu^{q,t}(F)$. Let $(E_i)_i$ be a countable family of subsets of $\mathbb{R}^n$ such that $F \subseteq \bigcup_i E_i$ and

and the E_i's are bounded, then

$$\overline{\mathcal{H}}_\mu^{q,t}(F) \leq \sum_i L_\mu^{q,t}(E_i) \quad \text{and} \quad \overline{\mathcal{H}}_\mu^{q,t}(F) \leq \overline{H}_\mu^{q,t}(F).$$

However, we conclude that $\mathcal{H}_\mu^{q,t}(E) \leq H_\mu^{q,t}(E)$.

Let $\left(B(x_i,r)\right)_i$ be a centered covering of $F \subset E$. Using Besicovitch's Covering Theorem (see [16]), we can construct ξ finite or countable sub-families $\left(B(x_{1j},r)\right)_j, \ldots, \left(B(x_{\xi j},r)\right)_j$ such that

$$\text{each } F \subseteq \bigcup_{i=1}^{\xi} \bigcup_j B(x_{ij},r) \text{ and } \Big(B(x_{ij},r)\Big)_j \text{ is a packing of } F.$$

Hence

$$\begin{aligned} N_{\mu,r}^q(F)(2r)^t &\leq \sum_{i=1}^{\xi} \sum_j \mu(B(x_i,r))^q (2r)^t \leq \sum_{i=1}^{\xi} M_{\mu,r}^q(F)(2r)^t \\ &\leq \xi M_{\mu,r}^q(F)(2r)^t. \end{aligned}$$

It follows immediately from the definitions that

$$L_\mu^{q,t}(F) \leq \xi C_\mu^{q,t}(F) \quad \text{and} \quad \overline{H}_\mu^{q,t}(F) \leq \xi P_\mu^{q,t}(F) \leq \xi P_\mu^{q,t}(E).$$

We therefore conclude

$$H_\mu^{q,t}(E) \leq \xi P_\mu^{q,t}(E).$$

Let E be a bounded subset of $\mathbb{R}^n$. Then $\overline{\mathcal{P}}_\mu^{q,t}(E) \geq C_\mu^{q,t}(E)$ and so,

$$\begin{aligned} \mathcal{P}_\mu^{q,t}(E) &= \inf_{E \subseteq \bigcup_i E_i} \left\{ \sum_i \overline{\mathcal{P}}_\mu^{q,t}(E_i), \;\; E_i \text{ are bounded} \right\} \\ &\geq \inf_{E \subseteq \bigcup_i E_i} \left\{ \sum_i C_\mu^{q,t}(E_i), \;\; E_i \text{ are bounded} \right\} \\ &= P_\mu^{q,t}(E). \end{aligned}$$

(3) We may clearly assume that $C_\mu^{q,t}(E) < +\infty$. Consider, for $s \in \mathbb{N}^*$, the set

$$E_s = \left\{ x \in E, \frac{\mu\big(B(x,5r)\big)}{\mu\big(B(x,r)\big)} < s \;\text{ for }\; 0 < r < \frac{1}{s} \right\}.$$

Fix $0 < r < \frac{1}{s}$ and let $\Theta = \Big\{ B(x,r), \quad x \in E_s \Big\}$. Then the set Θ is a Vitali covering (see [16]) of E_s. By Vitali's Lemma, there exists a ξ-countable or finite family $\Big(B(x_i,r)\Big)_i \subset \Theta$ such that $B(x_i,r) \cap B(x_j,r) = \emptyset$ for all $i \neq j$ and

$$E_s \setminus \bigcup_{i=1}^{k} B(x_i,r) \subset \bigcup_{i \geq k} B(x_i,5r) \quad \text{for all } k \geq 1.$$

However, since $x_i \in E_s$, then for $\varepsilon > 0$,

$$\begin{aligned}\sum_i \mu(B(x_i,5r))^q(10r)^t &\le s^q 5^t \sum_i \mu(B(x_i,r))^q(2r)^t\\ &\le s^q 5^t M^q_{\mu,r}(E_s)(2r)^t\\ &\le s^q 5^t \left(C^{q,t}_\mu(E)+\frac{\varepsilon}{2}\right) < +\infty.\end{aligned}$$

