

## A STUDY OF NEW CLASS OF INTEGRALS ASSOCIATED WITH GENERALIZED STRUVE FUNCTION AND POLYNOMIALS

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**ABSTRACT.** The main aim of this paper is to establish a new class of integrals involving the generalized Galuè-type Struve function with the different type of polynomials such as Jacobi, Legendre, and Hermite. Also, we derive the integral formula involving Legendre, Wright generalized Bessel and generalized Hypergeometric functions. The results obtained here are general in nature and can deduce many known and new integral formulas involving the various type of polynomials.

### 1. Introduction

The Struve function of order  $p$  is given by

$$(1.1) \quad H_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+3/2)\Gamma(k+p+3/2)} \left(\frac{x}{2}\right)^{2k+p}, \quad x \in \mathbb{R}, \quad a \in \mathbb{N}$$

is a particular solution of the non-homogeneous Bessel differential equation

$$(1.2) \quad x^2 y''(x) + xy'(x) + (x^2 - p^2)y(x) = \frac{4(x/2)^{p+1}}{\sqrt{\pi}\Gamma(p+1/2)},$$

where  $\Gamma$  is the classical gamma function whose Euler's integral is given by [17]

$$(1.3) \quad \Gamma(z) = \int_0^{\infty} \exp(-t) t^{z-1} dt, \quad \Re(z) > 0.$$

Galùè [6] introduced a generalization of the Bessel function of order  $p$  given by:

$$(1.4) \quad {}_a J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(ak+p+1)k!} \left(\frac{x}{2}\right)^{2k+p}, \quad x \in \mathbb{R}, \quad a \in \mathbb{N}.$$

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Baricz [1] investigated Galuè type generalization of modified Bessel function as:

$$(1.5) \quad {}_a I_p(x) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(ak + p + 1)k!} \left(\frac{x}{2}\right)^{2k+p}, \quad x \in \mathbb{R}, \quad a \in \mathbb{N}.$$

The Struve function and its more generalizations are found in many papers (see [2, 3, 7, 13–16]).

The generalized Struve function is given by Bhowmick [2]

$$(1.6) \quad H_l^\lambda(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k + 3/2)\Gamma(\lambda k + l + 3/2)} \left(\frac{x}{2}\right)^{2k+l+1}, \quad \lambda > 0$$

and by Kanth [7]

$$(1.7) \quad H_l^{\lambda,\alpha}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\alpha k + 3/2)\Gamma(\lambda k + l + 3/2)} \left(\frac{x}{2}\right)^{2k+l+1}, \quad \lambda > 0, \quad \alpha > 0.$$

Singh [13] gave another generalized form as

$$(1.8) \quad H_{l,\xi}^\lambda(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k + 3/2)\Gamma(\lambda k + l/\xi + 3/2)} \left(\frac{x}{2}\right)^{2k+l+1}, \quad \lambda > 0.$$

The generalized Struve function of four parameters was given by Singh [18] as:

$$(1.9) \quad H_{p,\mu}^{\lambda,\alpha}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\lambda k + p + 3/2)\Gamma(\alpha k + \mu)} \left(\frac{x}{2}\right)^{2k+p+1}, \quad \lambda, p \in \mathbb{C}$$

where  $\lambda > 0$ ,  $\alpha > 0$  and  $\mu$  is an arbitrary parameter. Another generalization of Struve function by Orhan and Yagmur [10, 22] is,

$$(1.10) \quad H_{p,b,c}(z) = \sum_{k=0}^{\infty} \frac{(-c)^k}{\Gamma(k + 3/2)\Gamma(k + p + b/2 + 1)} \left(\frac{z}{2}\right)^{2k+p+1}, \quad p, b, c \in \mathbb{C}.$$

Motivated from the above definitions, Nisar *et al.* [9] defined a new generalized form of Struve function named as generalized Galuè type Struve function (GTSF) as:

$$(1.11) \quad {}_a W_{p,b,c,\xi}^{\alpha,\mu}(z) = \sum_{k=0}^{\infty} \frac{(-c)^k}{\Gamma(ak + p/\xi + \frac{b+2}{2})\Gamma(\alpha k + \mu)} \left(\frac{z}{2}\right)^{2k+p+1}, \quad a \in \mathbb{N}, \quad p, b, c \in \mathbb{C},$$

where  $\alpha > 0$ ,  $\xi > 0$  and  $\mu$  is an arbitrary parameter.

The aim of this paper to establish various integral formulae involving GTSF employing different polynomials.

## 2. Integrals with Jacobi polynomials

The Jacobi polynomials  $P_n^{\rho,\sigma}(t)$  [11, 19] may be defined by

$$(2.1) \quad P_n^{\rho,\sigma}(t) = \frac{(1+\rho)_n}{n!} {}_2F_1 \left[ \begin{matrix} -n, 1+\rho+\sigma+n; \\ \rho+1; \end{matrix} \frac{1-t}{2} \right].$$

