Commun. Korean Math. Soc. **0** (0), No. 0, pp. 1–0
https://doi.org/10.4134/CKMS.c180006
pISSN: 1225-1763 / eISSN: 2234-3024

EIGENVALUE MONOTONICITY OF (p, q)-LAPLACIAN ALONG THE RICCI-BOURGUIGNON FLOW

Shahroud Azami

Abstract. In this paper we study monotonicity the first eigenvalue for a class of (p, q)-Laplace operator acting on the space of functions on a closed Riemannian manifold. We find the first variation formula for the first eigenvalue of a class of (p, q)-Laplacians on a closed Riemannian manifold evolving by the Ricci-Bourguignon flow and show that the first eigenvalue on a closed Riemannian manifold along the Ricci-Bourguignon flow is increasing provided some conditions. At the end of paper, we find some applications in 2-dimensional and 3-dimensional manifolds.

1. Introduction

Given an n-dimensional closed Riemannian manifold (M, g_0), the Ricci-Bourguignon flow is the following evolution equation

$$\frac{d}{dt}g(t) = -2Ric(g(t)) + 2\rho R(g(t))g(t) = -2(Ric - \rho Rg), \tag{1}$$

with the initial condition

$$g(0) = g_0,$$

where Ric is the Ricci tensor of $g(t)$, R is the scalar curvature and ρ is a real constant. This evolution equation was introduced by Bourguignon for the first time in 1981 (see [4]) and it is a system of partial differential equations. Short time existence and uniqueness for solution to the Ricci-Bourguignon flow on $[0, T)$ have been shown by Catino and et al. in [7] for $\rho < \frac{1}{2(n-1)}$. When $\rho = 0$, the Ricci-Bourguignon flow is the Ricci flow.

At present, studying the eigenvalues of geometric operators is a very powerful tool for understanding Riemannian manifolds. In the past few years there

---

Received January 7, 2018; Revised April 3, 2018; Accepted April 25, 2018.
2010 *Mathematics Subject Classification.* 58C40, 53C44, 53C21.
Key words and phrases. Laplace, Ricci-Bourguignon flow, eigenvalue.

has been an increasing interest in geometric operators as p-Laplace and (p,q)-Laplace operators on Riemannain manifolds. There are many interesting properties about the eigenvalues of the geometric operator and geometrical invariants have been pointed out. In [22], Perelman introduced the energy functional

$$F=\int_M (R+|\nabla f|^2)e^{-f}\,d\mu$$

and showed that it is nondecreasing along the Ricci flow coupled to a backward heat-type equation, where R and $d\mu$ denote the scalar curvature and volume form of the metric $g=g(t)$, respectively. The nondecreasing of the functional F implies that the lowest eigenvalue of the geometric operator $-4\Delta+R$ is nondecreasing under the Ricci flow. Later, Li [19] and Cao [6] considered a general geometric operator $-\Delta+cR$, and both of them proved that the first eigenvalue of the geometric operator $-\Delta+cR$ for $c\geq\frac{1}{4}$ is nondecreasing along the Ricci flow without any curvature assumption. Then Wu [23], investigated the first eigenvalue monotonicity for the p-Laplace operator under the Ricci flow. On the other hand, Zeng and et al. [24] studied the monotonicity of eigenvalues of the operator $-\Delta+cR$ along the Ricci-Bourguignon flow. For the other recent research in this direction, see [8, 9, 13, 14, 18].

Let (M^n,g) be a closed Riemannian manifold. In this paper, we consider the nonlinear system introduced in [17], that is

$$\begin{cases}\Delta_p u=-\lambda|u|^{\alpha}|v|^{\beta}v \ \text{ in } M,\\ \Delta_q v=-\lambda|u|^{\alpha}|v|^{\beta}u \ \text{ in } M,\\ (u,v)\in W^{1,p}(M)\times W^{1,q}(M),\end{cases}\tag{2}$$

where $p>1,\ q>1$ and α,β are real numbers satisfying

$$\alpha>0,\ \beta>0,\quad \frac{\alpha+1}{p}+\frac{\beta+1}{q}=1.\tag{3}$$

The problem (2) has applications in mathematics and physics, for instance, if $p>2$, then (2) appears in the in the study of non-Newtonian fluids, pseudoplastics, if $1<p<2$, then it applies in reaction-diffusion problems, flows through porous media and if $p=\frac{4}{3}$ it arise in glaciology (see [10], [16]). Also, the eigenvalue problem for (2) was studied in several works for instance, the existence of a sequence of variational eigenvalues of problem (2) was proved in [11] by using the abstract theory developed by Amann in [1], and the existence of generalized eigenvalues was obtained in [12]. We refer the interested reader to [2, 3, 15, 20].

Motivated by the above works, in this paper we will study the first eigenvalue of a class of (p,q)-Laplace operator whose metric satisfies the Ricci-Bourguignon flow (1).

2. Preliminaries

Let M be a closed Riemannian manifold and $f : M \longrightarrow \mathbb{R}$ be a smooth function on M or $f \in W^{1,p}(M)$, the Sobolev space. The p-Laplacian of f for $1 < p < \infty$ is defined as

$$\begin{aligned} \triangle_p f &= div(|\nabla f|^{p-2}\nabla f) \\ &= |\nabla f|^{p-2}\Delta f + (p-2)|\nabla f|^{p-4}(Hessf)(\nabla f, \nabla f), \end{aligned} \tag{4}$$

where

$$(Hessf)(X,Y) = \nabla(\nabla f)(X,Y) = Y.(X.f) - (\nabla_Y X)f, \quad X, Y \in \mathcal{X}(M),$$

and

$$\Delta f = g^{ij}(\frac{\partial^2 f}{\partial x_i \partial x_j} - \Gamma^k_{ij}\frac{\partial f}{\partial x_k}).$$

