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REDHEFFER TYPE INEQUALITIES FOR THE FOX-WRIGHT FUNCTIONS

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Abstract. In this note, new sharpened Redheffer type inequalities related to the Fox–Wright functions are established. As consequence, we show new Redheffer type inequalities for hypergeometric functions and for the four–parametric Mittag–Leffler functions with best possible exponents.

1. Introduction and main results

In 1969, Redheffer [7] posed the problem of proving the inequality

$$\frac{\pi^2 - x^2}{\pi^2 + x^2} \leq \frac{\sin x}{x}, \; x \in (0, \pi]. \tag{1}$$

Williams [8] proved this inequality. Motivated by this inequality recently Zhu and Sun [11], using the fact that the hyperbolic functions $\sinh x$ and $\cosh x$ have no zeros in $(0, \infty)$ established the following Redheffer–type inequalities:

$$\left(\frac{r^2 + x^2}{r^2 - x^2}\right)^{\alpha} \leq \frac{\sinh x}{x} \leq \left(\frac{r^2 + x^2}{r^2 - x^2}\right)^{\beta} \tag{2}$$

and

$$\left(\frac{r^2 + x^2}{r^2 - x^2}\right)^{\alpha} \leq \cosh x \leq \left(\frac{r^2 + x^2}{r^2 - x^2}\right)^{\beta_1}, \tag{3}$$

where $0 < x < r$, $\alpha \leq 0$, $\beta \geq \frac{r^2}{12}$, and $\beta_1 \geq \frac{r^2}{4}$.

Recently, some extensions of inequalities (2) and (3) involving modified Bessel function have been shown by Zhu [10] and Mehrez [3], as follows:

Theorem A. *Let $0 < x < r$ and $\nu > -1$. Then the following inequalities*

$$\left(\frac{r^2 + z^2}{r^2 - z^2}\right)^{\alpha} \leq \mathcal{I}_\nu(z) \leq \left(\frac{r^2 + z^2}{r^2 - z^2}\right)^{\beta} \tag{4}$$

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hold if and only if $\alpha \le 0$ *and* $\beta \ge \frac{r^2}{8(\nu+1)}$, *where* $\mathcal{I}_\nu(z)$ *is the normalized modified Bessel function of the first kind, defined by*

$$\mathcal{I}_\nu(z) = \sum_{k=0}^{\infty} \frac{\Gamma(\nu+1) z^{2k}}{2^{2k} k! \Gamma(\nu+k+1)}.$$

Moreover, the author of this paper extended and sharpened the inequalities (4), as follows [4]:

Theorem B. *Let* $r > 0$ *and* $\alpha, \beta > 0$. *Then the following inequalities*

$$\left(\frac{r+z}{r-z}\right)^{\sigma_{\alpha,\beta}} \le \mathcal{W}_{\alpha,\beta}(z) \le \left(\frac{r+z}{r-z}\right)^{\gamma_{\alpha,\beta}} \tag{5}$$

hold for all $0 < z < r$, *where* $\sigma_{\alpha,\beta} = 0$ *and* $\gamma_{\alpha,\beta} = \frac{r\Gamma(\beta)}{2\Gamma(\beta+\alpha)}$ *are the best possible constants, and* $\mathcal{W}_{\alpha,\beta}(z)$ *is the normalized Wright function defined by*

$$\mathcal{W}_{\alpha,\beta}(z) = \Gamma(\beta) \sum_{k=0}^{\infty} \frac{z^k}{k! \Gamma(\beta + k\alpha)}, \ \alpha > -1, \ \beta \in \mathbb{C}.$$