When  $\rho = \sigma = 0$ , the polynomial in (1.6) becomes the Legendre polynomial [11]. From (1.6) it follows that  $P_n^{\rho,\sigma}(t)$  is a polynomial of degree precisely  $n$  and that

$$(2.2) \quad P_n^{\rho,\sigma}(t) = \frac{(1+\rho)_n}{n!}.$$

**First integral formula:**

$$(2.3) \quad \begin{aligned} I_1 &= \int_{-1}^1 t^\lambda (1-t)^\rho (1+t)^\delta P_n^{\rho,\sigma}(t) {}_aW_{p,b,c,\xi}^{\alpha,\mu} [z(1+t)^h] dt \\ &= \frac{(-1)^n 2^{\rho+\delta+1} \Gamma(\delta+2hk+hp+h+1) \Gamma(n+\rho+1) \Gamma(\delta+2hk+hp+h+\sigma+1)}{n! \Gamma(\eta+hk+\sigma+n+1) \Gamma(\eta+hk+\rho+n+2)} \\ &\quad \times {}_aW_{p,b,c,\xi}^{\alpha,\mu} (2^h z) {}_3F_2 \left[ \begin{matrix} -\lambda, \delta+2hk+hp+h+\sigma+1, \delta+2hk+hp+h+1; \\ \delta+2hk+hp+h\sigma+n+1, \delta+2hk+hp+h+\rho+n+2; \end{matrix} 1 \right], \end{aligned}$$

provided  $(\alpha) > 0, (\xi) > 0, a \in \mathbb{N}, p, b, c \in \mathbb{C}, \mu$  is an arbitrary parameter and  $\Re(\lambda) > -1, \rho > -1, \sigma > -1$ .

*Proof.* In dealing with the Jacobi polynomials, it is natural to make much use of our knowledge of the  ${}_2F_1$  function [11]

$$(2.4) \quad \begin{aligned} I_1 &= \int_{-1}^1 t^\lambda (1-t)^\rho (1+t)^\delta P_n^{\rho,\sigma}(t) {}_aW_{p,b,c,\xi}^{\alpha,\mu} [z(1+t)^h] dt \\ &= \int_{-1}^1 t^\lambda (1-t)^\rho (1+t)^\delta P_n^{\rho,\sigma}(t) \sum_{k=0}^{\infty} \frac{(-c)^k (z(1+t)^h/2)^{2k+p+1}}{\Gamma(\alpha k + \mu) \Gamma(ak + p/\xi + (b+2)/2)} dt. \end{aligned}$$

Interchanging the order of integration and summation which is permissible under the condition, then the above expression becomes

$$(2.5) \quad \begin{aligned} I_1 &= \sum_{k=0}^{\infty} \frac{(-c)^k (z/2)^{2k+p+1}}{\Gamma(\alpha k + \mu) \Gamma(ak + p/\xi + (b+2)/2)} \\ &\quad \times \int_{-1}^1 t^\lambda (1-t)^\rho (1+t)^{\delta+2hk+hp+h} P_n^{\rho,\sigma}(t) dt. \end{aligned}$$

Using the formula [12]

$$(2.6) \quad \begin{aligned} \int_{-1}^1 t^\lambda (1-t)^\rho (1+t)^\delta P_n^{\rho,\sigma}(t) dt &= \frac{(-1)^n 2^{\rho+\delta+1} \Gamma(\delta+1) \Gamma(n+\rho+1) \Gamma(\delta+\sigma+1)}{n! \Gamma(\delta+\sigma+n+1) \Gamma(\delta+\rho+n+2)} \\ &\quad \times {}_3F_2 \left[ \begin{matrix} -\lambda, \delta+\sigma+1, \delta+1; \\ \delta+\sigma+n+1, \delta+\rho+n+2; \end{matrix} 1 \right]. \end{aligned}$$

provided  $\rho > -1$  and  $\sigma > -1$ , we get the desired result.  $\square$

**Second integral formula:**

$$\begin{aligned}
& \int_{-1}^1 (1-t)^\delta (1+t)^\sigma P_n^{(\rho,\sigma)}(t) P_n^{(\eta,\theta)}(t) {}_aW_{p,b,c,\xi}^{\alpha,\mu}[z(1-t)^h] dt \\
&= \frac{2^{\delta+\sigma+1} \Gamma(1+\eta+m) \Gamma(1+\rho+n)}{m!n!} \\
& \times \sum_{k=0}^{\infty} \frac{(-n)_k (1+\rho+\sigma+n)_k}{\Gamma(1+\rho+k)(k!)} \sum_{l=0}^{\infty} \frac{(-m)_l (1+\eta+\theta+m)_l}{\Gamma(1+\eta+l)(l!)} \\
(2.7) \quad & \times {}_aW_{p,b,c,\xi}^{\alpha,\mu}(2^h z) \mathbb{B}(1+\delta+h(2k+p+1)+k+l, 1+\sigma),
\end{aligned}$$

provided  $(\alpha) > 0, (\xi) > 0, a \in \mathbb{N}, p, b, c \in \mathbb{C}, \mu$  is an arbitrary parameter and  $\Re(\lambda) > -1, \rho > -1, \sigma > -1$ .

*Proof.* The left-hand side of (2.7) is denoted by  $I_2$ ,

$$\begin{aligned}
I_2 &= \int_{-1}^1 (1-t)^\delta (1+t)^\sigma P_n^{(\rho,\sigma)}(t) P_n^{(\eta,\theta)}(t) {}_aW_{p,b,c,\xi}^{\alpha,\mu}[z(1-t)^h] dt \\
&= \sum_{k=0}^{\infty} \frac{(-c)^k (z/2)^{2k+p+1}}{\Gamma(\alpha k + \mu) \Gamma(ak + p/\xi + (b+2)/2)} \\
(2.8) \quad & \times \int_{-1}^1 (1-t)^{\delta+2hk+hp+h} (1+t)^\sigma P_n^{(\rho,\sigma)}(t) P_m^{(\eta,\theta)}(t) dt.
\end{aligned}$$