We say that λ is an eigenvalue of (2), whenever for some $u \in W_0^{1,p}(M)$ and $v \in W_0^{1,q}(M)$,

$$\int_M |\nabla u|^{p-2}\langle \nabla u, \nabla \phi\rangle d\mu = \lambda \int_M |u|^{\alpha}|v|^{\beta} v\phi d\mu, \tag{5}$$

$$\int_M |\nabla v|^{q-2}\langle \nabla v, \nabla \psi\rangle d\mu = \lambda \int_M |u|^{\alpha}|v|^{\beta} u\psi d\mu, \tag{6}$$

where $\phi \in W^{1,p}(M)$, $\psi \in W^{1,q}(M)$ and $W_0^{1,p}(M)$ is the closure of $C_0^{\infty}(M)$ in Sobolev space $W^{1,p}(M)$. The pair (u,v) is called eigenfunctions. A first positive eigenvalue of (2) obtained as

$$\inf\{A(u,v) : (u,v) \in W_0^{1,p}(M) \times W_0^{1,q}(M),\ B(u,v) = 1,\ C(u,v) = D(u,v) = 0\},$$

where

$$\begin{aligned} A(u,v) &= \frac{\alpha+1}{p}\int_M |\nabla u|^p d\mu + \frac{\beta+1}{q}\int_M |\nabla v|^q d\mu, \\ B(u,v) &= \int_M |u|^{\alpha}|v|^{\beta} uv d\mu, \\ C(u,v) &= \int_M |u|^{\alpha}|v|^{\beta} v d\mu, \\ D(u,v) &= \int_M |u|^{\alpha}|v|^{\beta} u d\mu. \end{aligned}$$

Let $(M^n, g(t))$ be a solution of the Ricci-Bourguignon flow on the smooth closed manifold (M^n, g_0) in the interval $[0,T)$. Then

$$\lambda(t) = \frac{\alpha+1}{p}\int_M |\nabla u|^p d\mu_t + \frac{\beta+1}{q}\int_M |\nabla v|^q d\mu_t \tag{7}$$

defines the evolution of the first eigenvalue of (2), under the variation of $g(t)$ where the eigenfunctions associated to $\lambda(t)$ are normalized that is $B(u,v) =$

1, $C(u,v)=0$, $D(u,v)=0$. We prove some facts about the spectrum variation under a deformation of the metric given by the Ricci-Bourguignon flow equation.

3. Variation of $\lambda(t)$

In this section, we will give evolution formulas for $\lambda(t)$ under the Ricci-Bourguignon flow. Now, we give a useful statement about the variation of the first eigenvalue of (2) along the Ricci-Bourguignon flow.

Lemma 3.1. *If g_1 and g_2 are two metrics on Riemannian manifold M^n which satisfy $(1+\epsilon)^{-1}g_1 < g_2 < (1+\epsilon)g_1$, then for any $p \geq q > 1$, we have*

$$\lambda(g_2)-\lambda(g_1) \leq \left((1+\epsilon)^{\frac{p+n}{2}}-(1+\epsilon)^{-\frac{n}{2}}\right)\lambda(g_1).$$

In particular, $\lambda(t)$ is a continuous function with respect to the t-variable.

Proof. The proof is straightforward. We get

$$(1+\epsilon)^{-\frac{n}{2}}d\mu_{g_1} < d\mu_{g_2} < (1+\epsilon)^{\frac{n}{2}}d\mu_{g_1}.$$

Let

$$G(g,u,v)=\frac{\alpha+1}{p}\int_M |\nabla u|_g^p d\mu_g+\frac{\beta+1}{q}\int_M |\nabla v|_g^q d\mu_g, \tag{8}$$

hence

$$\begin{aligned}
&\int_M |u|^\alpha |v|^\beta uv d\mu_{g_1} G(g_2,u,v)-\int_M |u|^\alpha |v|^\beta uv d\mu_{g_2} G(g_1,u,v)\\
=&\frac{\alpha+1}{p}\int_M |u|^\alpha|v|^\beta uv d\mu_{g_1}\left(\int_M |\nabla u|_{g_2}^p d\mu_{g_2}-\int_M |\nabla u|_{g_1}^p d\mu_{g_1}\right)\\
&+\frac{\alpha+1}{p}\left(\int_M |u|^\alpha|v|^\beta uv d\mu_{g_1}-\int_M |u|^\alpha|v|^\beta uv d\mu_{g_2}\right)\int_M |\nabla u|_{g_1}^p d\mu_{g_1}\\
&+\frac{\beta+1}{q}\int_M |u|^\alpha|v|^\beta uv d\mu_{g_1}\left(\int_M |\nabla v|_{g_2}^q d\mu_{g_2}-\int_M |\nabla v|_{g_1}^q d\mu_{g_1}\right)\\
&+\frac{\beta+1}{q}\left(\int_M |u|^\alpha|v|^\beta uv d\mu_{g_1}-\int_M |u|^\alpha|v|^\beta uv d\mu_{g_2}\right)\int_M |\nabla v|_{g_1}^q d\mu_{g_1}\\
\leq&\frac{\alpha+1}{p}\left((1+\epsilon)^{\frac{p+n}{2}}-(1+\epsilon)^{-\frac{n}{2}}\right)\int_M |u|^\alpha|v|^\beta uv d\mu_{g_1}\int_M |\nabla u|_{g_1}^p d\mu_{g_1}\\
&+\frac{\beta+1}{q}\left((1+\epsilon)^{\frac{q+n}{2}}-(1+\epsilon)^{-\frac{n}{2}}\right)\int_M |u|^\alpha|v|^\beta uv d\mu_{g_1}\int_M |\nabla v|_{g_1}^q d\mu_{g_1}\\
\leq&\left((1+\epsilon)^{\frac{p+n}{2}}-(1+\epsilon)^{-\frac{n}{2}}\right)G(g_1,u,v)\int_M |u|^\alpha|v|^\beta uv d\mu_{g_1}.
\end{aligned}$$

