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GENERALIZED DERIVATIONS WITH CENTRALIZING CONDITIONS IN PRIME RINGS

Priyadwip Das, Basudeb Dhara, and Sukhendu Kar

Abstract. Let R be a noncommutative prime ring of characteristic different from 2, U the Utumi quotient ring of R, C the extended centroid of R and $f(x_1,\ldots,x_n)$ a noncentral multilinear polynomial over C in n noncommuting variables. Denote by $f(R)$ the set of all the evaluations of $f(x_1,\ldots,x_n)$ on R. If d is a nonzero derivation of R and G a nonzero generalized derivation of R such that

$$d(G(u)u) \in Z(R)$$

for all $u \in f(R)$, then $f(x_1,\ldots,x_n)^2$ is central-valued on R and there exists $b \in U$ such that $G(x) = bx$ for all $x \in R$ with $d(b) \in C$.

As an application of this result, we investigate the commutator $[F(u)u, G(v)v] \in Z(R)$ for all $u, v \in f(R)$, where F and G are two nonzero generalized derivations of R.

1. Introduction

Throughout this paper R always denotes an associative prime ring. U denotes the Utumi ring of quotients of R. $C = Z(U)$ is called the extended centroid of R. For $x, y \in R$, the commutator of x and y is denoted and defined by $[x, y] = xy - yx$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. The concept of a derivation is extended to generalized derivation. An additive mapping $G : R \to R$ is called a generalized derivation, if $G(xy) = G(x)y + xd(y)$ holds for all $x, y \in R$, for some derivation d of R. $f(x_1,\ldots,x_n)$ is a noncentral multilinear polynomial over C in n noncommuting variables which is $f(x_1,\ldots,x_n) = x_1x_2\cdots x_n + \sum_{I\neq\sigma\in S_n} \alpha_\sigma x_{\sigma(1)}\cdots x_{\sigma(n)}$ for some $\alpha_\sigma \in C$.

In [7], Demir and Argac studied the situation $G(f(x_1,\ldots,x_n))f(x_1,\ldots,x_n) = 0$ for all $x_1,\ldots,x_n \in R$ and then obtains the forms of the map G.

In [10], Carini and Filippis proved that if R is of characteristic different from 2, d is a nonzero derivation of R, G is a nonzero generalized derivation

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of R such that $d(G(f(x_1,\ldots,x_n))f(x_1,\ldots,x_n)) = 0$ for all $x_1,\ldots,x_n \in R$, then $f(x_1,\ldots,x_n)^2$ is central-valued on R and there exists $a \in U$ such that $G(x) = ax$ for all $x \in R$, d is an inner derivation of R such that $d(a) = 0$.

In the present paper, our motivation is to extend the above result of [10] by considering central values. More precisely, we prove the following.

Main Theorem. *Let R be a noncommutative prime ring of characteristic different from 2, U the Utumi quotient ring of R, C the extended centroid of R and $f(x_1,\ldots,x_n)$ a noncentral multilinear polynomial over C in n noncommuting variables. Denote by $f(R)$ the set of all the evaluations of $f(x_1,\ldots,x_n)$ on R. If d is a nonzero derivation of R and G is a nonzero generalized derivation of R such that*

$$d(G(u)u) \in Z(R)$$

for all $u \in f(R)$, then $f(x_1,\ldots,x_n)^2$ is central-valued on R and there exists $b \in U$ such that $G(x) = bx$ for all $x \in R$ with $d(b) \in C$.

Recently, in [1], Ali et al. proved the following theorem:

Let R be a prime ring of characteristic different from 2, $Z(R)$ the center of R, U the two-sided Utumi quotient ring of R, $f(x_1,\ldots,x_n)$ a non-central multilinear polynomial over K and F a non-zero generalized derivation of R. Denote $f(R)$ the set of all evaluations of the polynomial $f(x_1,\ldots,x_n)$ in R. If $[F(u)u, F(v)v] = 0$ for all $u, v \in f(R)$, then there exists $c \in U$ such that $F(x) = cx$ for all $x \in R$ and one of the following holds: (1) $f(x_1,\ldots,x_n)^2$ *is central valued on R*; (2) *R satisfies s_4, the standard identity of degree* 4.