Thus, we may choose $K \in \mathbb{N}$ such that

$$\sum_{i=K+1}^{+\infty} \mu(B(x_i,5r))^q(10r)^t \le \frac{\varepsilon}{2}.$$

Hence,

$$\begin{aligned}N^q_{\mu,r}(E_s)(2r)^t &\le \sum_{i=1}^{K} \mu(B(x_i,r))^q(2r)^t + \sum_{i=K+1}^{+\infty} \mu(B(x_i,5r))^q(10r)^t\\ &\le \sum_{i=1}^{K} \mu(B(x_i,r))^q(2r)^t + \frac{\varepsilon}{2}\\ &\le M^q_{\mu,r}(E_s)(2r)^t + \frac{\varepsilon}{2}\\ &\le C^{q,t}_\mu(E) + \varepsilon.\end{aligned}$$

Making $r \to 0$ and $\varepsilon \to 0$ and since $E = \bigcup_s E_s$, we obtain $H^{q,t}_\mu \le P^{q,t}_\mu$. The case when $q \le 0$ followed by a similar argument, since in this case $\mu(B(x_i,5r))^q \le \mu(B(x_i,r))^q$ for all x_i and then we need not to assume that $\mu \in \mathcal{P}_D(\mathbb{R}^n)$.

3.3. Proof of Theorem 2.2

We will first begin by proving this elementary lemma.

Lemma 3.12. *Let E be a Borel subset of $\operatorname{supp}\mu$ and F be a $\mathcal{H}^{q,t}_\mu$-measurable set. Suppose that $\mathcal{H}^{q,t}_\mu(E) < \infty$. If $F \subseteq E$ and $\theta \in \mathcal{P}(\mathbb{R}^n)$, then*

$$\overline{d}^{q,t}_\mu(x,\theta\llcorner_E) = \overline{d}^{q,t}_\mu(x,\theta\llcorner_F) \text{ and } \underline{d}^{q,t}_\mu(x,\theta\llcorner_E) = \underline{d}^{q,t}_\mu(x,\theta\llcorner_F) \text{ for } \mathcal{H}^{q,t}_\mu\text{-a.a. on } F.$$

Proof. Let $\nu = \theta\llcorner_E$. Then, we have, for $\mathcal{H}^{q,t}_\mu$-a.a. on F,

$$\overline{d}^{q,t}_\mu(x,\nu) = \overline{d}^{q,t}_\mu(x,\nu\llcorner_F) \quad \text{and} \quad \underline{d}^{q,t}_\mu(x,\nu) = \underline{d}^{q,t}_\mu(x,\nu\llcorner_F). \tag{3.6}$$

In fact, it is clear that

$$\underline{d}^{q,t}_\mu(x,\nu) \ge \underline{d}^{q,t}_\mu(x,\nu\llcorner_F) \quad \text{and} \quad \overline{d}^{q,t}_\mu(x,\nu) \ge \overline{d}^{q,t}_\mu(x,\nu\llcorner_F).$$

Let's set $\lambda(A) = \nu(A \setminus F)$ for any Borel set A. Then,

$$\nu(A) = \nu\big(A \cap (F^c \cup F)\big) = \nu(A \setminus F) + \nu(A \cap F) = \lambda(A) + \nu\llcorner_F(A).$$

A simple calculation shows that

$$\underline{d}^{q,t}_\mu(x,\nu) \le \underline{d}^{q,t}_\mu(x,\nu\llcorner_F) + \overline{d}^{q,t}_\mu(x,\lambda) \quad \text{and} \quad \overline{d}^{q,t}_\mu(x,\nu) \le \overline{d}^{q,t}_\mu(x,\nu\llcorner_F) + \overline{d}^{q,t}_\mu(x,\lambda).$$

We must now show that $\overline{d}_\mu^{q,t}(x,\lambda)=0$. For any integer $k\neq 0$, let

$$F_k=\left\{x\in F;\quad \overline{d}_\mu^{q,t}(x,\lambda)\geq \frac{1}{k}\right\}.$$

Then $F_k\subset F$ for any $k\geq 1$. So, by (3.1), we immediately conclude that

$$0\leq \frac{1}{\xi k}\mathcal{H}_\mu^{q,t}(F_k)\leq \lambda(F_k)=\nu(F_k\setminus F)=\nu(\emptyset)=0\quad \text{for all}\quad k\geq 1.$$

We deduce from the previous inequality that $\mathcal{H}_\mu^{q,t}(F_k)=0$ for all $k\geq 1$ and $\overline{d}_\mu^{q,t}(x,\lambda)=0$ for $\mathcal{H}_\mu^{q,t}$-a.a. on F, which leads to (3.6). $\square$

Remark 3.13. Moreover, according to Proposition 3.1, if $d_\mu^{q,t}(x,\theta_{\llcorner E})$ exists, then

$$d_\mu^{q,t}(x,\theta_{\llcorner E})=d_\mu^{q,t}(x,\theta_{\llcorner F})\;\text{ for }\;\mathcal{P}_\mu^{q,t}\text{-a.a. on }\;F.$$

Proof of Theorem 2.2.