Therefore we have

$$\lambda(g_2)-\lambda(g_1) \leq \left((1+\epsilon)^{\frac{p+n}{2}}-(1+\epsilon)^{-\frac{n}{2}}\right)\lambda(g_1).$$

This completes the proof of the lemma. □

Proposition 3.2. *Let $g(t)$, $t \in [0,T)$, be a solution of the Ricci-Bourguignon flow on a closed manifold M^n for $\rho < \frac{1}{2(n-1)}$ and let $\lambda(t)$ be the first eigenvalue of the (p,q)-Laplacian along this flow. For any t_1, $t_2 \in [0,T)$ and $t_2 > t_1$, we have*

$$\lambda(t_2) \geq \lambda(t_1) + \int_{t_1}^{t_2} \mathcal{G}(g(\tau), u(\tau), v(\tau)) d\tau, \tag{9}$$

where

$$\begin{aligned} \mathcal{G}(g(t),u(t),v(t)) = (\alpha+1)&\int_M \left(Ric(\nabla u,\nabla u) + \langle \nabla u', \nabla u\rangle\right)|\nabla u|^{p-2} d\mu \\ &+(\beta+1)\int_M \left(Ric(\nabla v,\nabla v) + \langle \nabla v', \nabla v\rangle\right)|\nabla v|^{q-2} d\mu \\ &-(\alpha+1)(\rho + \frac{1-\rho n}{p})\int_M |\nabla u|^p R d\mu \\ &-(\beta+1)(\rho + \frac{1-\rho n}{q})\int_M |\nabla v|^q R d\mu. \end{aligned} \tag{10}$$

Proof. Assume that

$$G(g(t),u(t),v(t)) = \frac{\alpha+1}{p}\int_M |\nabla u(t)|^p_{g(t)} d\mu_{g(t)} + \frac{\beta+1}{q}\int_M |\nabla v(t)|^q_{g(t)} d\mu_{g(t)},$$

and at time t_2, let $(u_2, v_2) = (u(t_2), v(t_2))$ be the eigenfunctions for the eigenvalue $\lambda(t_2)$ of (p,q)-Laplacian (2). We consider the following smooth functions

$$h(t) = u_2 \left[\frac{\det[g_{ij}(t_2)]}{\det[g_{ij}(t)]}\right]^{\frac{1}{2(\alpha+\beta+1)}}, \qquad l(t) = v_2 \left[\frac{\det[g_{ij}(t_2)]}{\det[g_{ij}(t)]}\right]^{\frac{1}{2(\alpha+\beta+1)}},$$

along the Ricci-Bourguignon flow. Let

$$u(t) = \frac{h(t)}{\left(\int_M |h(t)|^\alpha |l(t)|^\beta h(t) l(t) d\mu\right)^{\frac{1}{p}}}, \quad u(t) = \frac{l(t)}{\left(\int_M |h(t)|^\alpha |l(t)|^\beta h(t) l(t) d\mu\right)^{\frac{1}{q}}}$$

which $u(t)$, $v(t)$ are smooth functions under the Ricci-Bourguignon flow, satisfy

$$\int_M |u|^\alpha |v|^\beta uv d\mu = 1, \quad \int_M |u|^\alpha |v|^\beta v d\mu = 0, \quad \int_M |u|^\alpha |v|^\beta u d\mu = 0,$$

and at time t_2, $(u(t_2), v(t_2))$ is the eigenfunctions for $\lambda(t_2)$ of (p,q)-Laplacian (2), i.e., $\lambda(t_2) = G(g(t_2), u(t_2), v(t_2))$. Under the Ricci-Bourguignon flow we have

$$\begin{aligned} &\frac{d}{dt}(|\nabla f|^p) = p|\nabla f|^{p-2}\Big(Ric(\nabla f,\nabla f) - \rho R|\nabla f|^2 + \langle \nabla f', \nabla f\rangle\Big), \\ &\frac{d}{dt}(d\mu) = (-1+\rho n) R\, d\mu. \end{aligned} \tag{11}$$

Since $u(t)$ and $v(t)$ are smooth functions, therefore $G(g(t),u(t),v(t))$ is a smooth function with respect to t. Suppose that

$$\mathcal{G}(g(t),u(t),v(t)) := \frac{d}{dt}G(g(t),u(t),v(t)) \tag{12}$$

then $\mathcal{G}(g(t),u(t),v(t))$ is as form (10) and taking integration on the both sides of (12) between t_1 and t_2, we get

$$G(g(t_2),u(t_2),v(t_2)) - G(g(t_1),u(t_1),v(t_1)) = \int_{t_1}^{t_2} \mathcal{G}(g(\tau),u(\tau),v(\tau))d\tau, \tag{13}$$

where $t_1 \in [0,T)$ and $t_2 > t_1$. Noticing that $G(g(t_1),u(t_1),v(t_1)) \geq \lambda(t_1)$ and replacing $\lambda(t_2) = G(g(t_2),u(t_2),v(t_2))$ in (13), yields (9) and $\mathcal{G}(g(t),u(t),v(t))$ satisfies in (10). □

Theorem 3.3. *Let $(M^n, g(t))$ be a solution of the Ricci-Bourguignon flow on the smooth closed manifold (M^n, g_0) for $\rho < \frac{1}{2(n-1)}$, $n > 1$ and $\lambda(t)$ denotes the evolution of the first eigenvalue of (p,q)-Laplacian (2) under the Ricci-Bourguignon flow. If $k = \min\{p,q\}$ and there exists a non-negative constant a such that*

$$Ric - (\frac{1-n\rho}{k} + \rho)Rg \geq -ag \ \text{in } M^n \times [0,T) \tag{14}$$

and

$$R > \frac{ak}{1-n\rho} \ \text{ in } M^n \times \{0\}, \tag{15}$$

then $\lambda(t)$ is strictly increasing and differentiable almost everywhere along the Ricci-Bourguignon flow on $[0,T)$.