The Fox-Wright function ${}_p\Psi_q$ is a generalization of the familiar hypergeometric function ${}_pF_q$ with p numerator and q denominator parameters (see [2]), defined by (cf., e.g., [9, p. 4, Eq. (2.4)]

$${}_p\Psi_q\left[{}^{(\alpha_1,A_1),\dots,(\alpha_p,A_p)}_{(\beta_1,B_1),\dots,(\beta_q,B_q)}\Big| z\right] = \sum_{k=0}^{\infty} \frac{\prod_{l=1}^{p} \Gamma(\alpha_l + kA_l)}{\prod_{l=1}^{q} \Gamma(\beta_l + kB_l)} \frac{z^k}{k!}, \tag{6}$$

where $A_l \ge 0$, $l = 1, \dots, p$; $B_l \ge 0$, $l = 1, \dots, q$; such that $1 + \sum_{l=1}^{q} B_l - \sum_{l=1}^{p} A_l > 0$ for suitably bounded values of $|z|$. The generalized hypergeometric function ${}_pF_q$ is defined by

$${}_pF_q\left[{}^{\alpha_1,\dots,\alpha_p}_{\beta_1,\dots,\beta_q}\Big| z\right] = \sum_{k=0}^{\infty} \frac{\prod_{l=1}^{p} (\alpha_l)_k}{\prod_{l=1}^{q} (\beta_l)_k} \frac{z^k}{k!}, \tag{7}$$

where, as usual, we make use of the following notation:

$$(\tau)_0 = 1, \text{and } \ (\tau)_k = \tau(\tau+1)\cdots(\tau+k-1) = \frac{\Gamma(\tau+k)}{\Gamma(\tau)}, \ k \in \mathbb{N},$$

to denote the shifted factorial or the Pochhammer symbol. Obviously, we find from the definitions (6) and (7) that

$${}_p\Psi_q\left[{}^{(\alpha_1,1),\dots,(\alpha_p,1)}_{(\beta_1,1),\dots,(\beta_q,1)}\Big| z\right] = \frac{\Gamma(\alpha_1)\cdots\Gamma(\alpha_p)}{\Gamma(\beta_1)\cdots\Gamma(\beta_q)} {}_pF_q\left[{}^{\alpha_1,\dots,\alpha_p}_{\beta_1,\dots,\beta_q}\Big| z\right]. \tag{8}$$

The Mittag–Leffler functions with $2n$ parameters are defined for $B_j \in \mathbb{R}$ $(B_1^2 + \cdots + B_n^2 \neq 0)$ and $\beta_j \in \mathbb{C}$ $(j = 1, \dots, n \in \mathbb{N})$ by the series

$$E_{(\beta,B)_n}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\prod_{j=1}^{n} \Gamma(\beta_j + kB_j)}, \ z \in \mathbb{C}. \tag{9}$$

When $n = 1$, the definition in (9) coincides with the definition of the two–parametric Mittag–Leffler function [5]

$$E_{(\beta,B)_1}(z) = E_{\beta,B}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta + kB)},\ z \in \mathbb{C}, \tag{10}$$

and similarly for $n = 2$, where $E_{(\beta,B)_2}(z)$ coincides with the four–parametric Mittag–Leffler function

$$E_{(\beta,B)_2}(z) = E_{\beta_1,B_1;\beta_2,B_2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta_1 + kB_1)\Gamma(\beta_2 + kB_2)},\ z \in \mathbb{C}. \tag{11}$$

The generalized $2n$-parametric Mittag-Leffler function $E_{(\beta,B)_n}(z)$ can be represented in terms of the Fox–Wright hypergeometric function ${}_p\Psi_q(z)$ by

$$E_{(\beta,B)_n}(z) = {}_1\Psi_n\Big[{}^{(1,1)}_{(\beta_1,B_1),\dots,(\beta_q,B_q)} \Big| z \Big],\ z \in \mathbb{C}. \tag{12}$$

In the following, we define the function $\Phi^{(\beta_1,B_1)}_{\alpha_1,\beta_2} : (0,\infty) \longrightarrow \mathbb{R}$ by

$$\Phi^{(\beta_1,B_1)}_{\alpha_1,\beta_2}(z) = \frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\alpha_1)}\, {}_1\Psi_2\Big[{}^{(\alpha_1,1)}_{(\beta_1,B_1),(\beta_2,1)} \Big| z \Big],$$

where $\alpha_1, \beta_1, \beta_2 > 0$ and $B_1 > 0$.

