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A STUDY OF NEW CLASS OF INTEGRALS ASSOCIATED WITH GENERALIZED STRUVE FUNCTION AND POLYNOMIALS

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Abstract. The main aim of this paper is to establish a new class of integrals involving the generalized Galuè-type Struve function with the different type of polynomials such as Jacobi, Legendre, and Hermite. Also, we derive the integral formula involving Legendre, Wright generalized Bessel and generalized Hypergeometric functions. The results obtained here are general in nature and can deduce many known and new integral formulas involving the various type of polynomials.

1. Introduction

The Struve function of order p is given by

$$H_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+3/2)\Gamma(k+p+3/2)} \left(\frac{x}{2}\right)^{2k+p}, \; x \in \mathbb{R}, \; a \in \mathbb{N} \tag{1.1}$$

is a particular solution of the non-homogeneous Bessel differential equation

$$x^2 y''(x) + xy'(x) + (x^2 - p^2)y(x) = \frac{4(x/2)^{p+1}}{\sqrt{\pi}\Gamma(p+1/2)}, \tag{1.2}$$

where Γ is the classical gamma function whose Euler's integral is given by [17]

$$\Gamma(z) = \int_0^{\infty} \exp(-t)\; t^{z-1} dt, \; \Re(z) > 0. \tag{1.3}$$

Galuè [6] introduced a generalization of the Bessel function of order p given by:

$$ {}_aJ_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(ak+p+1)k!} \left(\frac{x}{2}\right)^{2k+p}, \; x \in \mathbb{R}, \; a \in \mathbb{N}. \tag{1.4}$$

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Baricz [1] investigated Galuè type generalization of modified Bessel function as:

$$ {}_aI_p(x) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(ak+p+1)k!} \left(\frac{x}{2}\right)^{2k+p}, \ x \in \mathbb{R},\ a \in \mathbb{N}. \tag{1.5}$$

The Struve function and its more generalizations are found in many papers (see [2, 3, 7, 13–16]).

The generalized Struve function is given by Bhowmick [2]

$$ H_l^{\lambda}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+3/2)\Gamma(\lambda k+l+3/2)} \left(\frac{x}{2}\right)^{2k+l+1}, \ \lambda > 0 \tag{1.6}$$

and by Kanth [7]

$$ H_l^{\lambda,\alpha}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\alpha k+3/2)\Gamma(\lambda k+l+3/2)} \left(\frac{x}{2}\right)^{2k+l+1}, \ \lambda > 0,\ \alpha > 0. \tag{1.7}$$

Singh [13] gave another generalized form as

$$ H_{l,\xi}^{\lambda}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+3/2)\Gamma(\lambda k+l/\xi+3/2)} \left(\frac{x}{2}\right)^{2k+l+1}, \ \lambda > 0. \tag{1.8}$$

The generalized Struve function of four parameters was given by Singh [18] as:

$$ H_{p,\mu}^{\lambda,\alpha}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\lambda k+p+3/2)\Gamma(\alpha k+\mu)} \left(\frac{x}{2}\right)^{2k+p+1}, \ \lambda, p \in \mathbb{C} \tag{1.9}$$

where $\lambda > 0,\ \alpha > 0$ and μ is an arbitrary parameter. Another generalization of Struve function by Orhan and Yagmur [10, 22] is,

$$ H_{p,b,c}(z) = \sum_{k=0}^{\infty} \frac{(-c)^k}{\Gamma(k+3/2)\Gamma(k+p+b/2+1)} \left(\frac{z}{2}\right)^{2k+p+1}, \ p, b, c \in \mathbb{C}. \tag{1.10}$$

Motivated from the above definitions, Nisar *et al.* [9] defined a new generalized form of Struve function named as generalized Galuè type Struve function (GTSF) as:

$$ {}_aW_{p,b,c,\xi}^{\alpha,\mu}(z) = \sum_{k=0}^{\infty} \frac{(-c)^k}{\Gamma(ak+p/\xi+\frac{b+2}{2})\Gamma(\alpha k+\mu)} \left(\frac{z}{2}\right)^{2k+p+1}, \ a \in \mathbb{N},\ p, b, c \in \mathbb{C}, \tag{1.11}$$

where $\alpha > 0$, $\xi > 0$ and μ is an arbitrary parameter.

The aim of this paper to establish various integral formulae involving GTSF employing different polynomials.

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2. Integrals with Jacobi polynomials

The Jacobi polynomials $P_n^{\rho,\sigma}(t)$ [11, 19] may be defined by

$$P_n^{(\rho,\sigma)}(t) = \frac{(1+\rho)_n}{n!}\,{}_2F_1\left[\begin{matrix} -n, 1+\rho+\sigma+n; \\ \rho+1; \end{matrix}\ \frac{1-t}{2}\right]. \tag{2.1}$$

When $\rho = \sigma = 0$, the polynomial in (1.6) becomes the Legendre polynomial[11]. From (1.6) it follows that $P_n^{(\rho,\sigma)}(t)$ is a polynomial of degree precisely n and that

$$P_n^{(\rho,\sigma)}(t) = \frac{(1+\rho)_n}{n!}. \tag{2.2}$$

First integral formula:

$$\begin{aligned} I_1 &= \int_{-1}^{1} t^{\lambda}(1-t)^{\rho}(1+t)^{\delta} P_n^{(\rho,\sigma)}(t)\,{}_aW_{p,b,c,\xi}^{\alpha,\mu}[z(1+t)^h]dt \\ &= \frac{(-1)^n 2^{\rho+\delta+1}\Gamma(\delta+2hk+hp+h+1)\Gamma(n+\rho+1)\Gamma(\delta+2hk+hp+h+\sigma+1)}{n!\Gamma(\eta+hk+\sigma+n+1)\Gamma(\eta+hk+\rho+n+2)} \\ &\quad \times {}_aW_{p,b,c,\xi}^{\alpha,\mu}(2^h z)\ {}_3F_2\left[\begin{matrix} -\lambda, \delta+2hk+hp+h+\sigma+1, \delta+2hk+hp+h+1; \\ \delta+2hk+hp+h\sigma+n+1, \delta+2hk+hp+h+\rho+n+2; \end{matrix}\ 1\right], \end{aligned} \tag{2.3}$$

provided $(\alpha) > 0, (\xi) > 0, a \in \mathbb{N},\ p, b, c \in \mathbb{C},\ \mu$ is an arbitrary parameter and $\Re(\lambda) > -1, \rho > -1, \sigma > -1$.