Proof. At time t_2, $u(t_2)$ and $v(t_2)$ are the eigenfunctions for $\lambda(t_2)$ of (p,q)-Laplacian (2), then $\int_M |u(t_2)|^{\alpha}|v(t_2)|^{\beta}u(t_2)v(t_2)d\mu_{g(t_2)} = 1$. Therefore

$$\begin{aligned} \mathcal{G}(g(t_2),u(t_2),v(t_2)) = &(\alpha+1)\int_M \left(Ric(\nabla u,\nabla u) + \langle\nabla u', \nabla u\rangle\right)|\nabla u|^{p-2}d\mu \\ &+(\beta+1)\int_M \left(Ric(\nabla v,\nabla v) + \langle\nabla v', \nabla v\rangle\right)|\nabla v|^{q-2}d\mu \\ &-(\alpha+1)(\rho + \frac{1-\rho n}{p})\int_M |\nabla u|^p R d\mu \\ &-(\beta+1)(\rho + \frac{1-\rho n}{q})\int_M |\nabla v|^q R d\mu. \end{aligned} \tag{16}$$

Now, the time derivative of the condition

$$\int_M |u|^{\alpha}|v|^{\beta}uv d\mu = 1,$$

yields

$$(\alpha+1)\int_M |u|^\alpha |v|^\beta u' v d\mu + (\beta+1)\int_M |u|^\alpha |v|^\beta u v' d\mu \\ = (1-n\rho)\int_M R|u|^\alpha |v|^\beta uv d\mu. \tag{17}$$

(5) and (6) imply that

$$\int_M \langle \nabla u', \nabla u\rangle |\nabla u|^{p-2} d\mu = \lambda \int_M |u|^\alpha |v|^\beta u' v d\mu, \tag{18}$$

$$\int_M \langle \nabla v', \nabla v\rangle |\nabla v|^{q-2} d\mu = \lambda \int_M |u|^\alpha |v|^\beta u v' d\mu. \tag{19}$$

Therefore from (17), (18) and (19) we have

$$(\alpha+1)\int_M \langle \nabla u', \nabla u\rangle |\nabla u|^{p-2} d\mu + (\beta+1)\int_M \langle \nabla v', \nabla v\rangle |\nabla v|^{q-2} d\mu \\ = (1-n\rho)\lambda \int_M R|u|^\alpha |v|^\beta uv d\mu. \tag{20}$$

Replacing (20) in (16), results that

$$\begin{aligned} \mathcal{G}(g(t_2), u(t_2), v(t_2)) = {} & (1-n\rho)\lambda(t_2)\int_M R|u|^\alpha |v|^\beta uv d\mu \\ & + (\alpha+1)\int_M Ric(\nabla u, \nabla u)|\nabla u|^{p-2} d\mu \\ & + (\beta+1)\int_M Ric(\nabla v, \nabla v)|\nabla v|^{q-2} d\mu \\ & - (\alpha+1)(\rho + \frac{1-\rho n}{p})\int_M |\nabla u|^p R d\mu \\ & - (\beta+1)(\rho + \frac{1-\rho n}{q})\int_M |\nabla v|^q R d\mu. \end{aligned} \tag{21}$$

From (21) and (14) we have

$$\begin{aligned} \mathcal{G}(g(t_2), u(t_2), v(t_2)) \geq {} & (1-n\rho)\lambda(t_2)\int_M R|u|^\alpha |v|^\beta uv d\mu \\ & + (1-n\rho)(\alpha+1)(\frac{1}{k} - \frac{1}{p})\int_M |\nabla u|^p R d\mu \\ & + (1-n\rho)(\beta+1)(\frac{1}{k} - \frac{1}{q})\int_M |\nabla v|^q R d\mu \\ & - (\alpha+1)a\int_M |\nabla u|^p d\mu - (\beta+1)a\int_M |\nabla v|^q d\mu. \end{aligned} \tag{22}$$

It is well-known that $R \geq \frac{ka}{1-n\rho}$ is preserved by the Ricci-Bourguignon flow. Also, by the strong maximum principle, we conclude that

$$R \geq \frac{ka}{1-n\rho} \quad \text{in } M^n \times [0,T).$$

Plugin this into (22) implies $\mathcal{G}(g(t_2), u(t_2), v(t_2)) > 0$ thus in any small enough neighborhood of t_2 we get $\mathcal{G}(g(t), u(t), v(t)) > 0$. Hence

$$\int_{t_1}^{t_2} \mathcal{G}(g(\tau), u(\tau), v(\tau)) d\tau > 0$$

for any $t_1 < t_2$ sufficiently close to t_1. Since $t_2 \in [0,T)$ is arbitrary, Proposition 3.2 completes the proof of the first part of theorem. For the differentiability for $\lambda(t)$, since $\lambda(t)$ is increasing and continuous on the interval $[0,T)$ by the classical Lebesgue's theorem (see [21]), $\lambda(t)$ is differentiable almost everywhere on $[0,T)$. □

Motivated by the works of X.-D. Cao [5, 6] and J. Y. Wu [23], like in the proof of Proposition 3.2, we first define a new smooth eigenvalue function and then we give an evolution formula for it. Let M be an n-dimensional closed Riemannian manifold and $g(t)$ be a smooth solution of the Ricci-Bourguignon flow. We define a smooth eigenvalue function

$$\lambda(u, v, t) := \frac{\alpha+1}{p} \int_M |\nabla u|^p d\mu + \frac{\beta+1}{q} \int_M |\nabla v|^q d\mu,$$

where u, v are smooth functions and satisfy

$$\int_M |u|^\alpha |v|^\beta uv d\mu = 1, \quad \int_M |u|^\alpha |v|^\beta v d\mu = 0, \quad \int_M |u|^\alpha |v|^\beta u d\mu = 0.$$

If (u, v) are the corresponding eigenfunctions of the first eigenvalue $\lambda(t)$ at t_0, then $\lambda(u, v, t_0) = \lambda(t_0)$. Like in the proof of Proposition 3.2 and Theorem 3.3, we get the following propositions.