Recently, Sharma et al. [12] investigated the above identity with left annihilator conditions in prime rings. Moreover, in [8] Dhara et al. studied the above situation considering two different generalized derivations, that is, $[F(u)u, G(v)v] = 0$ for all $u, v \in f(R)$, where F and G are two generalized derivations of R.

As an application of the Main Theorem of the present paper, we investigate the commutator with central values, that is, $[F(u)u, G(v)v] \in Z(R)$ for all $u, v \in f(R)$, where F and G are two nonzero generalized derivations of R.

2. Main results

We need the following lemma:

Lemma 2.1 ([6, Lemma 1.5])**.** *Let $R = M_m(F)$, $m \geq 2$, be the ring of all $m \times m$ matrices over an infinite field F. If $A_1,\ldots,A_k$ are not scalar matrices in $M_m(F)$, then there exists some invertible matrix $P \in M_m(F)$ such that any matrices $PA_1P^{-1},\ldots,PA_kP^{-1}$ have all non-zero entries.*

Proposition 2.2. *Let $R = M_m(C)$ be the ring of all $m \times m$ matrices over the infinite field C with char $(R) \neq 2$ and $f(x_1,\ldots,x_n)$ a non-central multilinear polynomial over C and $a, p, q \in R$. If*

$$[a, (pf(r) + f(r)q)f(r)] \in Z(R)$$

for all $r = (r_1, \dots, r_n) \in R^n$, *then one of the following holds*: (i) $a \in Z(R)$; (ii) $p = -q \in Z(R)$; (iii) $q \in Z(R)$, $f(x_1, \dots, x_n)^2$ *is central valued in* R *with* $[a, p+q] \in Z(R)$.

Proof. Here e_{ij} denotes the usual matrix unit with 1 in (i, j)-entry and zero elsewhere. First, we assume that $a \notin Z(R)$ and $q \notin Z(R)$. Then by Lemma 2.1, there exists an invertible matrix P such that PaP^{-1} and PqP^{-1} have all nonzero entries. We consider an F-automorphism ϕ of $M_m(C)$ such that $\phi(x) = PxP^{-1}$. Then by hypothesis R satisfies

$$[\phi(a), (\phi(p)f(x_1, \dots, x_n) + f(x_1, \dots, x_n)\phi(q))f(x_1, \dots, x_n)] \in Z(R). \tag{1}$$

Moreover, since $f(x_1, \dots, x_n)$ is not central valued, by [11], there exists a sequence of matrices $u_1, \dots, u_n$ and $0 \neq \gamma \in C$ such that $f(u_1, \dots, u_n) = \gamma e_{kl}$, with $k \neq l$. Since the set $\{f(r_1, \dots, r_n) : r_1, \dots, r_n \in M_m(C)\}$ is invariant under the action of all automorphisms of $M_m(C)$, then for any $i \neq j$ there exists a sequence of matrices $r_1, \dots, r_n$ such that $f(r_1, \dots, r_n) = \gamma e_{ij}$, where $0 \neq \gamma \in C$. Thus by (1), we have

$$[\phi(a), (\phi(p)e_{ij} + e_{ij}\phi(q))e_{ij}] \in Z(R). \tag{2}$$

This implies

$$e_{ij}[[\phi(a), (\phi(p)e_{ij} + e_{ij}\phi(q))e_{ij}], e_{ij}] = 0 \tag{3}$$

that is $2\phi(a)_{ji}\phi(q)_{ji} = 0$. Since char $(R) \neq 2$, we have $\phi(a)_{ji}\phi(q)_{ji} = 0$ which contradicts our assumption that $\phi(a) = PaP^{-1}$ and $\phi(q) = PqP^{-1}$ have all nonzero entries. Therefore, we conclude that either $a \in Z(R)$ or $q \in Z(R)$. If $a \in Z(R)$, then conclusion (i) is obtained. So assume that $a \notin Z(R)$.

If $q \in Z(R)$, by hypothesis,

$$[a, (p+q)f(r)^2] \in Z(R)$$

for all $r = (r_1, \dots, r_n) \in R^n$. Let A be the additive subgroup generated by the polynomial $f(r_1, \dots, r_n)^2$. By [3] either $f(r_1, \dots, r_n)^2$ is central in R or the non-central Lie ideal $[R, R]$ of R is contained in A.