Proof. In dealing with the Jacobi polynomials, it is natural to make much use of our knowledge of the ${}_2F_1$ function [11]

$$\begin{aligned} I_1 &= \int_{-1}^{1} t^{\lambda}(1-t)^{\rho}(1+t)^{\delta} P_n^{(\rho,\sigma)}(t)\,{}_aW_{p,b,c,\xi}^{\alpha,\mu}[z(1+t)^h]dt \\ &= \int_{-1}^{1} t^{\lambda}(1-t)^{\rho}(1+t)^{\delta} P_n^{(\rho,\sigma)}(t) \sum_{k=0}^{\infty} \frac{(-c)^k \left(z(1+t)^h/2\right)^{2k+p+1}}{\Gamma(\alpha k+\mu)\Gamma(ak+p/\xi+(b+2)/2)}dt. \end{aligned} \tag{2.4}$$

Interchanging the order of integration and summation which is permissible under the condition, then the above expression becomes

$$\begin{aligned} I_1 &= \sum_{k=0}^{\infty} \frac{(-c)^k (z/2)^{2k+p+1}}{\Gamma(\alpha k+\mu)(\Gamma(ak+p/\xi+(b+2)/2)} \\ &\quad \times \int_{-1}^{1} t^{\lambda}(1-t)^{\rho}(1+t)^{\delta+2hk+hp+h} P_n^{(\rho,\sigma)}(t)dt. \end{aligned} \tag{2.5}$$

Using the formula [12]

$$\begin{aligned} \int_{-1}^{1} t^{\lambda}(1-t)^{\rho}(1+t)^{\delta} P_n^{(\rho,\sigma)}(t)dt &= \frac{(-1)^n 2^{\rho+\delta+1}\Gamma(\delta+1)\Gamma(n+\rho+1)\Gamma(\delta+\sigma+1)}{n!\Gamma(\delta+\sigma+n+1)\Gamma(\delta+\rho+n+2)} \\ &\quad \times {}_3F_2\left[\begin{matrix} -\lambda, \delta+\sigma+1, \delta+1; \\ \delta+\sigma+n+1, \delta+\rho+n+2; \end{matrix}\ 1\right]. \end{aligned} \tag{2.6}$$

provided $\rho > -1$ and $\sigma > -1$, we get the desired result. □

Second integral formula:

$$\int_{-1}^{1}(1-t)^{\delta}(1+t)^{\sigma}P_n^{(\rho,\sigma)}(t)P_n^{(\eta,\theta)}(t)\,{}_aW_{p,b,c,\xi}^{\alpha,\mu}[z(1-t)^h]dt$$
$$=\frac{2^{\delta+\sigma+1}\Gamma(1+\eta+m)\Gamma(1+\rho+n)}{m!n!}$$
$$\times\sum_{k=0}^{\infty}\frac{(-n)_k(1+\rho+\sigma+n)_k}{\Gamma(1+\rho+k)(k!)}\sum_{l=0}^{\infty}\frac{(-m)_l(1+\eta+\theta+m)_l}{\Gamma(1+\eta+l)(l!)}$$
$$\times\,{}_aW_{p,b,c,\xi}^{\alpha,\mu}(2^hz)\mathbb{B}(1+\delta+h(2k+p+1)+k+l,1+\sigma), \tag{2.7}$$

provided $(\alpha)>0,(\xi)>0,a\in\mathbb{N},\ p,b,c\in\mathbb{C},\ \mu$ is an arbitrary parameter and $\Re(\lambda)>-1,\rho>-1,\sigma>-1$.

Proof. The left-hand side of (2.7) is denoted by I_2,

$$I_2=\int_{-1}^{1}(1-t)^{\delta}(1+t)^{\sigma}P_n^{(\rho,\sigma)}(t)P_n^{(\eta,\theta)}(t)\,{}_aW_{p,b,c,\xi}^{\alpha,\mu}[z(1-t)^h]dt$$
$$=\sum_{k=0}^{\infty}\frac{(-c)^k(z/2)^{2k+p+1}}{\Gamma(\alpha k+\mu)\Gamma(ak+p/\xi+(b+2)/2)}$$
$$\times\int_{-1}^{1}(1-t)^{\delta+2hk+hp+h}(1+t)^{\sigma}P_n^{(\rho,\sigma)}(t)P_m^{(\eta,\theta)}(t)dt. \tag{2.8}$$

Now using (2.1) in (2.8), we get

$$I_2=\sum_{k=0}^{\infty}\frac{(-c)^k(z/2)^{2k+p+1}}{\Gamma(\alpha k+\mu)\Gamma(ak+p/\xi+(b+2)/2)}\frac{(1+\rho)_n}{n!}\sum_{k=0}^{\infty}\frac{(-n)_k(1+\rho+\sigma+n)_k}{(1+\rho)_k2^kk!}$$
$$\times\int_{-1}^{1}(1-t)^{\delta+h(2k+p+1)+k}(1+t)^{\sigma}P_n^{(\eta,\theta)}(t)dt. \tag{2.9}$$

Again using (2.1) in (2.9), we obtain

$$I_2=\sum_{k=0}^{\infty}\frac{((-c)^k(z/2)^{2k+p+1}}{\Gamma(\alpha k+\mu)\Gamma(ak+p/\xi+(b+2)/2)}\frac{\Gamma(1+\eta+m)\Gamma(1+\rho+n)}{m!n!}$$
$$\times\sum_{k=0}^{\infty}\frac{(-n)_k(1+\rho+\sigma+n)_k}{\Gamma(1+\rho+k)2^k(k!)}\sum_{l=0}^{\infty}\frac{(-m)_l(1+\eta+\theta+m)_l}{\Gamma(1+\eta+l)2^l(l!)}$$
$$\times\int_{-1}^{1}(1-t)^{\delta+h(2k+p+1)+k+l}(1+t)^{\sigma}P_n^{(\rho,\sigma)}(t)dt. \tag{2.10}$$