Proposition 3.4. *Let $(M^n, g(t))$ be a solution of the Ricci-Bourguignon flow on the smooth closed manifold (M^n, g_0). If $\lambda(t)$ denotes the evolution of the first eigenvalue under the Ricci-Bourguignon flow, then*

$$\begin{aligned} \frac{d}{dt}\lambda(u, v, t)|_{t=t_0} = {} & (1-n\rho)\lambda(t_0) \int_M R|u|^\alpha |v|^\beta uv d\mu \\ & + (\alpha+1) \int_M Ric(\nabla u, \nabla u)|\nabla u|^{p-2} d\mu \\ & + (\beta+1) \int_M Ric(\nabla v, \nabla v)|\nabla v|^{q-2} d\mu \\ & - (\alpha+1)(\rho + \frac{1-\rho n}{p}) \int_M |\nabla u|^p R d\mu \\ & - (\beta+1)(\rho + \frac{1-\rho n}{q}) \int_M |\nabla v|^q R d\mu, \end{aligned} \tag{23}$$

where (u, v) is the associated normalized evolving eigenfunctions.

Theorem 3.5. *Let $(M^n, g(t))$ be a solution of the Ricci-Bourguignon flow on the smooth closed manifold (M^n, g_0), and let $\lambda(t)$ denote the evolution of the first eigenvalue of (p, q)-Laplacian (2) under the Ricci-Bourguignon flow. If $k = \min\{p, q\}$ and*

$$Ric - \left(\frac{1-n\rho}{k} + \rho\right) Rg > 0 \quad in \ \ M^n \times [0, T), \tag{24}$$

then $\lambda(t)(b-2(a-\rho)t)^{\frac{1-n\rho}{2(a-\rho)}}$ is strictly increasing under the Ricci-Bourguignon flow on $[0, T')$, where $a := \max\{\frac{1}{n}, \frac{n}{k^2}\}$, $\frac{1}{b} = \inf_M R(0)$ and $T' := \min\{\frac{b}{2(a-\rho)}, T\}$.

Proof. According to (23) and (24), we have

$$\begin{aligned} \frac{d}{dt}\lambda(u, v, t)|_{t=t_0} > (1-n\rho)\lambda(t_0) \int_M R|u|^{\alpha}|v|^{\beta} uv d\mu \\ + (1-n\rho)(\alpha+1)(\frac{1}{k} - \frac{1}{p}) \int_M |\nabla u|^p R d\mu \\ + (1-n\rho)(\beta+1)(\frac{1}{k} - \frac{1}{q}) \int_M |\nabla v|^q R d\mu. \end{aligned} \tag{25}$$

The evolution of the scalar curvature R under the Ricci-Bourguignon flow is

$$\frac{\partial R}{\partial t} = (1 - 2(n-1)\rho)\Delta R + 2|Ric|^2 - 2\rho R^2,$$

and the inequality $|Ric|^2 \geq aR^2$ $(a := \max\{\frac{1}{n}, \frac{n}{k^2}\})$ implies

$$\frac{\partial R}{\partial t} \geq (1 - 2(n-1)\rho)\Delta R + 2(a-\rho)R^2. \tag{26}$$

Since the solutions to $\frac{dy(t)}{dt} = 2(a-\rho)y^2(t)$ are $y(t) = \frac{1}{b-2(a-\rho)t}$, $t \in [0, T')$, where $\frac{1}{b} = \inf_M R(0)$ and $T' := \min\{\frac{b}{2(a-\rho)}, T\}$, using maximum principle to (26), we get $R(x, t) \geq \rho(t)$. Therefore (25) becomes

$$\frac{d}{dt}\lambda(u, v, t)|_{t=t_0} > (1-n\rho)\lambda(t_0)y(t_0), \tag{27}$$

and in any sufficiently small neighborhood of t_0 as I, we get

$$\frac{d}{dt}\lambda(u, v, t) > (1-n\rho)\lambda(u, v, t)\frac{1}{b-2(a-\rho)t}.$$

Integrating both sides of the last inequality with respect to t on $[t_1, t_0] \subset I$, we have

$$\ln \frac{\lambda(u(t_0), v(t_0), t_0)}{\lambda(u(t_1), v(t_1), t_1)} \geq \ln \left(\frac{b-2(a-\rho)t_0}{b-2(a-\rho)t_1}\right)^{\frac{-(1-n\rho)}{2(a-\rho)}}.$$

Since $\lambda(u(t_0), v(t_0), t_0) = \lambda(t_0)$ and $\lambda(u(t_1), v(t_1), t_1) \geq \lambda(t_1)$, we conclude that

$$\ln \frac{\lambda(t_0)}{\lambda(t_1)} \geq \ln \left(\frac{b-2(a-\rho)t_0}{b-2(a-\rho)t_1}\right)^{\frac{-(1-n\rho)}{2(a-\rho)}},$$

that is, $\lambda(t)(b-2(a-\rho)t)^{\frac{1-n\rho}{2(a-\rho)}}$ is strictly increasing closed to t_0. But t_0 is arbitrary, therefore $\lambda(t)(b-2(a-\rho)t)^{\frac{1}{2(a-\rho)}}$ is strictly increasing on $[0,T')$. □

Remark 3.6. If the function $(b-2(a-\rho)t)^{\frac{1-n\rho}{2(a-\rho)}}$ is decreasing, Theorem 3.5 also implies that $\lambda(t)$ is strictly increasing along the Ricci-Bourguignon flow on $[0,T')$.