In case $f(r_1, \dots, r_n)^2$ is central valued in R, $[a, (p+q)f(r)^2] \in Z(R)$ implies $[a, p+q]f(r)^2 \in Z(R)$ for all $r = (r_1, \dots, r_n) \in R^n$. Since $f(r_1, \dots, r_n)^2 = 0$ implies $f(r_1, \dots, r_n) = 0$, a contradiction, $f(r_1, \dots, r_n)^2$ has some nonzero central valued for some $r_1, \dots, r_n \in R$. Thus we have $[a, p+q] \in Z(R)$. This is our conclusion (iii).

In the last case we have that $[a, (p+q)[x_1, x_2]] \in Z(R)$ for all $x_1, x_2 \in R$. Again, if $p+q \notin Z(R)$, there exists an invertible matrix Q such that $Q(p+q)Q^{-1}$ and QaQ^{-1} have all nonzero entries. By considering an F-automorphism φ of $M_m(C)$ such that $\varphi(x) = QxQ^{-1}$, we have

$$[\varphi(a), \varphi(p+q)[x_1, x_2]] \in Z(R)$$

for all $x_1, x_2 \in R$. For $[x_1, x_2] = [e_{ij}, e_{jj}] = e_{ij}$, it follows that $(\varphi(a)\varphi(p+q)e_{ij} - \varphi(p+q)e_{ij}\varphi(a)) \in Z(R)$. This implies $e_{ij}[\varphi(a)\varphi(p+q)e_{ij} - \varphi(p+$

$q)e_{ij}\varphi(a), e_{ij}] = 0$ that is $\varphi(p+q)_{ji}\varphi(a)_{ji} = 0$, which contradicts our assumption. Therefore $p+q \in Z(R)$ and hence $p \in Z(R)$, Then by hypothesis

$$(p+q)[a, f(r)^2] \in Z(R)$$

for all $r = (r_1, \ldots, r_n) \in R^n$. This implies either $p+q = 0$, that is $p = -q \in C$ conclusion (ii), or

$$[a, f(r)^2] \in Z(R),$$

it gives

$$[a, f(r)]f(r) + f(r)[a, f(r)] \in Z(R)$$

for all $r = (r_1, \ldots, r_n) \in R^n$. Then by ([13]) we have one of followings (i) $a \in Z(R)$, a contradiction, (ii) $f(x_1, \ldots, x_n)^2 \in Z(R)$, which is our conclusion (iii). □

Proposition 2.3. *Let $R = M_m(C)$ be the ring of all $m \times m$ matrices over the field C with char $(R) \neq 2$ and $f(x_1, \ldots, x_n)$ a noncentral multilinear polynomial over C and $a, p, q \in R$. If*

$$[a, (pf(r) + f(r)q)f(r)] \in Z(R)$$

for all $r = (r_1, \ldots, r_n) \in R^n$, then one of the following holds: (i) $a \in Z(R)$; (ii) $p = -q \in Z(R)$; (iii) $q \in Z(R)$, *$f(x_1, \ldots, x_n)^2$ is central valued in R with $[a, p+q] \in Z(R)$.*

Proof. If C is infinite, conclusions follow by Proposition 2.2. Thus we assume that C be finite. Let K be an infinite field which is an extension of the field C. Let $\overline{R} = M_m(K) \cong R \otimes_C K$. Notice that the multilinear polynomial $f(r_1, \ldots, r_n)$ is central-valued on R if and only if it is central-valued on $\overline{R}$. Consider the generalized polynomial identity for R

$$\begin{aligned}\psi(r_1, \ldots, r_n, r_{n+1}) &= [[a, (pf(r_1, \ldots, r_n) + f(r_1, \ldots, r_n)q)f(r_1, \ldots, r_n)], r_{n+1}]\\ &= 0.\end{aligned}$$

Moreover, it is a multi-homogeneous of multi-degree $(2, \ldots, 2)$ in the indeterminates $r_1, \ldots, r_n$. Hence, the complete linearization of $P(r_1, \ldots, r_n, r_{n+1})$ yields a multilinear generalized polynomial $\Theta(r_1, \ldots, r_n, s_1, \ldots, s_n, r_{n+1})$ in $2n+1$ indeterminates, such that

$$\Theta(r_1, \ldots, r_n, r_1, \ldots, r_n, r_{n+1}) = 2^n \psi(r_1, \ldots, r_n, r_{n+1}).$$