Now using the formula [11, 19]

$$\int_{-1}^{1}(1-t)^{\delta+n}(1+t)^{\sigma+n}dt=2^{2n+\sigma+\delta+1}\mathbb{B}(1+\delta+n,1+\sigma+n), \tag{2.11}$$

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(2.10) becomes,

$$I_2 = \frac{2^{\delta+\sigma+1}\Gamma(1+\eta+m)\Gamma(1+\rho+n)}{m!n!}$$
$$\times \sum_{k=0}^{\infty} \frac{(-n)_k(1+\rho+\sigma+n)_k}{\Gamma(1+\rho+k)(k!)} \sum_{l=0}^{\infty} \frac{(-m)_l(1+\eta+\theta+m)_l}{\Gamma(1+\eta+l)(l!)}$$
$$\times {}_aW^{\alpha,\mu}_{p,b,c,\xi}(2^h z)\mathbb{B}(1+\delta+h(2k+p+1)+k+l, 1+\sigma). \tag{2.12}$$
□

Third integral formula:

$$I_3 = \frac{2^{\eta+\theta+1}}{n!} \frac{(1+\rho)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k(1+\rho+\sigma+n)_k}{(1+\rho)_k(k!)}$$
$$\times {}_aW^{\alpha,\mu}_{p,b,c,\xi}(2^{h+l}z)\mathbb{B}(1+\eta+h(2k+p+1)+k, 1+\theta+l(2k+p+1)+l), \tag{2.13}$$

provided $(\alpha) > 0, (\xi) > 0, a \in \mathbb{N},\ p, b, c \in \mathbb{C}$, μ is an arbitrary parameter and $\rho > -1$, $\sigma > -1$.

Proof. Assume that the left-hand side of (2.13) is denoted by I_3, then

$$I_3 = \int_{-1}^{1} (1-t)^{\eta}(1+t)^{\theta} P_n^{(\rho,\sigma)}(t)_a W^{\alpha,\mu}_{p,b,c,\xi}[z(1-t)^h(1+t)^t]dt$$
$$= \sum_{k=0}^{\infty} \frac{((-c)^k(z/2)^{2k+p+1}}{\Gamma(\alpha k+\mu)\Gamma(ak+p/\xi+(b+2)/2)}$$
$$\times \int_{-1}^{1} (1-t)^{\eta+h(2k+p+1)}(1+t)^{\theta+l(2k+p+1)} P_n^{(\rho,\sigma)}(t)dt. \tag{2.14}$$

Now using (2.1) in (2.14), we obtain

$$I_3 = \sum_{k=0}^{\infty} \frac{((-c)^k(z/2)^{2k+p+1}}{\Gamma(\alpha k+\mu)\Gamma(ak+p/\xi+(b+2)/2)} \frac{(1+\rho)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k(1+\rho+\sigma+n)_k}{(1+\rho)_k 2^k k!}$$
$$\times \int_{-1}^{1} (1-t)^{\eta+h(2k+p+1)+k}(1+t)^{\theta+l(2k+p+1)+l}dt. \tag{2.15}$$

Further using (2.11) in (2.15) we get the required result. □

Fourth integral formula:

$$\int_{-1}^{1} (1-t)^{\eta}(1+t)^{\theta} P_n^{(\rho,\sigma)}(t)_a W^{\alpha,\mu}_{p,b,c,\xi}[z(1+t)^{-h}]dt$$
$$= \frac{2^{\eta+\theta+1}}{n!} \frac{(1+\rho)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k(1+\rho+\sigma+n)_k}{(1+\rho)_k(k!)}$$
$$\times {}_aW^{\alpha,\mu}_{p,b,c,\xi}(2^{-h}z)\mathbb{B}(1+\eta+k, 1+\theta-h(2k+p+1)), \tag{2.16}$$

provided $(\alpha) > 0, (\xi) > 0, a \in \mathbb{N},\ p, b, c \in \mathbb{C}$, μ is an arbitrary parameter and $\rho > -1, \sigma > -1$.

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Proof. The left-hand side of (2.16) is denoted by I_4,

$$\begin{aligned}I_4 &= \int_{-1}^{1}(1-t)^{\eta}(1+t)^{\theta}P_n^{(\rho,\sigma)}(t)_aW^{\alpha,\mu}_{p,b,c,\xi}[z(1+t)^{-h}]dt\\ &= \sum_{k=0}^{\infty}\frac{(-c)^k(z/2)^{2k+p+1}}{\Gamma(\alpha k+\mu)\Gamma(ak+p/\xi+(b+2)/2)}\\ &\quad\times\int_{-1}^{1}(1-t)^{\eta}(1+t)^{\theta-h(2k+p+1)}P_n^{(\rho,\sigma)}(t)dt.\end{aligned}\tag{2.17}$$