3.1. Variation of $\lambda(t)$ on a surface

Now, we write Proposition 3.4 in some remarkable particular cases.

Corollary 3.7. *Let $(M^2,g(t))$ be a solution of the Ricci-Bourguignon flow on a closed surface (M^2,g_0). If $\lambda(t)$ denotes the evolution of the first eigenvalue of (p,q)-Laplacian (2) under the Ricci-Bourguignon flow, then*

$$\begin{aligned}\frac{d}{dt}\lambda(u,v,t)|_{t=t_0} &= (1-n\rho)\lambda(t_0)\int_M R|u|^\alpha|v|^\beta uv d\mu \\ &+\frac{(p-2)(1-2\rho)}{2p}(\alpha+1)\int_M|\nabla u|^p R d\mu \\ &+\frac{(q-2)(1-2\rho)}{2q}(\beta+1)\int_M|\nabla v|^q R d\mu,\end{aligned}\tag{28}$$

where (u,v) is the associated normalized evolving eigenfunctions.

Proof. In dimension $n=2$, we have $Ric=\frac{1}{2}Rg$. Then (23) implies (28). □

Lemma 3.8. *Let (M^2,g_0) be a closed surface with nonnegative scalar curvature. Then the first eigenvalue of (2) for $p\geq 2$ and $q\geq 2$ are increasing under the Ricci-Bourguignon flow for $\rho<\frac{1}{2}$.*

Proof. From [7], under the Ricci-Bourguignon flow on a surface, we have

$$\frac{\partial}{\partial t}R=(1-2\rho)\Delta R+(1-2\rho)R^2.$$

By the scalar maximum principle, the nonnegativity of the scalar curvature is preserved along the Ricci-Bourguignon flow. (28) implies that $\frac{d\lambda}{dt}(u,v,t)|_{t=t_0}>0$. Since t_0 is arbitrary then $\lambda(t)$ is increasing. □

3.2. Variation of $\lambda(t)$ on homogeneous manifolds

In this section, we consider the behavior of the spectrum when we evolve an initial homogeneous metric.

Proposition 3.9. *Let $(M^n,g(t))$ be a solution of the Ricci-Bourguignon flow on the smooth closed homogeneous manifold (M^n,g_0). If $\lambda(t)$ denote the evaluation of the first eigenvalue under the Ricci-Bourguignon flow, then*

$$\frac{d}{dt}\lambda(u,v,t)|_{t=t_0}=(\alpha+1)\int_M Ric(\nabla u,\nabla u)|\nabla u|^{p-2}d\mu$$

$$+ (\beta+1)\int_M Ric(\nabla v, \nabla v)|\nabla v|^{q-2}d\mu$$
$$- \rho(\alpha+1)\int_M |\nabla u|^p R d\mu - \rho(\beta+1)\int_M |\nabla v|^q R d\mu. \tag{29}$$

Proof. The evolving metric remains homogeneous and a homogeneous manifold has constant scalar curvature. Therefore (23) implies (29). ☐

3.3. Variation of $\lambda(t)$ on 3-dimensional manifolds

In this section, we consider the behavior of $\lambda(t)$ on 3-dimensional manifolds.

Proposition 3.10. *Let $(M^3, g(t))$ be a solution of the Ricci-Borguignon flow (1) on a closed manifold M^3 whose Ricci curvature is initially positive and there exists $0 \le \epsilon \le \frac{1}{3}$ such that*

$$Ric \ge \epsilon R g.$$

Then the quantity $e^{-\int_0^t A(\tau)d\tau}\lambda(t)$ is nondecreasing along the Ricci-Borguignon flow (1) for $0 < \rho < \frac{1}{4}$ on closed manifold M^3, where

$$A(t) = \frac{3\beta(1-3\rho+q\epsilon)}{3-2(1-3\rho)\beta t} + (3\rho - 1 - p\rho)\left(-2(1-\rho)t + \frac{1}{\alpha}\right)^{-1},$$

$\alpha = \max_{x\in M} R(0)$, $\beta = \min_{x\in M} R(0)$ and $q \le p$.

Proof. In [7], it has been shown that the pinching inequality $Ric \ge \epsilon R g$ and nonnegative scalar curvature are preserved along the Ricci-Borguignon flow (1) on closed manifold M^3, then using (23) we obtain

$$\begin{aligned}\frac{d}{dt}\lambda(u,v,t)|_{t=t_0} \ge{}& (1-3\rho)\lambda(t_0)\int_M R\,|u|^\alpha|v|^\beta uv d\mu + (\alpha+1)\epsilon\int_M R|\nabla u|^p d\mu \\ &+ (\beta+1)\epsilon\int_M R|\nabla v|^q d\mu - \rho(\alpha+1)\int_M |\nabla u|^p R d\mu \\ &+ (-1+3\rho)\frac{\alpha+1}{p}\int_M |\nabla u|^p R d\mu - \rho(\beta+1)\int_M |\nabla v|^q R d\mu \\ &+ (-1+3\rho)\frac{\beta+1}{q}\int_M |\nabla v|^q R d\mu.\end{aligned}$$

On the other hand, the scalar curvature under the Ricci-Bourguignon flow evolves by

$$\frac{\partial R}{\partial t} = (1-4\rho)\Delta R + 2|\mathrm{Ric}|^2 - 2\rho R^2.$$