Clearly the multilinear polynomial $\Theta(r_1, \ldots, r_n, s_1, \ldots, s_n, r_{n+1})$ is a generalized polynomial identity for R and $\overline{R}$ also. Since $char(R) \neq 2$, we obtain $\psi(r_1, \ldots, r_n, r_{n+1}) = 0$ for all $r_1, \ldots, r_n, r_{n+1} \in \overline{R}$ and thus conclusion follows by Proposition 2.2. □

Lemma 2.4. *Let R be a noncommutative prime ring of characteristic different from 2 and C be its extended centroid. Suppose that $f(x_1, \ldots, x_n)$ is a* *noncentral multilinear polynomial over C, $a \in R - C$ and $G(\neq 0)$ is an inner generalized derivation of R such that*

$$[a, G(f(r))f(r)] \in Z(R)$$

for all $r = (r_1, \ldots, r_n) \in R^n$, then $f(x_1, \ldots, x_n)^2$ is central-valued on R and there exists $b \in U$ such that $G(x) = bx$ for all $x \in R$ with $[a, b] \in C$.

Proof. Let $G(x) = px + xq$ for all $x \in R$, where $p, q \in U$. By hypothesis

$$\begin{aligned} \Psi(r_1, \ldots, r_n, y) &= [[a, (pf(r_1, \ldots, r_n) + f(r_1, \ldots, r_n)q)f(r_1, \ldots, r_n)], y] \\ &= 0 \end{aligned} \tag{4}$$

for all $r_1, \ldots, r_n, y \in R$. If $[a, G(x)x] = 0$ for all $x \in S = f(R)$, then by ([10]), $f(x_1, \ldots, x_n)^2$ is central-valued on R and there exists $b \in U$ such that $G(x) = bx$ for all $x \in R$ and $[a, b] = 0 \in C$. This is our conclusion. Thus we assume that there exists $x_0 \in S$ such that $0 \neq [a, G(x_0)x_0] \in C$. Thus R satisfies a central generalized polynomial identity $[a, G(x)x] \in C$ for all $x \in S$. By [2, Theorem 1], RC is a finite dimensional central simple C-algebra, so that $Soc(U) = RC = U$, in particular R is a PI-ring. Denote by K the algebraic closure of C, if C is infinite, otherwise let $K = C$. Then $RC \otimes_C K \cong M_l(K)$ for some $l \geq 2$. Moreover, $RC \otimes_C K$ satisfies the same generalized polynomial identities of $RC = U$, in particular $[a, G(x)x]$ is central in $RC \otimes_C K$ for any $x \in f(RC \otimes_C K)$. Therefore, by Proposition 2.3 we have one of the following holds: (i) $[a, RC \otimes_C K] = (0)$; (ii) $[q, RC \otimes_C K] = (0)$ and $p = -q$; (iii) $[q, RC \otimes_C K] = (0)$, $f(x_1, \ldots, x_n)^2 \in Z(RC \otimes_C K)$ with $[a, p+q] \in Z(RC \otimes_C K)$. Since $RC \otimes_C K$ satisfies the same generalized polynomial identities of $RC = U$, it follows one of these (i) $[a, U] = (0)$; (ii) $[q, U] = (0)$, $p = -q$ (iii) $[q, U] = (0)$, $f(x_1, \ldots, x_n)^2 \in Z(U)$ with $[a, p + q] \in Z(U)$. In other words, since $a \notin C$ and G in nonzero, we have (iii) $q \in C$, $f(x_1, \ldots, x_n)^2 \in C$ with $[a, p+q] \in C$. Then $G(x) = (p + q)x$ for all $x \in R$, which is our conclusion. □

Proof of Main Theorem. Lee proved in [13, Theorem 3] that every generalized derivation G on a dense right ideal of R can be uniquely extended to a generalized derivation of U with the form $G(x) = bx + \delta(x)$ for some $b \in U$ and δ is a derivation of U.