Now using (2.1) in (2.17) we have

$$\begin{aligned}I_4 = \sum_{k=0}^{\infty}&\frac{(-c)^k(z/2)^{2k+p+1}}{\Gamma(\alpha k+\mu)\Gamma(ak+p/\xi+(b+2)/2)}\frac{(1+\rho)_n}{n!}\sum_{k=0}^{\infty}\frac{(-n)_k(1+\rho+\sigma+n)_k}{(1+\rho)_k2^kk!}\\ &\times\int_{-1}^{1}(1-t)^{\eta+k}(1+t)^{\theta-h(2k+p+1)}dt\end{aligned}\tag{2.18}$$

further using (2.11) in (2.18) we attain the required result. □

Fifth integral formula:

$$\begin{aligned}&\int_{-1}^{1}(1-t)^{\rho}(1+t)^{\theta}P_n^{(\rho,\sigma)}(t)_aW^{\alpha,\mu}_{p,b,c,\xi}[z(1-t)^{h}(1+t)^{-l}]dt\\ &= 2^{\eta+\theta+1}\frac{(1+\rho)_n}{n!}\sum_{k=0}^{\infty}\frac{(-n)_k(1+\rho+\sigma+n)_k}{(1+\rho)_k(k!)}\\ &\quad\times{}_aW^{\alpha,\mu}_{p,b,c,\xi}(2^{h-l}z)\mathbb{B}(1+\mu+h(2k+p+1)+k,1+\theta-l(2k+p+1)),\end{aligned}\tag{2.19}$$

provided $(\alpha)>0,(\xi)>0,a\in\mathbb{N},\ p,b,c\in\mathbb{C},\ \mu$ is an arbitrary parameter and $\rho>-1,\sigma>-1$.

Proof. The left-hand side of (2.19) is denoted by I_5, then we have

$$\begin{aligned}I_5 &= \int_{-1}^{1}(1-t)^{\rho}(1+t)^{\theta}P_n^{(\rho,\sigma)}(t)_aW^{\alpha,\mu}_{p,b,c,\xi}[z(1-t)^{h}(1+t)^{-l}]dt\\ &= \sum_{k=0}^{\infty}\frac{(-c)^k(z/2)^{2k+p+1}}{\Gamma(\alpha k+\mu)(ak+p/\xi+(b+2)/2)}\\ &\quad\times\int_{-1}^{1}(1-t)^{\eta+h(2k+p+1)}(1+t)^{\theta-l(2k+p+1)}P_n^{(\rho,\sigma)}(t)dt,\end{aligned}\tag{2.20}$$

now using (2.1) in (2.20) we have

$$\begin{aligned}I_5 = \sum_{k=0}^{\infty}&\frac{(-c)^k(z/2)^{2k+p+1}}{\Gamma(\alpha k+\mu)\Gamma(ak+p/\xi+(b+2)/2)}\frac{(1+\rho)_n}{n!}\\ &\times\sum_{k=0}^{\infty}\frac{(-n)_k(1+\rho+\sigma+n)_k}{(1+\rho)_k2^kk!}\end{aligned}$$

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$$\times \int_{-1}^{1} (1-t)^{\eta+h(2k+p+1)+k}(1+t)^{\theta-l(2k+p+1)}dt \tag{2.21}$$

further using (2.11) in (2.21) we get the desired result. □

3. Special cases:

Sixth integral formula:

If we replace δ by $\lambda-1$ and put $\rho=\sigma=\mu=\theta=0$, then the integral I_2 transforms into the following integral involving Legendre polynomial [11].

$$\begin{aligned} I_6 &= \int_{-1}^{1} (1-t)^{\lambda-1} P_n(t)_a W^{\alpha,\mu}_{p,b,c,\xi}[z(1-t)^h]dt \\ &= \sum_{k=0}^{\infty} \frac{2^\lambda(-n)_k(1+n)_k}{(k!)^2} \times \sum_{l=0}^{\infty} \frac{(-m)_l(1+m)_l}{l!^2} \\ &\quad \times {}_aW^{\alpha,\mu}_{p,b,c,\xi}(2^h z)\mathbb{B}(1+\lambda+h(2k+p+1)+k+l,1). \end{aligned} \tag{3.1}$$

Seventh integral formula:

If $\sigma=\rho=0$, η is replaced by $\eta-1$ and θ by $\theta-1$ then the integral I_3 transforms into the following integral involving Legendre polynomial [11].

$$\begin{aligned} I_7 &= \int_{-1}^{1} (1-t)^{\eta-1}(1+t)^{\theta-1} P_n(t) E^{\alpha,\mu}_{p,b,c\xi}[z(1-t)^h(1+t)^l]dy, \\ &= \sum_{k=0}^{\infty} \frac{2^{\mu+\theta-1}(-n)_k(1+n)_k}{(k!)^2} \\ &\quad \times {}_aW^{\alpha,\mu}_{p,b,c,\xi}(2^{h+l} z)\mathbb{B}(1+\eta+h(2k+p+1)+k,\theta+l(2k+p+1)). \end{aligned} \tag{3.2}$$

Eighth integral formula:

If $\rho=\sigma=0$, η is replaced by $\eta-1$ and θ by $\theta-1$ then the integral I_3 transforms into the following integral involving Legendre polynomial [11]

$$\begin{aligned} I_8 &= \int_{-1}^{1} (1-t)^{\eta-1}(1+t)^{\theta-1} P_n(t)_a W^{\alpha,\mu}_{p,b,c,\xi}[z(1-t)^h(1+t)^{-l}]dt, \\ &= \sum_{k=0}^{\infty} \frac{2^{\eta+\theta-1}(-n)_k(1+n)_k}{(k!)^2} \\ &\quad \times {}_aW^{\alpha,\mu}_{p,b,c,\xi}(2^{h-l} z)\mathbb{B}(1+\mu+h(2k+p+1)+k,\theta-l(2k+p+1)). \end{aligned} \tag{3.3}$$

4. Integral with Wright generalized Bessel function

The special case of the Wright function [5] and (see the ref. [20, 21]) in the form

$$\phi(A,a;z) = {}_0\psi_1 \left[\begin{matrix} -; \\ (A,a); \end{matrix} \; z \right] = \sum_{k=0}^{\infty} \frac{1}{\Gamma(Ak+a)} \frac{z^k}{k!}, \tag{4.1}$$

with complex $z, a \in \mathbb{C}$ and real $A \in \mathbb{R}$ when $A = \eta$, $a = \nu + 1$ and z is replaced by $-z$, then the function $\phi(\eta, \nu + 1; -z)$ is defined by $J_\nu^\eta(z)$

$$\phi(\eta, \nu + 1; -z) = J_\nu^\eta(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\eta k + \nu + 1)} \frac{(-z)^k}{k!}, \tag{4.2}$$

and such a function is known as the Bessel-Maitland function or the Wright generalized Bessel function see [8].