Since $|\mathrm{Ric}|^2 \le R^2$, we have

$$\frac{\partial R}{\partial t} \le (1-4\rho)\Delta R + 2(1-\rho)R^2.$$

Let $\sigma(t)$ be the solution of the ODE $y' = 2(1-\rho)y^2$ with initial value $\alpha = \max_{x\in M} R(0)$. By the maximum principle, we have

$$R(t) \leq \sigma(t) = \left(-2(1-\rho)t + \frac{1}{\alpha}\right)^{-1} \tag{30}$$

on $[0, T')$, where $T' = \min\{T, \frac{1}{2(1-\rho)\alpha}\}$. Also, the inequality $|\mathrm{Ric}|^2 \geq \frac{R^2}{3}$ implies that

$$\frac{\partial R}{\partial t} \geq (1-4\rho)\Delta R + 2(\frac{1}{3}-\rho)R^2.$$

We assume that $\gamma(t)$ be the solution to the ODE $y' = 2(\frac{1}{3}-\rho)y^2$ with initial value $\beta = \min_{x\in M} R(0)$. Then the maximum principle implies that

$$R(t) \geq \gamma(t) = \frac{3\beta}{3-2(1-3\rho)\beta t} \quad \text{on} \quad [0, T). \tag{31}$$

Hence

$$\begin{aligned}\frac{d}{dt}\lambda(u,v,t)|_{t=t_0} &\geq (1-3\rho+q\epsilon)\lambda(t_0)\frac{3\beta}{3-2(1-3\rho)\beta t_0}\\ &\quad + (3\rho-1-p\rho)\lambda(t_0)\left(-2(1-\rho)t_0+\frac{1}{\alpha}\right)^{-1}\\ &= \lambda(t_0)A(t_0).\end{aligned}$$

It follows that in any sufficiently small neighborhood of t_0 as I_0, we get

$$\frac{d}{dt}\lambda(u,v,t) \geq \lambda(u,v,t)A(t).$$

Integrating the last inequality with respect to t on $[t_1, t_0] \subset I_0$, we have

$$\ln\frac{\lambda(u(t_0),v(t_0),t_0)}{\lambda(u(t_1),v(t_1),t_1)} > \int_{t_1}^{t_0} A(\tau)d\tau.$$

Since $\lambda(u(t_0), v(t_0)), t_0) = \lambda(t_0)$ and $\lambda(u(t_1), v(t_1), t_1) \geq \lambda(t_1)$, we conclude that

$$\ln\frac{\lambda(t_0)}{\lambda(t_1)} > \int_{t_1}^{t_0} A(\tau)d\tau;$$

that is, the quantity $\lambda(t)e^{-\int_0^t A(\tau)d\tau}$ is strictly increasing in any sufficiently small neighborhood of t_0. Since t_0 is arbitrary, we get that $\lambda(t)e^{-\int_0^t A(\tau)d\tau}$ is strictly increasing along the Ricci-Bourguignon flow on $[0, T)$. □

Remark 3.11. In Proposition 3.10, if we consider $\rho < 0$ instead of $\rho > 0$, then the quantity $e^{-\int_0^t B(\tau)d\tau}\lambda(t)$ is nondecreasing along the Ricci-Borguignon flow (1) on closed manifold M^3, where

$$B(t) = \frac{3\beta(1-3\rho+q\epsilon-q\rho)}{3-2(1-3\rho)\beta t} + (3\rho-1)\left(-2(1-\rho)t+\frac{1}{\alpha}\right)^{-1}.$$

Proposition 3.12. *Let $(M^3, g(t))$ be a solution to the Ricci-Bourguignon flow for $\rho < 0$ on a closed homogeneous 3-manifold whose Ricci curvature is initially nonnegative. Then the first eigenvalues of the (p,q)-Laplacian (2) is increasing.*

Proof. In dimension three, the nonnegativity of the Ricci curvature is preserved under the Ricci-Bourguignon flow [7]. From (29), it follows that $\lambda(t)$ is increasing. □

4. Example

In this section, we show that the variational formula is effective to derive some properties of the evolving the first eigenvalue of the (p,q)-Laplacian (2) on some of Riemannian manifolds.

Example 4.1. Let (M^n, g_0) be an Einstein manifold; i.e., there exists a constant a such that $Ric(g_0) = ag_0$. Assume that we have a solution of the Ricci-Bourguignon flow which is of the form

$$g(t) = u(t)g_0, \quad u(0) = 1,$$

where $u(t)$ is a positive function. We get

$$\frac{\partial g}{\partial t} = u'(t)g_0, \;\; Ric(g(t)) = Ric(g_0) = ag_0 = \frac{a}{u(t)}g(t), \;\; R_{g(t)} = \frac{an}{u(t)}.$$

For this to be a solution of the Ricci-Bourguignon flow, we require

$$u'(t)g_0 = -2Ric(g(t)) + 2\rho R_{g(t)}g(t) = (-2a + 2\rho an)g_0.$$

This shows that

$$u'(t) = -2a + 2\rho an,$$

and $u(t)$ satisfies

$$u(t) = 2a(-1 + \rho n)t + 1.$$

So $g(t)$ is an Einstein metric. If $p \geq q$, using equation (23), we obtain the following relation

$$\frac{d}{dt}\lambda(u, v, t)|_{t=t_0} \geq (1 - n\rho)\frac{qa}{u(t_0)}\lambda(t_0).$$

Thus, in any sufficiently small neighborhood of t_0 as I_0, we get

$$\frac{d}{dt}\lambda(u, v, t) \geq (1 - n\rho)\frac{aq}{u(t)}\lambda(u, v, t).$$