(1) Since $\mathcal{H}_\mu^{q,t}(E)<\infty$ it follows from Lemma 3.12 that $\overline{d}_\mu^{q,t}(x,F)=\overline{d}_\mu^{q,t}(x,E)$ and $\underline{d}_\mu^{q,t}(x,F)=\underline{d}_\mu^{q,t}(x,E)$ for $\mathcal{H}_\mu^{q,t}$-a.a. on F. We therefore conclude that $\underline{d}_\mu^{q,t}(x,F)=\overline{d}_\mu^{q,t}(x,F)$ for $\mathcal{H}_\mu^{q,t}$-a.a. on F. Finally, it follows from Proposition 3.3 that $\underline{d}_\mu^{q,t}(x,F)=\overline{d}_\mu^{q,t}(x,F)$ for $H_\mu^{q,t}$-a.a. on F.

(2) The proof of this statement is similar to the proof of the statement in (1).

(3) Since $\mathcal{P}_\mu^{q,t}(E)<\infty$, it follows immediately from Lemma 3.12 that $\overline{D}_\mu^{q,t}(x,G)=\overline{D}_\mu^{q,t}(x,E)$ and $\underline{D}_\mu^{q,t}(x,G)=\underline{D}_\mu^{q,t}(x,E)$ for $\mathcal{H}_\mu^{q,t}$-a.a. on G. We deduce from Proposition 3.3 that $\underline{D}_\mu^{q,t}(x,G)=\overline{D}_\mu^{q,t}(x,G)$ for $P_\mu^{q,t}$-a.a. on G.

(4) The proof is similar to the one of (3). $\square$

3.4. Proof of Theorem 2.3

It follows from [20, Corollary 2.16] that (1) $\iff$ (4) $\iff$ (5).

(1) $\implies$ (2) : If $\mathcal{P}_\mu^{q,t}(E)<\infty$ and since $\mathcal{H}_\mu^{q,t}(E)=H_\mu^{q,t}(E)=P_\mu^{q,t}(E)=\mathcal{P}_\mu^{q,t}(E)$, then

$$\mathcal{H}_\mu^{q,t}(F)=H_\mu^{q,t}(F)=P_\mu^{q,t}(F)=\mathcal{P}_\mu^{q,t}(F)\quad\text{for any}\quad F\subset E. \tag{3.7}$$

Put the set $F=\left\{x\in E;\;\overline{d}_\mu^{q,t}(x,E)>1\right\}$, and for $m\in\mathbb{N}^*$

$$F_m=\left\{x\in E;\;\overline{d}_\mu^{q,t}(x,E)>1+\frac{1}{m}\right\}.$$

We therefore deduce from (3.2) and (3.7) that

$$\left(1+\frac{1}{m}\right)\mathcal{H}_\mu^{q,t}(F_m)\leq H_\mu^{q,t}(F_m)=\mathcal{H}_\mu^{q,t}(F_m).$$

This implies that $\mathcal{H}_\mu^{q,t}(F_m)=0$. As $F=\bigcup_m F_m$, we obtain $\mathcal{H}_\mu^{q,t}(F)=0$ and so, $\mathcal{P}_\mu^{q,t}(F)=0$, i.e.,

$$\overline{d}_\mu^{q,t}(x,E)\leq 1 \quad \text{for} \quad \mathcal{P}_\mu^{q,t}\text{-a.a. } x\in E. \tag{3.8}$$

Now consider the set $\tilde{F}=\Big\{x\in E;\ \underline{d}_\mu^{q,t}(x,E)<1\Big\}$, and for $m\in\mathbb{N}^*$

$$\tilde{F}_m=\left\{x\in E;\ \underline{d}_\mu^{q,t}(x,E)<1-\frac{1}{m}\right\}.$$