Now using (2.1) in (2.8), we get

$$\begin{aligned}
I_2 &= \sum_{k=0}^{\infty} \frac{(-c)^k (z/2)^{2k+p+1}}{\Gamma(\alpha k + \mu) \Gamma(ak + p/\xi + (b+2)/2)} \frac{(1+\rho)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k (1+\rho+\sigma+n)_k}{(1+\rho)_k 2^k k!} \\
(2.9) \quad & \times \int_{-1}^1 (1-t)^{\delta+h(2k+p+1)+k} (1+t)^\sigma P_n^{(\eta,\theta)}(t) dt.
\end{aligned}$$

Again using (2.1) in (2.9), we obtain

$$\begin{aligned}
I_2 &= \sum_{k=0}^{\infty} \frac{((-c)^k (z/2)^{2k+p+1}}{\Gamma(\alpha k + \mu) \Gamma(ak + p/\xi + (b+2)/2)} \frac{\Gamma(1+\eta+m) \Gamma(1+\rho+n)}{m!n!} \\
& \times \sum_{k=0}^{\infty} \frac{(-n)_k (1+\rho+\sigma+n)_k}{\Gamma(1+\rho+k) 2^k (k!)} \sum_{l=0}^{\infty} \frac{(-m)_l (1+\eta+\theta+m)_l}{\Gamma(1+\eta+l) 2^l (l!)} \\
(2.10) \quad & \times \int_{-1}^1 (1-t)^{\delta+h(2k+p+1)+k+l} (1+t)^\sigma P_n^{(\rho,\sigma)}(t) dt.
\end{aligned}$$

Now using the formula [11, 19]

$$(2.11) \quad \int_{-1}^1 (1-t)^{\delta+n} (1+t)^{\sigma+n} dt = 2^{2n+\sigma+\delta+1} \mathbb{B}(1+\delta+n, 1+\sigma+n),$$

(2.10) becomes,

$$\begin{aligned}
 I_2 &= \frac{2^{\delta+\sigma+1}\Gamma(1+\eta+m)\Gamma(1+\rho+n)}{m!n!} \\
 &\times \sum_{k=0}^{\infty} \frac{(-n)_k(1+\rho+\sigma+n)_k}{\Gamma(1+\rho+k)(k!)} \sum_{l=0}^{\infty} \frac{(-m)_l(1+\eta+\theta+m)_l}{\Gamma(1+\eta+l)(l!)} \\
 (2.12) \quad &\times {}_aW_{p,b,c,\xi}^{\alpha,\mu}(2^h z)\mathbb{B}(1+\delta+h(2k+p+1)+k+l,1+\sigma). \quad \square
 \end{aligned}$$

**Third integral formula:**

$$\begin{aligned}
 I_3 &= \frac{2^{\eta+\theta+1}}{n!} \frac{(1+\rho)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k(1+\rho+\sigma+n)_k}{(1+\rho)_k(k!)} \\
 (2.13) \quad &\times {}_aW_{p,b,c,\xi}^{\alpha,\mu}(2^{h+l}z)\mathbb{B}(1+\eta+h(2k+p+1)+k,1+\theta+l(2k+p+1)+l),
 \end{aligned}$$

provided  $(\alpha) > 0, (\xi) > 0, a \in \mathbb{N}, p, b, c \in \mathbb{C}, \mu$  is an arbitrary parameter and  $\rho > -1, \sigma > -1$ .

*Proof.* Assume that the left-hand side of (2.13) is denoted by  $I_3$ , then

$$\begin{aligned}
 I_3 &= \int_{-1}^1 (1-t)^\eta(1+t)^\theta P_n^{(\rho,\sigma)}(t) {}_aW_{p,b,c,\xi}^{\alpha,\mu}[z(1-t)^h(1+t)^t] dt \\
 &= \sum_{k=0}^{\infty} \frac{((-c)^k(z/2)^{2k+p+1}}{\Gamma(\alpha k + \mu)\Gamma(ak + p/\xi + (b+2)/2)} \\
 (2.14) \quad &\times \int_{-1}^1 (1-t)^{\eta+h(2k+p+1)}(1+t)^{\theta+l(2k+p+1)} P_n^{(\rho,\sigma)}(t) dt.
 \end{aligned}$$

Now using (2.1) in (2.14), we obtain

$$\begin{aligned}
 I_3 &= \sum_{k=0}^{\infty} \frac{((-c)^k(z/2)^{2k+p+1}}{\Gamma(\alpha k + \mu)\Gamma(ak + p/\xi + (b+2)/2)} \frac{(1+\rho)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k(1+\rho+\sigma+n)_k}{(1+\rho)_k 2^k k!} \\
 (2.15) \quad &\times \int_{-1}^1 (1-t)^{\eta+h(2k+p+1)+k}(1+t)^{\theta+l(2k+p+1)+l} dt.
 \end{aligned}$$

Further using (2.11) in (2.15) we get the required result.  $\square$

**Fourth integral formula:**

$$\begin{aligned}
 &\int_{-1}^1 (1-t)^\eta(1+t)^\theta P_n^{(\rho,\sigma)}(t) {}_aW_{p,b,c,\xi}^{\alpha,\mu}[z(1+t)^{-h}] dt \\
 &= \frac{2^{\eta+\theta+1}}{n!} \frac{(1+\rho)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k(1+\rho+\sigma+n)_k}{(1+\rho)_k(k!)} \\
 (2.16) \quad &\times {}_aW_{p,b,c,\xi}^{\alpha,\mu}(2^{-h}z)\mathbb{B}(1+\eta+k,1+\theta-h(2k+p+1)),
 \end{aligned}$$

provided  $(\alpha) > 0, (\xi) > 0, a \in \mathbb{N}, p, b, c \in \mathbb{C}, \mu$  is an arbitrary parameter and  $\rho > -1, \sigma > -1$ .