In this paper, we shall extend and sharpen the inequalities (4) and (5) and obtain a general refinement of Redheffer type inequality involving the normalized Fox-Wright functions $\Phi^{(\beta_1,B_1)}_{\alpha_1,\beta_2}(z)$. As consequence, we show new Redheffer type inequalities for the hypergeometric function ${}_1F_2$ and for the four–parametric Mittag–Leffler function $\tilde{E}_{\beta_1,B_1;\beta_2,1}(z) = \Gamma(\beta_1)\Gamma(\beta_2)E_{\beta_1,B_1;\beta_2,1}(z)$ as follows.

Theorem 1.1. *Let $r, \alpha_1, \beta_1, \beta_2, B_1 > 0$. If $\alpha_1 \geq \beta_2$, then the following inequalities*

$$\left(\frac{r+z}{r-z}\right)^{\lambda^{(\beta_1,B_1)}_{\alpha_1,\beta_2}} \leq \Phi^{(\beta_1,B_1)}_{\alpha_1,\beta_2}(z) \leq \left(\frac{r+z}{r-z}\right)^{\mu^{(\beta_1,B_1)}_{\alpha_1,\beta_2}} \tag{13}$$

hold for all $z \in (0, r)$, where $\lambda^{(\beta_1,B_1)}_{\alpha_1,\beta_2} = 0$, and $\mu^{(\beta_1,B_1)}_{\alpha_1,\beta_2} = \frac{\alpha_1\Gamma(\beta_1)r}{2\beta_2\Gamma(\beta_1+B_1)}$, are the best possible constants.

Taking in (13) the value $B_1 = 1$ and using the identities (8), we obtain the Redheffer type inequalities for hypergeometric function ${}_1F_2$.

Corollary 1.2. *Let $r, \alpha_1, \beta_1, \beta_2 > 0$. If $\alpha_1 \geq \beta_2$, then the following inequalities*

$$\left(\frac{r+z}{r-z}\right)^{\lambda^{(\beta_1,B_1)}_{\alpha_1,\beta_2}} \leq {}_1F_2(\alpha_1; \beta_1, \beta_2; z) \leq \left(\frac{r+z}{r-z}\right)^{\mu^{(\beta_1,1)}_{\alpha_1,\beta_2}} \tag{14}$$

hold for all $z \in (0, r)$, where $\lambda^{(\beta_1,1)}_{\alpha_1,\beta_2} = 0$, and $\mu^{(\beta_1,1)}_{\alpha_1,\beta_2} = \frac{\alpha_1 r}{2\beta_2\beta_1}$, are the best possible constants.

Letting in (13) the value $\alpha_1 = 1$ and using the identities (12), we obtain the Redheffer type inequalities for the four–parametric Mittag–Leffler function $\tilde{E}_{\beta_1,B_1;\beta_2,1}(z)$.

Corollary 1.3. *Let* $r, \beta_1, B_1 > 0$*, If* $0 < \beta_2 \leq 1$*, then the following inequalities*

$$\left(\frac{r+z}{r-z}\right)^{\lambda_{1,\beta_2}^{(\beta_1,B_1)}} \leq \tilde{E}_{\beta_1,B_1;\beta_2,1}(z) \leq \left(\frac{r+z}{r-z}\right)^{\mu_{1,\beta_2}^{(\beta_1,B_1)}} \tag{15}$$

hold for all $z \in (0,r)$*, where* $\lambda_{1,\beta_2}^{(\beta_1,B_1)} = 0$*, and* $\mu_{1,\beta_2}^{(\beta_1,B_1)} = \frac{r\Gamma(\beta_1)}{2\beta_2\Gamma(\beta_1+B_1)}$*, are the best possible constants.*

Remarks. 1. We note that Theorem B is obtained by choosing $\alpha_1 = \beta_2,\ \beta_1 = \beta$ and $B_1 = \alpha$ in (13).