Since I, R and U satisfy the same generalized polynomial identities (GPIs) (see [4]) as well as the same differential identities (see [11]), we can write that U satisfies

$$d(bf(x_1, \ldots, x_n)^2 + \delta(f(x_1, \ldots, x_n))f(x_1, \ldots, x_n)) \in C. \tag{5}$$

If both d and δ are inner derivations of U, then conclusions follow by Lemma 2.4. Thus we assume that not both of d and δ are inner. Hence, we rewrite above equation as

$$\begin{aligned} &d(b)f(x_1, \ldots, x_n)^2 + bd(f(x_1, \ldots, x_n))f(x_1, \ldots, x_n) \\ &+ bf(x_1, \ldots, x_n)d(f(x_1, \ldots, x_n)) + d(\delta(f(x_1, \ldots, x_n)))f(x_1, \ldots, x_n) \end{aligned}$$

$$+\delta(f(x_1,\dots,x_n))d(f(x_1,\dots,x_n))\in C \tag{6}$$

for all $x_1,\dots,x_n\in U$. Now we consider the following cases:

Case-1: Let d and δ be linear C-dependent modulo inner derivations of U.

In this case, there exist $\alpha,\beta\in C$ and $p\in U$ such that $\alpha d+\beta\delta=ad_p$, the inner derivation induced by p.

First assume that $\alpha=0$. Then $\delta=\beta^{-1}ad_p$, that is $\delta(x)=[q,x]$ for all $x\in R$, where $q=\beta^{-1}p$. Obviously d is not an inner derivation of U. In this case by (6), U satisfies

$$\begin{aligned}&d(b)f(x_1,\dots,x_n)^2+bd(f(x_1,\dots,x_n))f(x_1,\dots,x_n)\\&+bf(x_1,\dots,x_n)d(f(x_1,\dots,x_n))+d([q,f(x_1,\dots,x_n)])f(x_1,\dots,x_n)\\&+[q,f(x_1,\dots,x_n)]d(f(x_1,\dots,x_n))\in C,\end{aligned}\tag{7}$$

that is

$$\begin{aligned}&d(b)f(x_1,\dots,x_n)^2+bd(f(x_1,\dots,x_n))f(x_1,\dots,x_n)\\&+bf(x_1,\dots,x_n)d(f(x_1,\dots,x_n))+([d(q),f(x_1,\dots,x_n)]\\&+[q,d(f(x_1,\dots,x_n))])f(x_1,\dots,x_n)+[q,f(x_1,\dots,x_n)]d(f(x_1,\dots,x_n))\in C.\end{aligned}$$

Let $f^d(x_1,\dots,x_n)$ be the polynomials obtained from $f(x_1,\dots,x_n)$ replacing each coefficients α_σ with $d(\alpha_\sigma)$. Then

$$d(f(x_1,\dots,x_n))=f^d(x_1,\dots,x_n)+\sum_i f(x_1,\dots,d(x_i),\dots,x_n),$$

and hence from above

$$\begin{aligned}&d(b)f(x_1,\dots,x_n)^2\\&+b\Big(f^d(x_1,\dots,x_n)+\sum_i f(x_1,\dots,d(x_i),\dots,x_n)\Big)f(x_1,\dots,x_n)\\&+bf(x_1,\dots,x_n)\Big(f^d(x_1,\dots,x_n)+\sum_i f(x_1,\dots,d(x_i),\dots,x_n)\Big)\\&+[d(q),f(x_1,\dots,x_n)]\\&+[q,f^d(x_1,\dots,x_n)+\sum_i f(x_1,\dots,d(x_i),\dots,x_n))]f(x_1,\dots,x_n)\\&+[q,f(x_1,\dots,x_n)]\Big(f^d(x_1,\dots,x_n)+\sum_i f(x_1,\dots,d(x_i),\dots,x_n)\Big)\in C\end{aligned}$$

for all $x_1,\dots,x_n\in U$. By Kharchenko's Theorem [9], U satisfies

$$d(b)f(x_1,\dots,x_n)^2+b\Big(f^d(x_1,\dots,x_n)+\sum_i f(x_1,\dots,y_i,\dots,x_n)\Big)f(x_1,\dots,x_n)$$