*Proof.* The left-hand side of (2.16) is denoted by  $I_4$ ,

$$\begin{aligned} I_4 &= \int_{-1}^1 (1-t)^\eta (1+t)^\theta P_n^{(\rho, \sigma)}(t) {}_aW_{p, b, c, \xi}^{\alpha, \mu} [z(1+t)^{-h}] dt \\ &= \sum_{k=0}^{\infty} \frac{(-c)^k (z/2)^{2k+p+1}}{\Gamma(\alpha k + \mu) \Gamma(ak + p/\xi + (b+2)/2)} \\ (2.17) \quad &\times \int_{-1}^1 (1-t)^\eta (1+t)^{\theta-h(2k+p+1)} P_n^{(\rho, \sigma)}(t) dt. \end{aligned}$$

Now using (2.1) in (2.17) we have

$$\begin{aligned} I_4 &= \sum_{k=0}^{\infty} \frac{(-c)^k (z/2)^{2k+p+1}}{\Gamma(\alpha k + \mu) \Gamma(ak + p/\xi + (b+2)/2)} \frac{(1+\rho)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k (1+\rho+\sigma+n)_k}{(1+\rho)_k 2^k k!} \\ (2.18) \quad &\times \int_{-1}^1 (1-t)^{\eta+k} (1+t)^{\theta-h(2k+p+1)} dt \end{aligned}$$

further using (2.11) in (2.18) we attain the required result.  $\square$

**Fifth integral formula:**

$$\begin{aligned} &\int_{-1}^1 (1-t)^\rho (1+t)^\theta P_n^{(\rho, \sigma)}(t) {}_aW_{p, b, c, \xi}^{\alpha, \mu} [z(1-t)^h (1+t)^{-l}] dt \\ &= 2^{\eta+\theta+1} \frac{(1+\rho)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k (1+\rho+\sigma+n)_k}{(1+\rho)_k (k!)} \\ (2.19) \quad &\times {}_aW_{p, b, c, \xi}^{\alpha, \mu} (2^{h-l} z) \mathbb{B}(1+\mu+h(2k+p+1)+k, 1+\theta-l(2k+p+1)), \end{aligned}$$

provided  $(\alpha) > 0, (\xi) > 0, a \in \mathbb{N}, p, b, c \in \mathbb{C}, \mu$  is an arbitrary parameter and  $\rho > -1, \sigma > -1$ .

*Proof.* The left-hand side of (2.19) is denoted by  $I_5$ , then we have

$$\begin{aligned} I_5 &= \int_{-1}^1 (1-t)^\rho (1+t)^\theta P_n^{(\rho, \sigma)}(t) {}_aW_{p, b, c, \xi}^{\alpha, \mu} [z(1-t)^h (1+t)^{-l}] dt \\ &= \sum_{k=0}^{\infty} \frac{(-c)^k (z/2)^{2k+p+1}}{\Gamma(\alpha k + \mu) \Gamma(ak + p/\xi + (b+2)/2)} \\ (2.20) \quad &\times \int_{-1}^1 (1-t)^{\eta+h(2k+p+1)} (1+t)^{\theta-l(2k+p+1)} P_n^{(\rho, \sigma)}(t) dt, \end{aligned}$$

now using (2.1) in (2.20) we have

$$\begin{aligned} I_5 &= \sum_{k=0}^{\infty} \frac{(-c)^k (z/2)^{2k+p+1}}{\Gamma(\alpha k + \mu) \Gamma(ak + p/\xi + (b+2)/2)} \frac{(1+\rho)_n}{n!} \\ &\times \sum_{k=0}^{\infty} \frac{(-n)_k (1+\rho+\sigma+n)_k}{(1+\rho)_k 2^k k!} \end{aligned}$$

$$(2.21) \quad \times \int_{-1}^1 (1-t)^{\eta+h(2k+p+1)+k} (1+t)^{\theta-l(2k+p+1)} dt$$

further using (2.11) in (2.21) we get the desired result.  $\square$

### 3. Special cases:

#### Sixth integral formula:

If we replace  $\delta$  by  $\lambda - 1$  and put  $\rho = \sigma = \mu = \theta = 0$ , then the integral  $I_2$  transforms into the following integral involving Legendre polynomial [11].

$$(3.1) \quad \begin{aligned} I_6 &= \int_{-1}^1 (1-t)^{\lambda-1} P_n(t) {}_aW_{p,b,c,\xi}^{\alpha,\mu} [z(1-t)^h] dt \\ &= \sum_{k=0}^{\infty} \frac{2^\lambda (-n)_k (1+n)_k}{(k!)^2} \times \sum_{l=0}^{\infty} \frac{(-m)_l (1+m)_l}{l!^2} \\ &\times {}_aW_{p,b,c,\xi}^{\alpha,\mu} (2^h z) \mathbb{B}(1 + \lambda + h(2k + p + 1) + k + l, 1). \end{aligned}$$

#### Seventh integral formula:

If  $\sigma = \rho = 0$ ,  $\eta$  is replaced by  $\eta - 1$  and  $\theta$  by  $\theta - 1$  then the integral  $I_3$  transforms into the following integral involving Legendre polynomial [11].