2. Taking in (13) the values $\alpha_1 = \beta_2,\ \beta_1 = \beta, B_1 = \alpha,\ z = x^2/4, \alpha = 1,\ \beta = \nu + 1$ where $\nu > -1$ and replacing r by $r^2/4$ in (5), we obtain Theorem A.

2. Proof of the main results

In the proof of the main result we will need the following two lemmas. The first lemma is about the monotonicity of two power series. For more details, one may see [6].

Lemma 2.1. *Let* $\{a_n\}_{n\geq 0}$ *and* $\{b_n\}_{n\geq 0}$ *be two sequences of real numbers, and let the power series* $f(x) = \sum_{n\geq 0} a_n x^n$ *and* $g(x) = \sum_{n\geq 0} b_n x^n$ *be convergent for* $|x| < r$*. If* $b_n > 0$ *for* $n \geq 0$ *and if the sequence* $\{a_n/b_n\}_{n\geq 0}$ *is* (*strictly*) *increasing* (*decreasing*)*, then the function* $x \mapsto f(x)/g(x)$ *is* (*strictly*) *increasing* (*decreasing*) *on* $(0,r)$*.*

The second lemma is the so-called monotone form of l'Hospital's rule, see [1] for a proof.

Lemma 2.2. *Let* $f, g : [a,b] \longrightarrow \mathbb{R}$ *be two continuous functions which are differentiable on* (a,b)*. Further, let* $g' \neq 0$ *on* (a,b)*. If* f'/g' *is increasing* (*decreasing*) *on* (a,b)*, then the functions*

$$x \mapsto \frac{f(x)-f(a)}{g(x)-g(a)} \quad \text{and} \quad x \mapsto \frac{f(x)-f(b)}{g(x)-g(b)}$$

are also increasing (*decreasing*) *on* (a,b)*.*

Now, we are ready to prove the main result.

Proof of Theorem 1.1. By using the definition of the function $\Phi_{\alpha_1,\beta_2}^{(\beta_1,B_1)}(z)$ we have

$$\left(\Phi_{\alpha_1,\beta_2}^{(\beta_1,B_1)}(z)\right)' = \frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\alpha_1)} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha_1+k+1)z^k}{k!\Gamma(\beta_2+k+1)\Gamma(\beta_1+(k+1)B_1)}. \tag{16}$$

Let

$$K(z) = \frac{\log \Phi_{\alpha_1,\beta_2}^{(\beta_1,B_1)}(z)}{\log\left(\frac{r+z}{r-z}\right)} = \frac{f(z)}{g(z)},$$

where $f(z) = \log \Phi_{\alpha_1,\beta_2}^{(\beta_1,B_1)}(z)$ and $g(z) = \log\left(\frac{r+z}{r-z}\right)$. Then

$$\frac{f'(z)}{g'(z)} = \frac{(r^2-z^2)\left(\Phi_{\alpha_1,\beta_2}^{(\beta_1,B_1)}(z)\right)'}{2r\Phi_{\alpha_1,\beta_2}^{(\beta_1,B_1)}(z)} = \frac{A(z)}{2rB(z)},$$

where $A(z) = (r^2-z^2)\left(\Phi_{\alpha_1,\beta_2}^{(\beta_1,B_1)}(z)\right)'$ and $B(z) = \Phi_{\alpha_1,\beta_2}^{(\beta_1,B_1)}(z)$. By computation we get