Ninth integral formula:

$$\begin{aligned}
&\int_{-\infty}^{\infty} (t)^\theta J_\eta^\tau(t)\,{}_aW_{p,b,c,\xi}^{\alpha,\mu}[z(t)^\rho]dt \\
&= \sum_{k=0}^{\infty} \frac{(-c)^k (z/2)^{2k+p+1}}{\Gamma(\alpha k + \mu)(\Gamma(ak + p/\xi + (b+2)/2))} \frac{\Gamma(\theta + \rho(2k+p+1)+1)}{\Gamma(1 + \eta - \tau - \tau(\eta + \rho(2k+p+1))} \\
&= \frac{\Gamma(\theta + \rho(2k+p+1)+1)}{\Gamma(1 + \eta - \tau - \tau(\eta + \rho(2k+p+1)))}\ {}_aW_{p,b,c,\xi}^{\alpha,\mu}(z),
\end{aligned} \tag{4.3}$$

provided $(\alpha) > 0, (\xi) > 0, a \in \mathbb{N},\ p, b, c \in \mathbb{C},\ \mu$ is an arbitrary parameter, $\rho - \rho\tau > -1$ and$\rho > 0$ and $0 < \tau < 1$ and $\Re(\theta + 1) > 0$.

Proof. The left-hand side of (4.3) is denoted by I_9

$$\begin{aligned}
I_9 &= \int_{-\infty}^{\infty} (t)^\theta J_\eta^\tau(t)\,{}_aW_{p,b,c,\xi}^{\alpha,\mu}[z(t)^\rho]dt \\
&= \sum_{k=0}^{\infty} \frac{(-c)^k (z/2)^{2k+p+1}}{\Gamma(\alpha k + \mu)\Gamma(ak + p/\xi + (b+2)/2)} \int_0^\infty (t^{\theta + \rho(2k+p+1)}) J_\eta^\tau(t) dt.
\end{aligned} \tag{4.4}$$

Now we use the following formula (see the ref.[12])

$$\int_0^\infty (t^\theta) J_\eta^\tau(t) dt = \frac{\Gamma(\theta + 1)}{\Gamma(1 + \eta - \tau - \tau\eta)}, \tag{4.5}$$

provided $\Re(\theta) > -1, 0 < \tau < 1$.

Now using (4.5) in (4.4), we arrive the required result. □

5. Integral with Legendre functions

The Legendre functions are solution of Legendres differential equation (see the ref. [4])

$$(1 - z^2)\frac{d^2w}{dz^2} - 2z\frac{dw}{dz} + [(\eta(\eta + 1)) - \mu^2(1 - z^2)^{-1}]w = 0, \tag{5.1}$$

where z, η, μ unrestricted.

Under the substitution $w = (z^2 - 1)^{\mu/2}\eta$ in (5.1) becomes

$$(1 - z^2)\frac{d^2\eta}{dz^2} - 2(\mu + 1)z\frac{d\eta}{dz} + [(\eta + \mu)(\eta + \mu + 1)]\eta = 0, \tag{5.2}$$

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and with $\xi = 1/2 - z/2$ as the independent variable, this differential equation becomes.

$$\xi(1-\xi)\frac{d^2\eta}{d\xi^2} + (\nu+1)(1-2\xi)\frac{d\eta}{d\xi} + [(\eta-\mu)(\eta+\mu+1)]\eta = 0. \tag{5.3}$$

This is the Gauss hypergeometric type equation with $a = \mu - \eta, b = \eta + \mu + 1, c = \mu + 1$.

Hence it follows that the function

$$W = P_\eta^{(\mu}(z) = \frac{1}{\Gamma(1-\mu)}\left(\frac{z+1}{z-1}\right)^{\beta/2} {}_2F_1\left[\begin{array}{l} -\eta, \eta+1; \\ 1-\mu; \end{array} 1/2 - z/2\right], \tag{5.4}$$

$\mid 1 - z \mid < 2$ is a solution of (5.1).

The function $P_\nu^\beta(z)$ is known as the Legendre function of first kind [4]. It is one valued and regular z-plane supposed cut along the real axis from 1 to $-\infty$.

Tenth integral formula:

$$\begin{aligned} &\int_0^1 (t)^{\theta-1}(1-t^2)^{\delta/2} P_\eta^\delta(t)\,{}_aW_{p,b,c,\xi}^{\alpha,\mu}[z(t)^\rho]dt \\ &= \frac{(-1)^\delta\sqrt{\pi}2^{-\theta-\delta}\Gamma(\eta+\delta+1)}{\Gamma(1-\delta+\eta)} \\ &\times \sum_{k=0}^{\infty} \frac{\Gamma(\theta+\rho(2k+p+1))}{\Gamma(1/2+\frac{\theta+\rho(2k+p+1)}{2}+\delta/2-\eta/2)\Gamma(1+\frac{\theta+\rho(2k+p+1)}{2}+\delta/2+\eta/2)}\,{}_aW_{p,b,c,\xi}^{\alpha,\mu}(z/2^\rho), \end{aligned} \tag{5.5}$$

provided $(\alpha) > 0, (\xi) > 0, a \in \mathbb{N},\ p, b, c \in \mathbb{C}$, μ is an arbitrary parameter and $\theta > 0$ and δ is positive integer.