$$(3.2) \quad \begin{aligned} I_7 &= \int_{-1}^1 (1-t)^{\eta-1} (1+t)^{\theta-1} P_n(t) E_{p,b,c,\xi}^{\alpha,\mu} [z(1-t)^h (1+t)^l] dy, \\ &= \sum_{k=0}^{\infty} \frac{2^{\mu+\theta-1} (-n)_k (1+n)_k}{(k!)^2} \\ &\times {}_aW_{p,b,c,\xi}^{\alpha,\mu} (2^{h+l} z) \mathbb{B}(1 + \eta + h(2k + p + 1) + k, \theta + l(2k + p + 1)). \end{aligned}$$

#### Eighth integral formula:

If  $\rho = \sigma = 0$ ,  $\eta$  is replaced by  $\eta - 1$  and  $\theta$  by  $\theta - 1$  then the integral  $I_3$  transforms into the following integral involving Legendre polynomial [11]

$$(3.3) \quad \begin{aligned} I_8 &= \int_{-1}^1 (1-t)^{\eta-1} (1+t)^{\theta-1} P_n(t) {}_aW_{p,b,c,\xi}^{\alpha,\mu} [z(1-t)^h (1+t)^{-l}] dt, \\ &= \sum_{k=0}^{\infty} \frac{2^{\eta+\theta-1} (-n)_k (1+n)_k}{(k!)^2} \\ &\times {}_aW_{p,b,c,\xi}^{\alpha,\mu} (2^{h-l} z) \mathbb{B}(1 + \mu + h(2k + p + 1) + k, \theta - l(2k + p + 1)). \end{aligned}$$

### 4. Integral with Wright generalized Bessel function

The special case of the Wright function [5] and (see the ref. [20, 21]) in the form

$$(4.1) \quad \phi(A, a; z) = {}_0\psi_1 \left[ \begin{matrix} -; \\ (A, a); \end{matrix} z \right] = \sum_{k=0}^{\infty} \frac{1}{\Gamma(Ak + a)} \frac{z^k}{k!},$$

with complex  $z, a \in \mathbb{C}$  and real  $A \in \mathbb{R}$  when  $A = \eta, a = \nu + 1$  and  $z$  is replaced by  $-z$ , then the function  $\phi(\eta, \nu + 1; -z)$  is defined by  $J_\nu^\eta(z)$

$$(4.2) \quad \phi(\eta, \nu + 1; -z) = J_\nu^\eta(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\eta k + \nu + 1)} \frac{(-z)^k}{k!},$$

and such a function is known as the Bessel-Maitland function or the Wright generalized Bessel function see [8].

**Ninth integral formula:**

$$(4.3) \quad \begin{aligned} & \int_{-\infty}^{\infty} (t)^\theta J_\eta^\tau(t) {}_a W_{p,b,c,\xi}^{\alpha,\mu} [z(t)^\rho] dt \\ &= \sum_{k=0}^{\infty} \frac{(-c)^k (z/2)^{2k+p+1}}{\Gamma(\alpha k + \mu) \Gamma(\alpha k + p/\xi + (b+2)/2)} \frac{\Gamma(\theta + \rho(2k+p+1) + 1)}{\Gamma(1 + \eta - \tau - \tau(\eta + \rho(2k+p+1)))} \\ &= \frac{\Gamma(\theta + \rho(2k+p+1) + 1)}{\Gamma(1 + \eta - \tau - \tau(\eta + \rho(2k+p+1)))} {}_a W_{p,b,c,\xi}^{\alpha,\mu}(z), \end{aligned}$$

provided  $(\alpha) > 0, (\xi) > 0, a \in \mathbb{N}, p, b, c \in \mathbb{C}, \mu$  is an arbitrary parameter,  $\rho - \rho\tau > -1$  and  $\rho > 0$  and  $0 < \tau < 1$  and  $\Re(\theta + 1) > 0$ .

*Proof.* The left-hand side of (4.3) is denoted by  $I_9$

$$(4.4) \quad \begin{aligned} I_9 &= \int_{-\infty}^{\infty} (t)^\theta J_\eta^\tau(t) {}_a W_{p,b,c,\xi}^{\alpha,\mu} [z(t)^\rho] dt \\ &= \sum_{k=0}^{\infty} \frac{(-c)^k (z/2)^{2k+p+1}}{\Gamma(\alpha k + \mu) \Gamma(\alpha k + p/\xi + (b+2)/2)} \int_0^{\infty} (t^{\theta+\rho(2k+p+1)}) J_\eta^\tau(t) dt. \end{aligned}$$

Now we use the following formula (see the ref.[12])

$$(4.5) \quad \int_0^{\infty} (t)^\theta J_\eta^\tau(t) dt = \frac{\Gamma(\theta + 1)}{\Gamma(1 + \eta - \tau - \tau\eta)},$$

provided  $\Re(\theta) > -1, 0 < \tau < 1$ .

Now using (4.5) in (4.4), we arrive the required result.  $\square$

## 5. Integral with Legendre functions

The Legendre functions are solution of Legendres differential equation (see the ref. [4])

$$(5.1) \quad (1 - z^2) \frac{d^2 w}{dz^2} - 2z \frac{dw}{dz} + [(\eta(\eta + 1)) - \mu^2(1 - z^2)^{-1}] w = 0,$$

where  $z, \eta, \mu$  unrestricted.