$$+ bf(x_1,\ldots,x_n)\Big(f^d(x_1,\ldots,x_n) + \sum_i f(x_1,\ldots,y_i,\ldots,x_n)\Big)$$
$$+ [d(q), f(x_1,\ldots,x_n)]$$
$$+ [q, f^d(x_1,\ldots,x_n) + \sum_i f(x_1,\ldots,y_i,\ldots,x_n)]f(x_1,\ldots,x_n)$$
$$+ [q, f(x_1,\ldots,x_n)]\Big(f^d(x_1,\ldots,x_n) + \sum_i f(x_1,\ldots,y_i,\ldots,x_n)\Big) \in C$$

for all $x_1,\ldots,x_n \in U$. In particular, U satisfies the blended component

$$\begin{aligned} & b\sum_i f(x_1,\ldots,y_i,\ldots,x_n)f(x_1,\ldots,x_n) \\ & + bf(x_1,\ldots,x_n)\sum_i f(x_1,\ldots,y_i,\ldots,x_n) \\ & + [q, \sum_i f(x_1,\ldots,y_i,\ldots,x_n)]f(x_1,\ldots,x_n) \\ & + [q, f(x_1,\ldots,x_n)]\sum_i f(x_1,\ldots,y_i,\ldots,x_n) \in C. \end{aligned} \tag{8}$$

In particular, for $y_1 = x_1$ and $y_2 = \cdots = y_n = 0$, we have

$$2(bf(x_1,\ldots,x_n)^2 + qf(x_1,\ldots,x_n)^2 - f(x_1,\ldots,x_n)qf(x_1,\ldots,x_n)) \in C.$$

This implies $((b+q)f(x_1,\ldots,x_n) - f(x_1,\ldots,x_n)q)f(x_1,\ldots,x_n) \in C$ for all $x_1,\ldots,x_n \in U$. Then by Lemma 6 in [5], $b, q \in C$. Then $bf(x_1,\ldots,x_n)^2 \in C$ for all $x_1,\ldots,x_n \in U$, implying either $b = 0$ or $f(x_1,\ldots,x_n)^2 \in C$ for all $x_1,\ldots,x_n \in U$. In case $b = 0$, $G(x) = bx + [q,x] = 0$ for all $x \in U$, a contradiction. In another case, $f(x_1,\ldots,x_n)^2 \in C$ for all $x_1,\ldots,x_n \in U$ and $G(x) = bx + [q,x] = bx$ for all $x \in U$, with $b \in C$. Hence $d(b) \in C$. This gives our conclusion.

Consider now the case when $\beta = 0$. Thus $d = \alpha^{-1}ad_p$, that is $d(x) = [q,x]$ for all $x \in R$, where $q = \alpha^{-1}p$. In this case δ is not an inner derivation. By (5) and by using Kharchenko's Theorem [9], U satisfies

$$\begin{aligned} & [q, bf(x_1,\ldots,x_n)^2 + f^\delta(x_1,\ldots,x_n)f(x_1,\ldots,x_n) \\ & + \sum_i f(x_1,\ldots,y_i,\ldots,x_n)f(x_1,\ldots,x_n)] \in C. \end{aligned} \tag{9}$$

In particular, U satisfied the blended component

$$[q, \sum_i f(x_1,\ldots,y_i,\ldots,x_n)f(x_1,\ldots,x_n)] \in C. \tag{10}$$

Let $p^{'}$ be a non-central element of U. Then for all $y_i = [p^{'}, x_i]$, we have

$$[q, [p^{'}, f(x_1,\ldots,x_n)]f(x_1,\ldots,x_n)] \in C.$$

By Lemma 2.4, it implies $p^{'} \in C$, a contradiction.

Finally we consider the case when both $\alpha \neq 0$ and $\beta \neq 0$. In this situation $d = \alpha^{-1} ad_p - \alpha^{-1}\beta\delta$, that is $d(x) = [q, x] + \gamma\delta(x)$ for all $x \in U$, where $q = \alpha^{-1}p$ and $\gamma = -\alpha^{-1}\beta \neq 0$. Here δ is not an inner derivation by our assumption. Then by (5), U satisfies

$$\begin{aligned} & [q, bf(x_1,\ldots,x_n)^2 + \delta(f(x_1,\ldots,x_n))f(x_1,\ldots,x_n)] \\ & + \gamma\{\delta(b)f(x_1,\ldots,x_n)^2 + b\delta(f(x_1,\ldots,x_n)^2) \\ & + \delta^2(f(x_1,\ldots,x_n))f(x_1,\ldots,x_n) \\ & + \delta(f(x_1,\ldots,x_n))\delta(f(x_1,\ldots,x_n))\} \in C. \end{aligned} \tag{11}$$