Using (3.3), we clearly have

$$H_\mu^{q,t}(\tilde{F}_m)=\mathcal{P}_\mu^{q,t}(\tilde{F}_m)\leq\left(1-\frac{1}{m}\right)\mathcal{P}_\mu^{q,t}(\tilde{F}_m).$$

This implies that $\mathcal{P}_\mu^{q,t}(\tilde{F}_m)=0$. As $\tilde{F}=\bigcup_m \tilde{F}_m$, we obtain $\mathcal{P}_\mu^{q,t}(\tilde{F})=0$, i.e.,

$$\underline{d}_\mu^{q,t}(x,E)\geq 1 \quad \text{for} \quad \mathcal{P}_\mu^{q,t}\text{-a.a. } x\in E. \tag{3.9}$$

The statement in (2) now follows from (3.8) and (3.9).

(2) $\Longrightarrow$ (1) : Consider the set

$$F=\Big\{x\in E;\ \underline{d}_\mu^{q,t}(x,E)=1=\overline{d}_\mu^{q,t}(x,E)\Big\}.$$

It therefore follows (3.2) and (3.3) and since, $\underline{d}_\mu^{q,t}(x,E)=1=\overline{d}_\mu^{q,t}(x,E)$ for $\mathcal{P}_\mu^{q,t}$-a.a. $x\in E$ that

$$\begin{aligned}\mathcal{H}_\mu^{q,t}(E)\leq H_\mu^{q,t}(E)\leq P_\mu^{q,t}(E)\leq \mathcal{P}_\mu^{q,t}(E)=\mathcal{P}_\mu^{q,t}(F)\\ \leq H_\mu^{q,t}(F)\leq \mathcal{H}_\mu^{q,t}(F)\leq \mathcal{H}_\mu^{q,t}(E).\end{aligned}$$

(1) $\Longrightarrow$ (3) : Since $\mathcal{H}_\mu^{q,t}(E)=\mathcal{P}_\mu^{q,t}(E)$, we conclude that

$$\mathcal{H}_\mu^{q,t}(F)=H_\mu^{q,t}(F)=P_\mu^{q,t}(F)=\mathcal{P}_\mu^{q,t}(F) \quad \text{for any} \quad F\subset E.$$

Now, we consider the set $F=\Big\{x\in E;\ \underline{D}_\mu^{q,t}(x,E)<1\Big\}$, and for $m\in\mathbb{N}^*$

$$F_m=\left\{x\in E;\ \underline{D}_\mu^{q,t}(x,E)<1-\frac{1}{m}\right\}.$$

Then it follows from (3.3),

$$\mathcal{P}_\mu^{q,t}(F_m)=P_\mu^{q,t}(F_m)\leq\mathcal{P}_\mu^{q,t}(F_m)\left(1-\frac{1}{m}\right).$$

This implies that $\mathcal{P}_\mu^{q,t}(F_m)=0$. As $F=\bigcup_m F_m$, we obtain $\mathcal{P}_\mu^{q,t}(F)=0$ and so, $\underline{D}_\mu^{q,t}(x,E)\geq 1$ for $\mathcal{P}_\mu^{q,t}$-a.a. $x\in E$.

Next, put $\tilde{F}=\Big\{x\in E;\ \overline{D}_\mu^{q,t}(x,E)>1\Big\}$, and for $m\in\mathbb{N}^*$

$$\tilde{F}_m=\left\{x\in E;\ \overline{D}_\mu^{q,t}(x,E)>1+\frac{1}{m}\right\}.$$

We deduce from (3.2) that,

$$\left(1+\frac{1}{m}\right)\mathcal{H}_\mu^{q,t}(\tilde{F}_m) \le P_\mu^{q,t}(\tilde{F}_m) = \mathcal{H}_\mu^{q,t}(\tilde{F}_m).$$

This implies that $\mathcal{H}_\mu^{q,t}(\tilde{F}_m) = 0$. Finally, it follows from $\tilde{F} = \bigcup_m \tilde{F}_m$ that $\mathcal{P}_\mu^{q,t}(\tilde{F}) = 0$, i.e.,

$$\overline{D}_\mu^{q,t}(x, E) \le 1 \quad \text{for} \quad \mathcal{P}_\mu^{q,t}\text{-a.a. } x \in E.$$

(3) $\Longrightarrow$ (1) : We consider the set

$$F = \Big\{x \in E;\ \underline{D}_\mu^{q,t}(x,E) = 1 = \overline{D}_\mu^{q,t}(x,E)\Big\}.$$

Combining (3.2) and (3.3) shows that

$$\begin{aligned}\mathcal{H}_\mu^{q,t}(E) \le H_\mu^{q,t}(E) \le P_\mu^{q,t}(E) \le \mathcal{P}_\mu^{q,t}(E) = \mathcal{P}_\mu^{q,t}(F)\\ \le P_\mu^{q,t}(F) \le \mathcal{H}_\mu^{q,t}(F) \le \mathcal{H}_\mu^{q,t}(E),\end{aligned}$$

which proves the desired result.