Under the substitution  $w = (z^2 - 1)^{\mu/2} \eta$  in (5.1) becomes

$$(5.2) \quad (1 - z^2) \frac{d^2 \eta}{dz^2} - 2(\mu + 1)z \frac{d\eta}{dz} + [(\eta + \mu)(\eta + \mu + 1)] \eta = 0,$$



and with  $\xi = 1/2 - z/2$  as the independent variable, this differential equation becomes.

$$(5.3) \quad \xi(1-\xi)\frac{d^2\eta}{d\xi^2} + (\nu+1)(1-2\xi)\frac{d\eta}{d\xi} + [(\eta-\mu)(\eta+\mu+1)]\eta = 0.$$

This is the Gauss hypergeometric type equation with  $a = \mu - \eta, b = \eta + \mu + 1, c = \mu + 1$ .

Hence it follows that the function

$$(5.4) \quad W = P_\eta^\mu(z) = \frac{1}{\Gamma(1-\mu)} \left(\frac{z+1}{z-1}\right)^{\beta/2} {}_2F_1 \left[ \begin{matrix} -\eta, \eta+1; \\ 1-\mu; \end{matrix} 1/2 - z/2 \right],$$

$|1-z| < 2$  is a solution of (5.1).

The function  $P_\nu^\beta(z)$  is known as the Legendre function of first kind [4]. It is one valued and regular z-plane supposed cut along the real axis from 1 to  $-\infty$ .

**Tenth integral formula:**

$$(5.5) \quad \int_0^1 (t)^{\theta-1} (1-t^2)^{\delta/2} P_\eta^\delta(t) {}_aW_{p,b,c,\xi}^{\alpha,\mu}[z(t)^\rho] dt \\ = \frac{(-1)^\delta \sqrt{\pi} 2^{-\theta-\delta} \Gamma(\eta+\delta+1)}{\Gamma(1-\delta+\eta)} \\ \times \sum_{k=0}^{\infty} \frac{\Gamma(\theta+\rho(2k+p+1))}{\Gamma(1/2+\frac{\theta+\rho(2k+p+1)}{2}+\delta/2-\eta/2)\Gamma(1+\frac{\theta+\rho(2k+p+1)}{2}+\delta/2+\eta/2)} {}_aW_{p,b,c,\xi}^{\alpha,\mu}(z/2^\rho),$$

provided  $(\alpha) > 0, (\xi) > 0, a \in \mathbb{N}, p, b, c \in \mathbb{C}, \mu$  is an arbitrary parameter and  $\theta > 0$  and  $\delta$  is positive integer.

*Proof.* Taking the left-hand side of (5.5) by  $I_{10}$ , we have

$$(5.6) \quad I_{10} = \int_0^1 (t)^{\theta-1} (1-t^2)^{\delta/2} P_\eta^\delta(t) {}_aW_{p,b,c,\xi}^{\alpha,\mu}[z(t)^\rho] dt \\ = \sum_{k=0}^{\infty} \frac{(-c)^k (z/2)^{2k+p+1}}{\Gamma(\alpha k + \mu)\Gamma(ak + p/\xi + (b+2)/2)} \\ \times \int_0^1 t^{\theta-1+\rho(2k+p+1)} (1-t^2)^{\delta/2} P_\eta^\delta(t) dt.$$

Now the integral in (5.6) can be solved by using the following formula [4]

$$(5.7) \quad \int_0^1 t^{\theta-1} (1-t^2)^{\delta/2} P_\eta^\delta(t) dt \\ = \frac{(-1)^\delta \sqrt{\pi} 2^{-\theta-\delta} \Gamma(\theta)\Gamma(\eta+\delta+1)}{(\Gamma(1/2+\theta/2+\delta/2-\eta/2)\Gamma(1+\theta/2+\delta/2+\eta/2)(1-\delta+\eta))},$$

provided  $\Re(\theta) > 0, \delta = 1, 2, 3, \dots$

Now (5.6) becomes

$$I_{10} = \sum_{k=0}^{\infty} \frac{(-c)^k (z/2)^{2k+p+1}}{\Gamma(\alpha k + \mu)\Gamma(ak + p/\xi + (b+2)/2)}$$

$$\begin{aligned}
& \times \frac{(-1)^\delta \sqrt{\pi} 2^{-(\theta+\rho(2k+p+1))-\delta} \Gamma(\theta + \rho(2k+p+1)) \Gamma(\delta + \eta + 1)}{\Gamma(1/2 + \frac{\theta+\rho k}{2} + \delta/2 - \eta/2) \Gamma(1 + \frac{\theta+\rho k}{2} + \delta/2 + \eta/2) \Gamma(1 - \delta + \eta)} \\
& = \frac{(-1)^\delta \sqrt{\pi} 2^{-\theta-\delta} \Gamma(\eta + \delta + 1)}{\Gamma(1 - \delta + \eta)} \\
& \times \sum_{k=0}^{\infty} \frac{\Gamma(\theta + \rho(2k+p+1))}{\Gamma(1/2 + \frac{\theta+\rho(2k+p+1)}{2} + \delta/2 - \eta/2) \Gamma(1 + \frac{\theta+\rho(2k+p+1)}{2} + \delta/2 + \eta/2)} {}_a W_{p,b,c,\xi}^{\alpha,\mu}(z/2^\rho) \\
& = \sum_{k=0}^{\infty} \frac{(-c)^k (z/2)^{2k+p+1}}{\Gamma(\alpha k + \mu) \Gamma(ak + p/\xi + (b+2)/2)} \\
& \times \frac{(-1)^\delta \sqrt{\pi} 2^{-(\theta+\rho(2k+p+1))-\delta} \Gamma(\theta + \rho(2k+p+1)) \Gamma(\delta + \eta + 1)}{\Gamma(1/2 + \frac{\theta+\rho k}{2} + \delta/2 - \eta/2) \Gamma(1 + \frac{\theta+\rho k}{2} + \delta/2 + \eta/2) \Gamma(1 - \delta + \eta)}. \quad \square
\end{aligned}$$