$$\begin{aligned}
A(z) &= (r^2-z^2)\left(\Phi_{\alpha_1,\beta_2}^{(\beta_1,B_1)}(z)\right)' \\
&= \frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\alpha_1)}(r^2-z^2)\sum_{k=0}^{\infty}\frac{\Gamma(\alpha_1+k+1)z^k}{k!\Gamma(\beta_2+k+1)\Gamma(\beta_1+(k+1)B_1)} \\
&= \frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\alpha_1)}\left(\sum_{k=0}^{\infty}\frac{r^2\Gamma(\alpha_1+k+1)z^k}{k!\Gamma(\beta_2+k+1)\Gamma(\beta_1+(k+1)B_1)}\right. \\
&\quad \left.-\sum_{k=2}^{\infty}\frac{\Gamma(\alpha_1+k-1)z^k}{(k-2)!\Gamma(\beta_2+k-1)\Gamma(\beta_1+(k-1)B_1)}\right) \\
&= \frac{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\alpha_1+1)r^2}{\Gamma(\alpha_1)\Gamma(\beta_2+1)\Gamma(\beta_1+B_1)} + \frac{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\alpha_1+2)r^2}{\Gamma(\beta_2+2)\Gamma(\alpha_1)\Gamma(\beta_1+2B_1)}z \\
&\quad + \frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\alpha_1)}\sum_{k=2}^{\infty}\left(\frac{r^2\Gamma(\alpha_1+k+1)}{k!\Gamma(\beta_2+k+1)\Gamma(\beta_1+(k+1)B_1)}\right. \\
&\quad \left.-\frac{\Gamma(\alpha_1+k-1)}{(k-2)!\Gamma(\beta_2+k-1)\Gamma(\beta_1+(k-1)B_1)}\right)z^k \\
&:= \sum_{k=0}^{\infty}a_k z^k,
\end{aligned} \tag{17}$$

where $a_0 = \frac{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\alpha_1+1)r^2}{\Gamma(\alpha_1)\Gamma(\beta_2+1)\Gamma(\beta_1+B_1)}$ and $a_1 = \frac{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\alpha_1+2)r^2}{\Gamma(\beta_2+2)\Gamma(\alpha_1)\Gamma(\beta_1+2B_1)}$ and a_k is defined for $k \geq 2$ by

$$\begin{aligned}
a_k = \frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\alpha_1)}&\left(\frac{r^2\Gamma(\alpha_1+k+1)}{k!\Gamma(\beta_2+k+1)\Gamma(\beta_1+(k+1)B_1)}\right. \\
&\left.-\frac{\Gamma(\alpha_1+k-1)}{(k-2)!\Gamma(\beta_2+k-1)\Gamma(\beta_1+(k-1)B_1)}\right).
\end{aligned}$$

On the other hand, we write $B(z)$ in the following form:

$$B(z)=\sum_{k=0}^{\infty} b_k z^k,$$

where

$$b_k=\frac{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\alpha_1+k)}{k!\Gamma(\alpha_1)\Gamma(\beta_2+k)\Gamma(\beta_1+kB_1)} \text{ for all } \ k\geq 0.$$

Now, we consider the sequence $u_k=a_k/b_k$ by $u_0=a_0,\ u_1=a_1/b_1$ and for $k\geq 2$

$$\begin{aligned}u_k=&\frac{\Gamma(\beta_2+k)\Gamma(\beta_1+kB_1)\Gamma(\alpha_1+k+1)r^2}{\Gamma(\alpha_1+k)\Gamma(\beta_2+k+1)\Gamma(\beta_1+(k+1)B_1)}\\&-\frac{k!\Gamma(\alpha_1+k-1)\Gamma(\beta_2+k)\Gamma(\beta_1+kB_1)}{(k-2)!\Gamma(\beta_2+k-1)\Gamma(\alpha_1+k)\Gamma(\beta_1+(k-1)B_1)}.\end{aligned}$$

Since $\alpha_1\geq\beta_2$, we have

$$\begin{aligned}u_1-u_0&=\frac{\Gamma(\alpha_1+2)\Gamma(\beta_2+1)\Gamma(\beta_1+B_1)r^2}{\Gamma(\alpha_1+1)\Gamma(\beta_2+2)\Gamma(\beta_1+2B_1)}-\frac{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\alpha_1+1)r^2}{\Gamma(\alpha_1)\Gamma(\beta_2+1)\Gamma(\beta_1+B_1)}\\&=\frac{(\alpha_1+1)\Gamma(\beta_1+B_1)}{(\beta_2+1)\Gamma(\beta_1+2B_1)}-\frac{\alpha_1\Gamma(\beta_1)}{\beta_2\Gamma(\beta_1+B_1)}\qquad(18)\\&\leq\frac{\alpha_1}{\beta_2}\left(\frac{\Gamma(\beta_1+B_1)}{\Gamma(\beta_1+2B_1)}-\frac{\Gamma(\beta_1)}{\Gamma(\beta_1+B_1)}\right)\\&\leq 0.\end{aligned}$$