Proof. Taking the left-hand side of (5.5) by I_{10}, we have

$$\begin{aligned} I_{10} &= \int_0^1 (t)^{\theta-1}(1-t^2)^{\delta/2} P_\eta^\delta(t)\,{}_aW_{p,b,c,\xi}^{\alpha,\mu}[z(t)^\rho]dt \\ &= \sum_{k=0}^{\infty} \frac{(-c)^k (z/2)^{2k+p+1}}{\Gamma(\alpha k+\mu)\Gamma(ak+p/\xi+(b+2)/2)} \\ &\quad \times \int_0^1 t^{\theta-1+\rho(2k+p+1)}(1-t^2)^{\delta/2} P_\eta^\delta(t)dt. \end{aligned} \tag{5.6}$$

Now the integral in (5.6) can be solved by using the following formula [4]

$$\begin{aligned} &\int_0^1 t^{\theta-1}(1-t^2)^{\delta/2} P_\eta^\delta(t)dt \\ &= \frac{(-1)^\delta\sqrt{\pi}2^{-\theta-\delta}\Gamma(\theta)\Gamma(\eta+\delta+1)}{(\Gamma(1/2+\theta/2+\delta/2-\eta/2)(\Gamma(1+\theta/2+\delta/2+\eta/2)(1-\delta+\eta)}, \end{aligned} \tag{5.7}$$

provided $\Re(\theta) > 0, \delta = 1, 2, 3, \ldots$.

Now (5.6) becomes

$$I_{10} = \sum_{k=0}^{\infty} \frac{(-c)^k (z/2)^{2k+p+1}}{\Gamma(\alpha k+\mu)(\Gamma(ak+p/\xi+(b+2)/2))}$$

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$$\times \frac{(-1)^{\delta}\sqrt{\pi}2^{-(\theta+\rho(2k+p+1))-\delta}\Gamma(\theta+\rho(2k+p+1))\Gamma(\delta+\eta+1)}{\Gamma(1/2+\frac{\theta+\rho k}{2}+\delta/2-\eta/2)\Gamma(1+\frac{\theta+\rho k}{2}+\delta/2+\eta/2)\Gamma(1-\delta+\eta)}$$

$$= \frac{(-1)^{\delta}\sqrt{\pi}2^{-\theta-\delta}\Gamma(\eta+\delta+1)}{\Gamma(1-\delta+\eta)}$$

$$\times \sum_{k=0}^{\infty}\frac{\Gamma(\theta+\rho(2k+p+1))}{\Gamma(1/2+\frac{\theta+\rho(2k+p+1)}{2}+\delta/2-\eta/2)\Gamma(1+\frac{\theta+\rho(2k+p+1)}{2}+\delta/2+\eta/2)}{}_aW^{\alpha,\mu}_{p,b,c,\xi}(z/2^{\rho})$$

$$= \sum_{k=0}^{\infty}\frac{(-c)^k(z/2)^{2k+p+1}}{\Gamma(\alpha k+\mu)(\Gamma(ak+p/\xi+(b+2)/2))}$$

$$\times \frac{(-1)^{\delta}\sqrt{\pi}2^{-(\theta+\rho(2k+p+1))-\delta}\Gamma(\theta+\rho(2k+p+1))\Gamma(\delta+\eta+1)}{\Gamma(1/2+\frac{\theta+\rho k}{2}+\delta/2-\eta/2)\Gamma(1+\frac{\theta+\rho k}{2}+\delta/2+\eta/2)\Gamma(1-\delta+\eta)}. \qquad \square$$

Eleventh integral formula:

$$\int_0^1 (t)^{\theta-1}(1-t^2)^{-\delta/2}P^{\delta}_{\eta}(t)\,{}_aW^{\alpha,\mu}_{p,b,c,\xi}[z(t)^{\rho}]dt$$
$$= \sqrt{\pi}2^{-\theta+\delta}\sum_{k=0}^{\infty}\frac{\Gamma(\theta+\rho(2k+p+1))}{\Gamma(\frac{1}{2}+\frac{\theta+\rho(2k+p+1)}{2}-\frac{\delta}{2}-\frac{\eta}{2})\Gamma(1+\frac{\theta+\rho(2k+p+1)}{2}-\frac{\delta}{2}-\frac{\eta}{2})}\,{}_aW^{\alpha,\mu}_{p,b,c,\xi}(\frac{z}{2}^{\rho}), \tag{5.8}$$

provided $(\alpha)>0,(\xi)>0,a\in\mathbb{N},\ p,b,c\in\mathbb{C}$, μ is an arbitrary parameter and $\theta>0$ and δ is positive integer.

Proof. Taking the LHS by I_{11}, we have

$$I_{11} = \int_0^1 (t)^{\theta-1}(1-t^2)^{-\delta/2}P^{\delta}_{\eta}(t)\,{}_aW^{\alpha,\mu}_{p,b,c,\xi}[z(t)^{\rho}]dt$$
$$= \sum_{k=0}^{\infty}\frac{(-c)^k(z/2)^{2k+p+1}}{\Gamma(\alpha k+\mu)\Gamma(ak+p/\xi+(b+2)/2)}$$
$$\times \int_0^1 t^{\theta-1+\rho(2k+p+1)}(1-t^2)^{-\delta/2}P^{\delta}_{\eta}(t)dt. \tag{5.9}$$

Now the integral in (5.9) can be solved by using the formula [4]

$$\int_0^1 t^{\theta-1}(1-t^2)^{-\delta/2}P^{\delta}_{\eta}(t)dt = \frac{\sqrt{\pi}2^{-\theta+\delta}\Gamma(\theta)}{\Gamma(1/2+\theta/2-\delta/2-\eta/2)\Gamma(1+\theta/2-\delta/2-\eta/2)},$$

provided $\Re(\theta)>0,\delta=1,2,3,\ldots$.