**Eleventh integral formula:**

$$\begin{aligned}
& \int_0^1 (t)^{\theta-1} (1-t^2)^{-\delta/2} P_\eta^\delta(t) {}_a W_{p,b,c,\xi}^{\alpha,\mu}[z(t)^\rho] dt \\
(5.8) \quad & = \sqrt{\pi} 2^{-\theta+\delta} \sum_{k=0}^{\infty} \frac{\Gamma(\theta + \rho(2k+p+1))}{\Gamma(\frac{1}{2} + \frac{\theta+\rho(2k+p+1)}{2} - \frac{\delta}{2} - \frac{\eta}{2}) \Gamma(1 + \frac{\theta+\rho(2k+p+1)}{2} - \frac{\delta}{2} - \frac{\eta}{2})} {}_a W_{p,b,c,\xi}^{\alpha,\mu}(\frac{z^\rho}{2}),
\end{aligned}$$

provided  $(\alpha) > 0$ ,  $(\xi) > 0$ ,  $a \in \mathbb{N}$ ,  $p, b, c \in \mathbb{C}$ ,  $\mu$  is an arbitrary parameter and  $\theta > 0$  and  $\delta$  is positive integer.

*Proof.* Taking the LHS by  $I_{11}$ , we have

$$\begin{aligned}
I_{11} & = \int_0^1 (t)^{\theta-1} (1-t^2)^{-\delta/2} P_\eta^\delta(t) {}_a W_{p,b,c,\xi}^{\alpha,\mu}[z(t)^\rho] dt \\
& = \sum_{k=0}^{\infty} \frac{(-c)^k (z/2)^{2k+p+1}}{\Gamma(\alpha k + \mu) \Gamma(ak + p/\xi + (b+2)/2)} \\
(5.9) \quad & \times \int_0^1 t^{\theta-1+\rho(2k+p+1)} (1-t^2)^{-\delta/2} P_\eta^\delta(t) dt.
\end{aligned}$$

Now the integral in (5.9) can be solved by using the formula [4]

$$\int_0^1 t^{\theta-1} (1-t^2)^{-\delta/2} P_\eta^\delta(t) dt = \frac{\sqrt{\pi} 2^{-\theta+\delta} \Gamma(\theta)}{\Gamma(1/2 + \theta/2 - \delta/2 - \eta/2) \Gamma(1 + \theta/2 - \delta/2 - \eta/2)},$$

provided  $\Re(\theta) > 0$ ,  $\delta = 1, 2, 3, \dots$

Again (5.9) becomes

$$\begin{aligned}
I_{11} & = \sum_{k=0}^{\infty} \frac{(-c)^k (z/2)^{2k+p+1}}{\Gamma(\alpha k + \mu) \Gamma(ak + p/\xi + (b+2)/2)} \\
& \times \frac{\sqrt{\pi} 2^{-(\theta+\rho(2k+p+1))+\delta} \Gamma(\theta + \rho(2k+p+1))}{\Gamma(1/2 + \frac{\theta+\rho(2k+p+1)}{2} - \delta/2 - \eta/2) \Gamma(1 + \frac{\theta+\rho(2k+p+1)}{2} - \delta/2 - \eta/2)} \\
& = \sqrt{\pi} 2^{-\theta+\delta} \sum_{k=0}^{\infty} \frac{\Gamma(\theta + \rho(2k+p+1))}{\Gamma(1/2 + \frac{\theta+\rho(2k+p+1)}{2} - \delta/2 - \eta/2) \Gamma(1 + \frac{\theta+\rho(2k+p+1)}{2} - \delta/2 - \eta/2)}
\end{aligned}$$

$$(5.10) \quad \times_a W_{p,b,c,\xi}^{\alpha,\mu}(z/2^p). \quad \square$$

### 6. Integrals with Hermite polynomials

Hermite polynomials  $H_n(t)$  (see the ref. [11, 19]) may be defined by means of the relation

$$(6.1) \quad \exp(2tx - x^2) = \sum_{k=0}^{\infty} \frac{H_n(t)x^n}{n!},$$

valid for all finite  $t$  and  $x$ . Since

$$\begin{aligned} \exp(2tx - x^2) &= \exp(2tx) \exp(-x^2) \\ &= \sum_{n=0}^{\infty} \frac{(2t)^n x^n}{n!} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{k!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} \frac{(-1)^k (2t)^{n-2k} x^n}{(n-2k)!k!}. \end{aligned}$$

It follows from (6.1) that

$$(6.2) \quad H_n(t) = \sum_{k=0}^{[n/2]} \frac{(-1)^k (2t)^{n-2k} x^n}{(n-2k)!k!}.$$

Examination of equation (6.2) shows that  $H_n(t)$  is a polynomial of degree precisely  $n$  in  $t$  and that

$$(6.3) \quad H_n(t) = 2^n t^n + \pi_{n-2}(t)$$

in which  $\pi_{n-2}(t)$  is a polynomial of degree  $(n-2)$  in  $t$ .