Indeed, due to log-convexity property of the Gamma function $\Gamma(z)$, the ratio $z\mapsto\Gamma(z+a)/\Gamma(z)$ is increasing on $(0,\infty)$, when $a>0$. Thus implies that the following inequality:

$$\frac{\Gamma(z+a)}{\Gamma(z)}\leq\frac{\Gamma(z+a+b)}{\Gamma(z+b)}\qquad(19)$$

holds for all $a,b,z>0$. Let $z=\beta_1$ and $a=b=B_1$ in (19) and using the inequality (18) we deduce that $u_1\leq u_0$. On the other hand, we have

$$\begin{aligned}u_2-u_1=&\frac{\Gamma(\alpha_1+2)\Gamma(\beta_2+1)}{\Gamma(\alpha_1+1)\Gamma(\beta_2+2)}\left[\frac{(\alpha_1+2)(\beta_2+1)\Gamma(\beta_1+2B_1)}{(\alpha_1+1)(\beta_2+2)\Gamma(\beta_1+3B_1)}-\frac{\Gamma(\beta_1+B_1)}{\Gamma(\beta_1+2B_1)}\right]\\&-\frac{2\Gamma(\alpha_1+1)\Gamma(\beta_2+2)\Gamma(\beta_1+2B_1)}{\Gamma(\alpha_1+2)\Gamma(\beta_2+1)}\\\leq&\frac{\Gamma(\alpha_1+2)\Gamma(\beta_2+1)}{\Gamma(\alpha_1+1)\Gamma(\beta_2+2)}\left[\frac{\Gamma(\beta_1+2B_1)}{\Gamma(\beta_1+3B_1)}-\frac{\Gamma(\beta_1+B_1)}{\Gamma(\beta_1+2B_1)}\right]\qquad(20)\\&-\frac{2\Gamma(\alpha_1+1)\Gamma(\beta_2+2)\Gamma(\beta_1+2B_1)}{\Gamma(\alpha_1+2)\Gamma(\beta_2+1)}\\\leq&\ 0.\end{aligned}$$

Indeed, in view of inequality (19) when $z=\beta_1+B_1,\ a=b=B_1$ and inequality (20) we deduce that $u_2 \leq u_1$. Now, let $k \geq 2$, we have

$$\begin{aligned} u_{k+1}-u_k = {} & r^2\left[\frac{(\alpha_1+k+1)\Gamma(\beta_1+(k+1)B_1)}{(\beta_2+k+1)\Gamma(\beta_1+(k+2)B_1)}-\frac{(\alpha_1+k)\Gamma(\beta_1+kB_1)}{(\beta_2+k)\Gamma(\beta_1+(k+1)B_1)}\right] \\ & +\frac{k!}{(k-2)!}\left[\frac{(\beta_2+k-1)\Gamma(\beta_1+kB_1)}{(\alpha_1+k-1)\Gamma(\beta_1+(k-1)B_1)}\right. \\ & \left.-\frac{(k+1)(\beta_2+k)\Gamma(\beta_1+(k+1)B_1)}{(k-1)(\alpha_1+k)\Gamma(\beta_1+kB_1)}\right] \\ \leq {} & \frac{(\alpha_1+k)r^2}{(\beta_2+k)}\left[\frac{\Gamma(\beta_1+(k+1)B_1)}{\Gamma(\beta_1+(k+2)B_1)}-\frac{\Gamma(\beta_1+kB_1)}{\Gamma(\beta_1+(k+1)B_1)}\right] \\ & +\frac{k!(\beta_2+k-1)}{(k-2)!(\alpha_1+k-1)}\left[\frac{\Gamma(\beta_1+kB_1)}{\Gamma(\beta_1+(k-1)B_1)}-\frac{\Gamma(\beta_1+(k+1)B_1)}{\Gamma(\beta_1+kB_1)}\right]. \end{aligned} \tag{21}$$