Integrating the last inequality with respect to t on $[t_1, t_0] \subset I_0$, we have

$$\ln \frac{\lambda(u(t_0), v(t_0), t_0)}{\lambda(u(t_1), v(t_1), t_1)} \geq \ln \left(\frac{2a(-1 + \rho n)t_1 + 1}{2a(-1 + \rho n)t_0 + 1}\right)^{\frac{q}{2}}.$$

Since $\lambda(u(t_0), v(t_0), t_0) = \lambda(t_0)$ and $\lambda(u(t_1), v(t_1), t_1) > \lambda(t_1)$, we conclude that

$$\ln \frac{\lambda(t_0)}{\lambda(t_1)} > \ln \left(\frac{2a(-1 + \rho n)t_1 + 1}{2a(-1 + \rho n)t_0 + 1}\right)^{\frac{q}{2}};$$

that is, the quantity $\lambda(t)(2a(-1+\rho n)t+1)^{\frac{q}{2}}$ is strictly increasing along the Ricci-Bourguignon flow on $[0,T)$.

References

[1] H. Amann, *Lusternik-Schnirelman theory and non-linear eigenvalue problems*, Math. Ann. **199** (1972), 55–72.

[2] S. Azami, *The first eigenvalue of some (p,q)-Laplacian and geometric estimates*, Commun. Korean Math. Soc. **33** (2018), no. 1, 317–323.

[3] L. Boccardo and D. Guedes de Figueiredo, *Some remarks on a system of quasilinear elliptic equations*, NoDEA Nonlinear Differential Equations Appl. **9** (2002), no. 3, 309–323.

[4] J.-P. Bourguignon, *Ricci curvature and Einstein metrics*, in Global differential geometry and global analysis (Berlin, 1979), 42–63, Lecture Notes in Math., 838, Springer, Berlin, 1981.

[5] X. Cao, *Eigenvalues of $(-\Delta+\frac{R}{2})$ on manifolds with nonnegative curvature operator*, Math. Ann. **337** (2007), no. 2, 435–441.

[6] ———, *First eigenvalues of geometric operators under the Ricci flow*, Proc. Amer. Math. Soc. **136** (2008), no. 11, 4075–4078.

[7] G. Catino, L. Cremaschi, Z. Djadli, C. Mantegazza, and L. Mazzieri, *The Ricci-Bourguignon flow*, Pacific J. Math. **287** (2017), no. 2, 337–370.

[8] S. Y. Cheng, *Eigenfunctions and eigenvalues of Laplacian*, in Differential geometry (Proc. Sympos. Pure Math., Vol. XXVII, Stanford Univ., Stanford, Calif., 1973), Part 2, 185–193, Amer. Math. Soc., Providence, RI, 1975.

[9] Q.-M. Cheng and H. Yang, *Estimates on eigenvalues of Laplacian*, Math. Ann. **331** (2005), no. 2, 445–460.

[10] J. I. Díaz, *Nonlinear Partial Differential Equations and Free Boundaries. Vol. I*, Research Notes in Mathematics, **106**, Pitman (Advanced Publishing Program), Boston, MA, 1985.

[11] P. L. de Napoli and M. C. Mariani, *Quasilinear elliptic systems of resonant type and nonlinear eigenvalue problems*, Abstr. Appl. Anal. **7** (2002), no. 3, 155–167.

[12] P. L. de Napoli and J. P. Pinasco, *Estimates for eigenvalues of quasilinear elliptic systems*, J. Differential Equations **227** (2006), no. 1, 102–115.

[13] L. F. Di Cerbo, *Eigenvalues of the Laplacian under the Ricci flow*, Rend. Mat. Appl. (7) **27** (2007), no. 2, 183–195.

[14] E. M. Harrell, II and P. L. Michel, *Commutator bounds for eigenvalues, with applications to spectral geometry*, Comm. Partial Differential Equations **19** (1994), no. 11-12, 2037–2055.

[15] D. A. Kandilakis, M. Magiropoulos, and N. B. Zographopoulos, *The first eigenvalue of p-Laplacian systems with nonlinear boundary conditions*, Bound. Value Probl. **2005**, no. 3, 307–321.

[16] A. E. Khalil, *Autour de la première courbe propre du p-Laplacien*, Thèse de Doctorat, 1999.

[17] A. El Khalil, S. El Manouni, and M. Ouanan, *Simplicity and stability of the first eigenvalue of a nonlinear elliptic system*, Int. J. Math. Math. Sci. **2005**, no. 10, 1555–1563.

[18] P. F. Leung, *On the consecutive eigenvalues of the Laplacian of a compact minimal submanifold in a sphere*, J. Austral. Math. Soc. Ser. A **50** (1991), no. 3, 409–416.

[19] J.-F. Li, *Eigenvalues and energy functionals with monotonicity formulae under Ricci flow*, Math. Ann. **338** (2007), no. 4, 927–946.

[20] R. Manásevich and J. Mawhin, *The spectrum of p-Laplacian systems with various boundary conditions and applications*, Adv. Differential Equations **5** (2000), no. 10-12, 1289–1318.

[21] A. Mukherjea and K. Pothoven, *Real and Functional Analysis*, Plenum Press, New York, 1978.
[22] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, arXiv preprint math, 0211159, 2002.
[23] J. Y. Wu, *First eigenvalue monotonicity for the p-Laplace operator under the Ricci flow*, Acta Math. Sin. (Engl. Ser.) **27** (2011), no. 8, 1591–1598.
[24] B. Chen, Q. He, and F. Zeng, *Monotonicity of eigenvalues of geometric operators along the Ricci–Bourguignon flow*, Pacific J. Math. **296** (2018), no. 1, 1–20.

Shahroud Azami
Department of Mathematics
Faculty of Sciences
Imam Khomeini International University
Qazvin, Iran
Email address: azami@sci.ikiu.ac.ir