4. Application

As an application of Theorems 2.1, 2.2 and 2.3 we prove a density theorem for the multifractal measures $\mathcal{H}_\mu^{q,t}$ and $\mathcal{P}_\mu^{q,t}$ that is more refined than those found in [3]. Let $q, t \in \mathbb{R}$, $\mu \in \mathcal{P}_D(\mathbb{R}^n)$ and E be a Borel subset of $\operatorname{supp}\mu$. Assume that $\mathcal{P}_\mu^{q,t}(E) < +\infty$ and $\overline{\Delta}_\mu^{q,t}(x, E) < +\infty$. Consider, for a finite measure ν on $\mathbb{R}^n$, the set

$$F(\nu) = \Big\{x \in E;\ d_\mu^{q,t}(x,\nu) = 1\ \Big\}.$$

Definition. Let $(X, \mathcal{B}, \mu)$ be a measure space and E, F in $\mathcal{B}$. We will say that E is a subset of F μ-almost everywhere and write $E \subseteq F$ μ-a.e., if $\mu(F \backslash E) = 0$.

The following theorem deals with the strong regularity of a measurable subset $B \subset E$ which extends the density theorem in [3].

Theorem 4.1. *The following assertions are equivalent*

(1) $\mathcal{H}_\mu^{q,t}(B) = \mathcal{P}_\mu^{q,t}(B)$ *for a measurable subset* B *of* E.

(2) $B \subset F\left(H_\mu^{q,t}{\llcorner_E}\right)$ $\mathcal{P}_\mu^{q,t}$*-a.e..*

(3) $B \subset F\left(\mathcal{H}_\mu^{q,t}{\llcorner_E}\right)$ $\mathcal{P}_\mu^{q,t}$*-a.e..*

(4) $B \subset F\left(P_\mu^{q,t}{\llcorner_E}\right)$ $\mathcal{P}_\mu^{q,t}$*-a.e..*

(5) $B \subset F\left(\mathcal{P}_\mu^{q,t}{\llcorner_E}\right)$ $\mathcal{P}_\mu^{q,t}$*-a.e..*

Theorem 4.1 is a consequence from the following lemmas.

Lemma 4.2. *Let* E *a Borel subset of* $\operatorname{supp}\mu$.

(1) *Suppose that* $\mathcal{P}_{\mu}^{q,t}(E)<\infty$. *For* $F=\Big\{x\in E;\ \ \overline{D}_{\mu}^{q,t}(x,E)<+\infty\Big\}$, *we have*

If G *is a Borel subset of* F *such that* $H_{\mu}^{q,t}(G)=0$, *then* $P_{\mu}^{q,t}(G)=0$.

(2) *Suppose that* $\mathcal{P}_{\mu}^{q,t}(E)<\infty$. *For* $F=\Big\{x\in E;\ \ \overline{\Delta}_{\mu}^{q,t}(x,E)<+\infty\Big\}$, *we have*

If G *is a Borel subset of* F *such that* $P_{\mu}^{q,t}(G)=0$, *then* $\mathcal{P}_{\mu}^{q,t}(G)=0$.

Proof. It is sufficient to take $P_{\mu}^{q,t}{}_{\llcorner E}$ in (3.4) and $\mathcal{P}_{\mu}^{q,t}{}_{\llcorner E}$ in (3.5). □

Lemma 4.3. *Let* E *be a* $\mathcal{P}_{\mu}^{q,t}$*-measurable set with* $\mathcal{P}_{\mu}^{q,t}(E)<\infty$ *and* $\overline{\Delta}_{\mu}^{q,t}(x,E)$ $<+\infty$ *on* E, *then*

$$F\left(\mathcal{P}_{\mu}^{q,t}{}_{\llcorner E}\right)=F\left(\mathcal{H}_{\mu}^{q,t}{}_{\llcorner E}\right)=F\left(P_{\mu}^{q,t}{}_{\llcorner E}\right)=F\left(H_{\mu}^{q,t}{}_{\llcorner E}\right)\ \mathcal{P}_{\mu}^{q,t}\text{-a.e..}$$