Setting in (19) the values $z=\beta_1+kB_1$ and $a=b=B_1$, we obtain the following Turán type inequality for the gamma function $\Gamma(z)$

$$\Gamma(\beta_1+kB_1)\Gamma(\beta_1+(k+2)B_1)-\Gamma^2(\beta_1+(k+1)B_1)\geq 0. \tag{22}$$

Similarly, letting in (19) the values $z=\beta_1+(k-1)B_1$ and $a=b=B_1$, we get

$$\Gamma(\beta_1+(k-1)B_1)\Gamma(\beta_1+(k+1)B_1)-\Gamma^2(\beta_1+kB_1)\geq 0. \tag{23}$$

In view of (18), (20), (21), (22) and (23) we deduce that the sequence $(u_k)_{k\geq 0}$ is decreasing. By using Lemma 2.1 we clearly have that f'/g' is decreasing on $(0,r)$, and consequently the function $K(z)$ is also decreasing $(0,r)$, by means of Lemma 2.2. On the other hand, by using the Bernoulli–l'Hospital's rule we obtain

$$\lim_{z\to 0}K(z)=\frac{u_0}{2r}=\frac{\alpha_1\Gamma(\beta_1)r}{2\beta_2\Gamma(\beta_1+B_1)}, \quad \text{and} \quad \lim_{z\to r}K(z)=0.$$

It is important to mention here that there is another proof of the inequalities (13). Namely, if we consider the function $\chi:(0,r)\longrightarrow\mathbb{R}$, defined by

$$\chi(z)=\frac{\alpha_1\Gamma(\beta_1)r}{2\beta_2\Gamma(\beta_1+B_1)}\log\left(\frac{r+z}{r-z}\right)-\log\Phi_{\alpha_1,\beta_2}^{(\beta_1,B_1)}(z).$$

Then,

$$\begin{aligned} & \Phi_{\alpha_1,\beta_2}^{(\beta_1,B_1)}(z)\chi'(z) \\ = {} & \frac{\alpha_1\Gamma(\beta_1)r^2}{\beta_2\Gamma(\beta_1+B_1)(r^2-z^2)}\Phi_{\alpha_1,\beta_2}^{(\beta_1,B_1)}(z)-\left(\Phi_{\alpha_1,\beta_2}^{(\beta_1,B_1)}(z)\right)' \\ = {} & \frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\alpha_1)}\left[\frac{\alpha_1\Gamma(\beta_1)r^2}{\beta_2\Gamma(\beta_1+B_1)(r^2-z^2)}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha_1+k)z^k}{k!\Gamma(\beta_1+kB_1)\Gamma(\beta_2+k)}\right. \\ & \left.-\sum_{k=0}^{\infty}\frac{\Gamma(\alpha_1+k+1)z^k}{k!\Gamma(\beta_1+kB_1+B_1)\Gamma(\beta_2+k+1)}\right] \end{aligned}$$

$$\geq \frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\alpha_1)}\left[\frac{\alpha_1\Gamma(\beta_1)}{\beta_2\Gamma(\beta_1+B_1)}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha_1+k)z^k}{k!\Gamma(\beta_1+kB_1)\Gamma(\beta_2+k)}\right.$$
$$\left.-\sum_{k=0}^{\infty}\frac{\Gamma(\alpha_1+k+1)z^k}{k!\Gamma(\beta_1+kB_1+B_1)\Gamma(\beta_2+k+1)}\right]$$
$$=\frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\alpha_1)}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha_1+k)z^k}{k!\Gamma(\beta_2+k)}\left(\frac{\alpha_1\Gamma(\beta_1)}{\beta_2\Gamma(\beta_1+B_1)\Gamma(\beta_1+kB_1)}\right.$$
$$\left.-\frac{\alpha_1+k}{(\beta_2+k)\Gamma((\beta_1+kB_1+B_1)}\right).$$