We have

$$\delta(f(x_1,\ldots,x_n)) = f^{\delta}(x_1,\ldots,x_n) + \sum_i f(x_1,\ldots,\delta(x_i),\ldots,x_n)$$

and

$$\begin{aligned} & \delta^2(f(x_1,\ldots,x_n)) \\ = & f^{\delta^2}(x_1,\ldots,x_n) + 2\sum_i f^{\delta}(x_1,\ldots,\delta(x_i),\ldots,x_n) \\ & + \sum_i f(x_1,\ldots,\delta^2(x_i),\ldots,x_n) + \sum_{i\neq j} f(x_1,\ldots,\delta(x_i),\ldots,\delta(x_j),\ldots,x_n). \end{aligned}$$

By applying Kharchenko's theorem [9] to (11), we can replace $\delta(f(x_1,\ldots,x_n))$ with $f^{\delta}(x_1,\ldots,x_n) + \sum_i f(x_1,\ldots,y_i,\ldots,x_n)$ and $\delta^2(f(x_1,\ldots,x_n))$ with

$$\begin{aligned} & f^{\delta^2}(x_1,\ldots,x_n) + 2\sum_i f^{d}(x_1,\ldots,y_i,\ldots,x_n) \\ & + \sum_i f(x_1,\ldots,z_i,\ldots,x_n) + \sum_{i\neq j} f(x_1,\ldots,y_i,\ldots,y_j,\ldots,x_n) \end{aligned}$$

in (11) and then U satisfies blended component

$$\gamma \sum_i f(x_1,\ldots,z_i,\ldots,x_n)f(x_1,\ldots,x_n) \in C.$$

Since $\gamma \neq 0$, we have $\sum_i f(x_1,\ldots,z_i,\ldots,x_n)f(x_1,\ldots,x_n) \in C$. Let p be a non-central element of U. Then for all $z_i = [p, x_i]$, we have

$$[p, f(x_1,\ldots,x_n)]f(x_1,\ldots,x_n) \in C,$$

which implies $p \in C$ (see [13]), a contradiction.

Case-2: Let d and δ be linearly C-independent modulo inner derivations of U.

In this case by (6), U satisfies

$$\begin{aligned} & d(b)f(x_1,\ldots,x_n)^2 + bd(f(x_1,\ldots,x_n))f(x_1,\ldots,x_n) \\ & + bf(x_1,\ldots,x_n)d(f(x_1,\ldots,x_n)) \end{aligned}$$

$$+ d(\delta(f(x_1,\ldots,x_n)))f(x_1,\ldots,x_n) + \delta(f(x_1,\ldots,x_n))d(f(x_1,\ldots,x_n)) \in C,$$

which gives

$$\begin{gathered}
d(b)f(x_1,\ldots,x_n)^2 \\
+b\Big\{f^d(x_1,\ldots,x_n) + \sum_i f(x_1,\ldots,d(x_i),\ldots,x_n)\Big\}f(x_1,\ldots,x_n) \\
+bf(x_1,\ldots,x_n)\Big\{f^d(x_1,\ldots,x_n) + \sum_i f(x_1,\ldots,d(x_i),\ldots,x_n)\Big\} \\
+\Big\{f^{d\delta}(x_1,\ldots,x_n) + 2\sum_i f^d(x_1,\ldots,\delta(x_i),\ldots,x_n) \\
+\sum_i f(x_1,\ldots,d\delta(x_i),\ldots,x_n) + \sum_{i\neq j} f(x_1,\ldots,d(x_i),\ldots,\delta(x_j),\ldots,x_n)\Big\}f(x_1,\ldots,x_n) \\
+\Big\{f^\delta(x_1,\ldots,x_n) + \sum_i f(x_1,\ldots,\delta(x_i),\ldots,x_n)\Big\} \\
.\Big\{f^d(x_1,\ldots,x_n) + \sum_i f(x_1,\ldots,d(x_i),\ldots,x_n)\Big\} \in C. \qquad (12)
\end{gathered}$$