Proof. It follow from [3] that $F\left(\mathcal{P}_{\mu}^{q,t}{}_{\llcorner E}\right)=F\left(\mathcal{H}_{\mu}^{q,t}{}_{\llcorner E}\right)$ $\mathcal{P}_{\mu}^{q,t}$-a.e.. It suffices to prove that

$$F\left(\mathcal{H}_{\mu}^{q,t}{}_{\llcorner E}\right)\subset F\left(P_{\mu}^{q,t}{}_{\llcorner E}\right)\subset F\left(H_{\mu}^{q,t}{}_{\llcorner E}\right)\ \mathcal{P}_{\mu}^{q,t}\text{-a.e..}$$

The other inclusions are similar.

Without loss of generality, for $K=F\Big(\mathcal{H}_{\mu}^{q,t}{}_{\llcorner E}\Big)$, we may assume that $\mathcal{P}_{\mu}^{q,t}(K)>0$. Theorem 2.7 in [3] implies that $\Lambda_{\mu}^{q,t}(x,K)=1$ for $\mathcal{H}_{\mu}^{q,t}$ -a.a. on K. Since $\mathcal{H}_{\mu}^{q,t}(K)<\infty$ and $\overline{\Delta}_{\mu}^{q,t}(x,K)<\infty$, then, using Proposition 3.3, we obtain $\Lambda_{\mu}^{q,t}(x,K)=1$ for $\mathcal{P}_{\mu}^{q,t}$-a.a. on K. Hence, by Theorem 2.3, we have $D_{\mu}^{q,t}(x,K)=1$ for $\mathcal{P}_{\mu}^{q,t}$-a.a. on K. Next by using Lemma 3.12, we have

$$D_{\mu}^{q,t}(x,E)=1\ \text{ for }\ \mathcal{H}_{\mu}^{q,t}\text{-a.a. on }\ K.$$

Proposition 3.3 now implies that $D_{\mu}^{q,t}(x,E)=1$ for $\mathcal{P}_{\mu}^{q,t}$-a.a. on K.

For $K=F\Big(P_{\mu}^{q,t}{}_{\llcorner E}\Big)$, we may clearly assume that $\mathcal{P}_{\mu}^{q,t}(K)>0$. By Using Theorem 2.2, we have $D_{\mu}^{q,t}(x,K)=1$ for $P_{\mu}^{q,t}$-a.a. on K. From Lemma 4.2, we have $D_{\mu}^{q,t}(x,K)=1$ for $\mathcal{P}_{\mu}^{q,t}$-a.a. on K. From Theorem 2.3, we obtain $d_{\mu}^{q,t}(x,K)=1$ for $\mathcal{P}_{\mu}^{q,t}$-a.a. on K. Hence, $d_{\mu}^{q,t}(x,K)=1$ for $\mathcal{H}_{\mu}^{q,t}$-a.a. on K. Lemma 3.12 now yields

$$d_{\mu}^{q,t}(x,E)=1\ \text{ for }\ \mathcal{H}_{\mu}^{q,t}\text{-a.a. on }\ K.$$

Finally, Proposition 3.3 implies that $d_{\mu}^{q,t}(x,E)=1$ for $\mathcal{P}_{\mu}^{q,t}$-a.a. on K. □

Proof of Theorem 4.1. It's enough to prove that (1) $\iff$ (2). Suppose (1). By using Theorem 2.3, we have $d_{\mu}^{q,t}(x,B)=1$ for $\mathcal{P}_{\mu}^{q,t}$-a.a. on B. Lemma 3.12 implies that $d_{\mu}^{q,t}(x,E)=1$ for $\mathcal{H}_{\mu}^{q,t}$-a.a. on B. By Theorem 3.3, we obtain $d_{\mu}^{q,t}(x,E)=1$ for $\mathcal{P}_{\mu}^{q,t}$-a.a. on B.

Conversely, suppose (2) then $d_\mu^{q,t}(x,E)=1$ for $\mathcal{H}_\mu^{q,t}$-a.a. on B. Using Lemma 3.12 and Proposition 3.3 we get $d_\mu^{q,t}(x,B)=1$ for $\mathcal{P}_\mu^{q,t}$-a.a. on B. Finally we conclude by Theorem 2.3. $\square$

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