Again (5.9) becomes

$$I_{11} = \sum_{k=0}^{\infty}\frac{(-c)^k(z/2)^{2k+p+1}}{\Gamma(\alpha k+\mu)\Gamma(ak+p/\xi+(b+2)/2)}$$
$$\times \frac{\sqrt{\pi}2^{-(\theta+\rho(2k+p+1))+\delta}\Gamma(\theta+\rho(2k+p+1))}{\Gamma(1/2+\frac{\theta+\rho(2k+p+1)}{2}-\delta/2-\eta/2)\Gamma(1+\frac{\theta+\rho(2k+p+1)}{2}-\delta/2-\eta/2)}$$
$$= \sqrt{\pi}2^{-\theta+\delta}\sum_{k=0}^{\infty}\frac{\Gamma(\theta+\rho(2k+p+1))}{\Gamma(1/2+\frac{\theta+\rho(2k+p+1)}{2}-\delta/2-\eta/2)\Gamma(1+\frac{\theta+\rho(2k+p+1)}{2}-\delta/2-\eta/2)}$$

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$$\times_a W^{\alpha,\mu}_{p,b,c,\xi}(z/2^{\rho}). \tag{5.10}$$

□

6. Integrals with Hermite polynomials

Hermite polynomials $H_n(t)$ (see the ref. [11, 19]) may be defined by means of the relation

$$\exp(2tx - x^2) = \sum_{k=0}^{\infty} \frac{H_n(t)x^n}{n!}, \tag{6.1}$$

valid for all finite t and x. Since

$$\begin{aligned}\exp(2tx - x^2) &= \exp(2tx)\exp(-x^2)\\ &= \sum_{n=0}^{\infty} \frac{(2t)^n x^n}{n!} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{k!}\\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} \frac{(-1)^k (2t)^{n-2k} x^n}{(n-2k)!k!}.\end{aligned}$$

It follows from (6.1) that

$$H_n(t) = \sum_{k=0}^{[n/2]} \frac{(-1)^k (2t)^{n-2k} x^n}{(n-2k)!k!}. \tag{6.2}$$

Examination of equation (6.2) shows that $H_n(t)$ is a polynomial of degree precisely n in t and that

$$H_n(t) = 2^n t^n + \pi_{n-2}(t) \tag{6.3}$$

in which $\pi_{n-2}(t)$ is a polynomial of degree $(n-2)$ in t.

Twelfth integral formula:

$$\begin{aligned}&\int_{-\infty}^{\infty} (t)^{2\mu} \exp(-t^2) H_{2\eta}(t)_a W^{\alpha,\mu}_{p,b,c,\xi}[z(t)^{-2h}]dt\\ &= \sqrt{\pi} 2^{2(\eta-\mu)} \sum_{k=0}^{\infty} \frac{\Gamma(2\mu - h(2k+p+1)+1)}{\Gamma(\mu - h(2k+p+1) - \eta + 1)}\ {}_aW^{\alpha,\mu}_{p,b,c,\xi}(2^{2h}z),\end{aligned} \tag{6.4}$$

provided $(\alpha) > 0, (\xi) > 0, a \in \mathbb{N},\ p, b, c \in \mathbb{C},\ \mu$ is an arbitrary parameter and $\theta > 0$ and δ is positive integer.

Proof. The LHS of (6.4) is denoted by I_{12}

$$\begin{aligned}I_{12} &= \int_{-\infty}^{\infty} (t)^{2\mu} \exp(-t^2) H_{2\eta}(t)_a W^{\alpha,\mu}_{p,b,c,\xi}[z(t)^{-2h}]dt\\ &= \sum_{k=0}^{\infty} \frac{(-c)^k (z/2)^{2k+p+1}}{\Gamma(\alpha k + \mu)(\Gamma(ak + p/\xi + (b+2)/2))}\\ &\times \int_{-\infty}^{\infty} t^{2\mu - 2h(2k+p+1)} (\exp)^{-t^2} H_{2\eta}(t)dt.\end{aligned} \tag{6.5}$$

Now the integral in (6.5) can be solved by using the formula see [12]

$$\int_{-\infty}^{\infty} t^{2\mu}(\exp)^{-t^2} H_{2\eta}(t)dt = \frac{\sqrt{\pi}2^{2(\eta-\mu)}\Gamma(2\mu+1)}{\Gamma(\mu-\eta+1)}. \tag{6.6}$$

Again (6.5) becomes

$$I_{12} = \sum_{k=0}^{\infty} \frac{(-c)^k (z/2)^{2k+p+1}}{\Gamma(\alpha k+\mu)\Gamma(ak+p/\xi+(b+2)/2)} \times \frac{\sqrt{\pi}2^{2(\eta-(2\mu+2h(2k+p+1))}\Gamma(2\mu-2h(2k+p+1)+1)}{\Gamma(\mu-h(2k+p+1)-\eta+1)}.$$

Hence the desired result. □

Thirteenth integral formula:

$$\int_{-\infty}^{\infty}(t)^{2\mu}\exp(-t^2)H_{2\eta}(t)_a W^{\alpha;\mu}_{p,b,c,\xi}[z(t)^{2h}]dt$$
$$= \sqrt{\pi}2^{2(\eta-\mu)}\sum_{k=0}^{\infty}\frac{\Gamma(2\mu+2h(2k+p+1)+1)}{\Gamma(\mu+h(2k+p+1)-\eta+1)}\ {}_a W^{\alpha,\mu}_{p,b,c,\xi}(2^{-2h}z), \tag{6.7}$$

provided $(\alpha) > 0, (\xi) > 0, a \in \mathbb{N},\ p, b, c \in \mathbb{C}$, μ is an arbitrary parameter and $h > 0$ and $\Re(\mu) = 0, 1, 2, \ldots$.

Proof. Denoting the LHS of (6.7), we have

$$I_{13} = \int_{-\infty}^{\infty}(t)^{2\mu}\exp(-t^2)H_{2\eta}(t)_a W^{\alpha;\mu}_{p,b,c,\xi}[z(t)^{2h}]dt$$
$$= \sum_{k=0}^{\infty}\frac{(-c)^k(z/2)^{2k+p+1}}{\Gamma(\alpha k+\mu)\Gamma(ak+p/\xi+(b+2)/2)} \times \int_{-\infty}^{\infty} t^{2\mu+2h(2k+p+1)}(\exp)^{-t^2}H_{2\eta}(t)dt. \tag{6.8}$$

Using the formula given in (6.6) we reach the required proof. □

7. Integral with generalized hypergeometric function

A generalized hypergeometric function [11] may be defined by

$${}_pF_q\left[\begin{array}{c}(\rho)_1,(\rho)_2,\ldots,(\rho)_p;\\ (\sigma)_1,(\sigma)_2,\ldots,(\sigma)_q,;\end{array} z\right] = \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(\rho_i)_n}{\prod_{j=1}^{q}(\sigma_j)_n}\frac{z^n}{n!}, \tag{7.1}$$

in which no denominator parameter σ_j is allowed to be zero or a negative integer. If any numerator parameter ρ_i in (7.1) is zero or a negative integer, then the series terminates.