Since both d and δ are outer derivations of U, by Kharchenko's theorem [9], U satisfies

$$\begin{gathered}
d(b)f(x_1,\ldots,x_n)^2 \\
+b\Big\{f^d(x_1,\ldots,x_n) + \sum_i f(x_1,\ldots,z_i,\ldots,x_n)\Big\}f(x_1,\ldots,x_n) \\
+bf(x_1,\ldots,x_n)\Big\{f^d(x_1,\ldots,x_n) + \sum_i f(x_1,\ldots,z_i,\ldots,x_n)\Big\} \\
+\Big\{f^{d\delta}(x_1,\ldots,x_n) + 2\sum_i f^d(x_1,\ldots,y_i,\ldots,x_n) \\
+\sum_i f(x_1,\ldots,t_i,\ldots,x_n) + \sum_{i\neq j} f(x_1,\ldots,z_i,\ldots,y_j,\ldots,x_n)\Big\}f(x_1,\ldots,x_n) \\
+\Big\{f^\delta(x_1,\ldots,x_n) + \sum_i f(x_1,\ldots,y_i,\ldots,x_n)\Big\} \\
.\Big\{f^d(x_1,\ldots,x_n) + \sum_i f(x_1,\ldots,z_i,\ldots,x_n)\Big\} \in C. \qquad (13)
\end{gathered}$$

In particular, U satisfies the blended component

$$\sum_i f(x_1,\ldots,t_i,\ldots,x_n)f(x_1,\ldots,x_n) \in C.$$

Let p be a non-central element of U. Then for all $t_i = [p, x_i]$, we have

$$[p, f(x_1,\ldots,x_n)]f(x_1,\ldots,x_n) \in C,$$

which implies $p \in C$, a contradiction. Thus, the theorem is proved. □

As an immediate application of the main theorem, we have the following corollary.

Corollary 2.5. *Let R be a noncommutative prime ring of characteristic different from 2, U the Utumi quotient ring of R, C the extended centroid of R and $f(x_1, \dots, x_n)$ a noncentral multilinear polynomial over C in n noncommuting variables. Denote by $f(R)$ the set of all the evaluations of $f(x_1, \dots, x_n)$ on R. If F and G are two nonzero generalized derivations of R such that*

$$[F(u)u, G(v)v] \in Z(R),$$

for all $u, v \in f(R)$, then $f(x_1, \dots, x_n)^2$ is central-valued on R and

(i) there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$;

(ii) there exists $\mu \in C$ such that $G(x) = \mu x$ for all $x \in R$;

(iii) there exist $b, b' \in U$ such that $G(x) = bx$ and $F(x) = b'x$ for all $x \in R$, with $[b, b'] \in C$.

Proof. If $F(u)u \in Z(R)$ for all $u \in f(R)$, then by [7], we have $F(x) = \lambda x$ for all $x \in R$, for some $\lambda \in C$ and $f(x_1, \dots, x_n)^2$ is central-valued on R, which is our conclusion (i).

Similarly, if $G(v)v \in C$ for all $v \in f(R)$, we have our conclusions (ii).

If $F(f(x_1, \dots, x_n))f(x_1, \dots, x_n)$ is not central valued on R, then we choose $x_1, \dots, x_n \in R$ such that $F(f(x_1, \dots, x_n)f(x_1, \dots, x_n) = a \notin C$. By hypothesis

$$[a, G(f(r))f(r)] \in Z(R)$$

for all $r = (r_1, \dots, r_n) \in R^n$. By Main Theorem, $f(x_1, \dots, x_n)^2$ is central-valued on R and there exists $b \in U$ such that $G(x) = bx$ for all $x \in U$, with $[b, F(u)u] \in C$ for all $u \in f(R)$. Then again by Main Theorem, we conclude that either $b \in C$ or there exists $b' \in U$ such that $F(x) = b'x$ for all $x \in U$ with $[b, b'] \in C$. Thus we obtain our conclusion (iii). □

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Priyadwip Das
Department of Mathematics
Jadavpur University
Kolkata-700032, W.B., India
Email address: priyadip72@gmail.com

Basudeb Dhara
Department of Mathematics
Belda College
Belda, Paschim Medinipur, 721424, W.B., India
Email address: basu_dhara@yahoo.com

Sukhendu Kar
Department of Mathematics
Jadavpur University
Kolkata-700032, W.B., India
Email address: karsukhendu@yahoo.co.in