On the other hand, using the fact that $\alpha_1 \geq \beta_2$, we have $\frac{\alpha_1}{\beta_2} \geq \frac{\alpha_1+k}{\beta_2+k}$ for each $k \geq 0$, and consequently,

$$\Phi_{\alpha_1,\beta_2}^{(\beta_1,B_1)}(z)\chi'(z) \geq \frac{\alpha_1\Gamma(\beta_1)\Gamma(\beta_2)}{\beta_2\Gamma(\alpha_1)}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha_1+k)z^k}{k!\Gamma(\beta_2+k)}\left(\frac{\Gamma(\beta_1)}{\Gamma(\beta_1+B_1)\Gamma(\beta_1+kB_1)}\right.$$
$$\left.-\frac{1}{\Gamma((\beta_1+kB_1+B_1)}\right)z^k. \tag{24}$$

Now, taking in (19) the values $z=\beta_1,\ a=B_1$ and $b=kB_1$, we obtain

$$\Gamma(\beta_1)\Gamma(\beta_1+kB_1+B_1) \geq \Gamma(\beta_1+B_1)\Gamma(\beta_1+kB_1). \tag{25}$$

In view of inequalities (24) and (25) we deduce that the function $\chi(z)$ is increasing on $(0,r)$, and hence $\chi(z) \geq \chi(0) = 0$, which implies the right–hand side of inequalities (13). To prove the left–hand side of (13), by using (16) we deduce that the function $\Phi_{\alpha_1,\beta_2}^{(\beta_1,B_1)}(z)$ is increasing on $(0,\infty)$, and hence

$$\Phi_{\alpha_1,\beta_2}^{(\beta_1,B_1)}(z) \geq \Phi_{\alpha_1,\beta_2}^{(\beta_1,B_1)}(0) = 1.$$

This completes the proof of Theorem 1.1. □

References

[1] G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen, *Inequalities for quasiconformal mappings in space*, Pacific J. Math. **160** (1993), no. 1, 1–18.

[2] A. M. Mathai and R. K. Saxena, *The H-Function with Applications in Statistics and Other Disciplines*, Halsted Press, New York, 1978.

[3] K. Mehrez, *Redheffer type inequalities for modified Bessel functions*, Arab J. Math. Sci. **22** (2016), no. 1, 38–42.

[4] ———, *Functional inequalities for the Wright functions*, Integral Transforms Spec. Funct. **28** (2017), no. 2, 130–144.

[5] K. Mehrez and S. M. Sitnik, *Functional inequalities for the Mittag-Leffler functions*, Results Math. **72** (2017), no. 1-2, 703–714.

[6] S. Ponnusamy and M. Vuorinen, *Asymptotic expansions and inequalities for hypergeometric functions*, Mathematika **44** (1997), no. 2, 278–301.

[7] R. Redheffer, *Problems and Solutions*: *Advanced Problems*: *5642, 5665-5670*, Amer. Math. Monthly **76** (1969), no. 4, 422–423.

[8] R. Redheffer and J. P. Williams, *Problems and Solutions*: *Solutions of Advanced Problems*: *5642*, Amer. Math. Monthly **76** (1969), no. 10, 1153–1154.

[9] H. M. Srivastava and Tomovski, *Some problems and solutions involving Mathieu's series and its generalizations*, JIPAM. J. Inequal. Pure Appl. Math. **5** (2004), no. 2, Article 45, 13 pp.

[10] L. Zhu, *Extension of Redheffer type inequalities to modified Bessel functions*, Appl. Math. Comput. **217** (2011), no. 21, 8504–8506.

[11] L. Zhu and J. Sun, *Six new Redheffer-type inequalities for circular and hyperbolic functions*, Comput. Math. Appl. **56** (2008), no. 2, 522–529.

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