Fourteenth integral formula:

$$\int_0^x (t)^{\eta-1}(x-t)^{\theta-1}{}_pF_q[(l_p);(m_q): at^{\rho}(x-t)^{\sigma}]_a W^{\alpha,\mu}_{p,b,c}[zt^u(x-t)^v]dt$$

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$$= x^{\eta+\theta-1}B(\eta+u(2k+p+1)+\rho l, \theta+v(2k+p+1)+\sigma l)$$

$$\times_a W^{\alpha,\mu}_{p,b,c,\xi}(z\, x^{u+v})\sum_{l=0}^{\infty} f(l)t^{(\rho+\sigma)l}, \tag{7.2}$$

where

$$f(l) = \frac{(l_1)_l, \dots, (l_p)_l(a)^l}{(m_1)_l, \dots, (m_q)_l l!} \tag{7.3}$$

provided

(1) $\Re(\eta) > 0, \Re(\theta) > 0$;

(2) ρ and σ are to be positive integers not both zero;

(3) No m_q is to be zero or a negative integer;

(4) $p \leq\ q+1$, unless some l_q is a negative integer, in which case p may be positive integer.

Proof. Representing the LHS of (7.2) by I_{14}

$$I_{14} = \int_0^x (t)^{\eta-1}(x-t)^{\theta-1}{}_pF_q[(l_p);(m_q) : at^\rho(x-t)^\sigma]_a W^{\alpha,\mu}_{p,b,c}[zt^u(x-t)^v]dt$$

$$= \sum_{k=0}^{\infty} \frac{(-c)^k(z/2)^{2k+p+1}}{\Gamma(\alpha k+\mu)\Gamma(ak+p/\xi+(b+2)/2)}$$

$$\times \int_0^x t^{\eta+u(2k+p+1)-1}(x-t)^{\theta+v(2k+p+1)-1}{}_pF_q[(l_p);(m_q) : at^\rho(x-t)^\sigma]dt, \tag{7.4}$$

putting $t = sx$ and $dt = xds$, then we get

$$I_{14} = \sum_{k=0}^{\infty} \frac{(-c)^k(z/2x^{u+v})^{2k+p+1}(x)^{\eta+\theta-1}}{\Gamma(\alpha k+\mu)\Gamma(ak+p/\xi+(b+2)/2)}$$

$$\times \int_0^1 (s)^{\eta+u(2k+p+1)-1}(1-s)^{\theta+v(2k+p+1)-1}{}_pF_q[(l_p);(m_q) : ax^{\rho+\sigma}s^\rho(1-s)^\sigma]ds.$$

which gives the desired result. □

Similarly, we can find the following integrals.

Fifteenth integral formula:

$$\int_0^x (t)^{\eta-1}(x-t)^{\theta-1}{}_pF_q[(l_p);(m_q) : at^\rho(x-t)^\sigma]_a W^{\alpha,\mu}_{p,b,c}[zt^{-u}(x-t)^{-v}]dt$$

$$= x^{\eta+\theta-1}B(\eta-u(2k+p+1)+\rho l, \theta-v(2k+p+1)+\sigma l)$$

$$\times_a W^{\alpha,\mu}_{p,b,c,\xi}(z\, x^{-u-v})\sum_{l=0}^{\infty} f(l)t^{(\rho+\sigma)l}, \tag{7.5}$$

where $f(l)$ is defined in (7.3) provided

(1) $\Re(\eta) > 0, \Re(\theta) > 0$;

(2) ρ and σ are to be positive integers not both zero;

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(3) No m_q is to be zero or a negative integer;

(4) $p \leq \ q+1$, unless some l_q is a negative integer, in which case p may be positive integer.

Sixteenth integral formula:

$$\int_0^x (t)^{\eta-1}(x-t)^{\theta-1}{}_pF_q[(l_p);(m_q):at^\rho(x-t)^\sigma]_aW^{\alpha,\mu}_{p,b,c}[zt^u(x-t)^{-v}]dt$$

$$= x^{\eta+\theta-1}B(\eta+u(2k+p+1)+\rho l,\theta-v(2k+p+1)+\sigma l)$$

$$\times_a W^{\alpha,\mu}_{p,b,c,\xi}(z\ x^{u-v})\sum_{l=0}^{\infty} f(l)t^{(\rho+\sigma)l}, \tag{7.6}$$

where $f(l)$ is defined in (7.3) provided

(1) $\Re(\eta)>0, \Re(\theta)>0$;

(2) ρ and σ are to be positive integers not both zero;

(3) m_q is not to be zero or a negative integer;

(4) $p \leq \ q+1$, unless some l_q is a negative integer, in which case p may be positive integer.

Seventeenth integral formula:

$$\int_0^x (t)^{\eta-1}(x-t)^{\theta-1}{}_pF_q[(l_p);(m_q):at^\rho(x-t)^\sigma]_aW^{\alpha,\mu}_{p,b,c}[zt^{-u}(x-t)^{v}]dt$$

$$= x^{\eta+\theta-1}B(\eta-u(2k+p+1)+\rho l,\theta+v(2k+p+1)+\sigma l)$$

$$\times_a W^{\alpha,\mu}_{p,b,c,\xi}(z\ x^{-u+v})\sum_{l=0}^{\infty} f(l)t^{(\rho+\sigma)l}, \tag{7.7}$$

provided

(1) $\Re(\eta)>0, \Re(\theta)>0$;

(2) ρ and σ are to be positive integers not both zero;

(3) No m_q is to be zero or a negative integer;

(4) $p \leq \ q+1$, unless some l_q is a negative integer, in which case p may be positive integer.

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