**Twelfth integral formula:**

$$(6.4) \quad \begin{aligned} &\int_{-\infty}^{\infty} (t)^{2\mu} \exp(-t^2) H_{2\eta}(t)_a W_{p,b,c,\xi}^{\alpha,\mu}[z(t)^{-2h}] dt \\ &= \sqrt{\pi} 2^{2(\eta-\mu)} \sum_{k=0}^{\infty} \frac{\Gamma(2\mu - h(2k+p+1) + 1)}{\Gamma(\mu - h(2k+p+1) - \eta + 1)} {}_a W_{p,b,c,\xi}^{\alpha,\mu}(2^{2h}z), \end{aligned}$$

provided  $(\alpha) > 0, (\xi) > 0, a \in \mathbb{N}, p, b, c \in \mathbb{C}, \mu$  is an arbitrary parameter and  $\theta > 0$  and  $\delta$  is positive integer.

*Proof.* The LHS of (6.4) is denoted by  $I_{12}$

$$(6.5) \quad \begin{aligned} I_{12} &= \int_{-\infty}^{\infty} (t)^{2\mu} \exp(-t^2) H_{2\eta}(t)_a W_{p,b,c,\xi}^{\alpha,\mu}[z(t)^{-2h}] dt \\ &= \sum_{k=0}^{\infty} \frac{(-c)^k (z/2)^{2k+p+1}}{\Gamma(\alpha k + \mu)(\Gamma(ak + p/\xi + (b+2)/2))} \\ &\quad \times \int_{-\infty}^{\infty} t^{2\mu-2h(2k+p+1)} (\exp)^{-t^2} H_{2\eta}(t) dt. \end{aligned}$$

Now the integral in (6.5) can be solved by using the formula see [12]

$$(6.6) \quad \int_{-\infty}^{\infty} t^{2\mu} (\exp)^{-t^2} H_{2\eta}(t) dt = \frac{\sqrt{\pi} 2^{2(\eta-\mu)} \Gamma(2\mu+1)}{\Gamma(\mu-\eta+1)}.$$

Again (6.5) becomes

$$I_{12} = \sum_{k=0}^{\infty} \frac{(-c)^k (z/2)^{2k+p+1}}{\Gamma(\alpha k + \mu) \Gamma(ak + p/\xi + (b+2)/2)} \\ \times \frac{\sqrt{\pi} 2^{2(\eta-(2\mu+2h(2k+p+1)))} \Gamma(2\mu - 2h(2k+p+1) + 1)}{\Gamma(\mu - h(2k+p+1) - \eta + 1)}.$$

Hence the desired result.  $\square$

**Thirteenth integral formula:**

$$(6.7) \quad \int_{-\infty}^{\infty} (t)^{2\mu} \exp(-t^2) H_{2\eta}(t) {}_aW_{p,b,c,\xi}^{\alpha,\mu}[z(t)^{2h}] dt \\ = \sqrt{\pi} 2^{2(\eta-\mu)} \sum_{k=0}^{\infty} \frac{\Gamma(2\mu + 2h(2k+p+1) + 1)}{\Gamma(\mu + h(2k+p+1) - \eta + 1)} {}_aW_{p,b,c,\xi}^{\alpha,\mu}(2^{-2h}z),$$

provided  $(\alpha) > 0$ ,  $(\xi) > 0$ ,  $a \in \mathbb{N}$ ,  $p, b, c \in \mathbb{C}$ ,  $\mu$  is an arbitrary parameter and  $h > 0$  and  $\Re(\mu) = 0, 1, 2, \dots$

*Proof.* Denoting the LHS of (6.7), we have

$$(6.8) \quad I_{13} = \int_{-\infty}^{\infty} (t)^{2\mu} \exp(-t^2) H_{2\eta}(t) {}_aW_{p,b,c,\xi}^{\alpha,\mu}[z(t)^{2h}] dt \\ = \sum_{k=0}^{\infty} \frac{(-c)^k (z/2)^{2k+p+1}}{\Gamma(\alpha k + \mu) \Gamma(ak + p/\xi + (b+2)/2)} \\ \times \int_{-\infty}^{\infty} t^{2\mu+2h(2k+p+1)} (\exp)^{-t^2} H_{2\eta}(t) dt.$$

Using the formula given in (6.6) we reach the required proof.  $\square$

**7. Integral with generalized hypergeometric function**

A generalized hypergeometric function [11] may be defined by

$$(7.1) \quad {}_pF_q \left[ \begin{matrix} (\rho)_1, (\rho)_2, \dots, (\rho)_p; \\ (\sigma)_1, (\sigma)_2, \dots, (\sigma)_q; \end{matrix} ; z \right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^p (\rho_i)_n}{\prod_{j=1}^q (\sigma_j)_n} \frac{z^n}{n!},$$

in which no denominator parameter  $\sigma_j$  is allowed to be zero or a negative integer. If any numerator parameter  $\rho_i$  in (7.1) is zero or a negative integer, then the series terminates.

**Fourteenth integral formula:**

$$\int_0^x (t)^{\eta-1} (x-t)^{\theta-1} {}_pF_q[(l_p); (m_q) : at^\rho(x-t)^\sigma] {}_aW_{p,b,c}^{\alpha,\mu}[zt^u(x-t)^v] dt$$

$$\begin{aligned}
 &= x^{\eta+\theta-1} B(\eta + u(2k + p + 1) + \rho l, \theta + v(2k + p + 1) + \sigma l) \\
 (7.2) \quad &\times_a W_{p,b,c,\xi}^{\alpha,\mu}(z x^{u+v}) \sum_{l=0}^{\infty} f(l) t^{(\rho+\sigma)l},
 \end{aligned}$$

where

$$(7.3) \quad f(l) = \frac{(l_1)_l, \dots, (l_p)_l (a)^l}{(m_1)_l, \dots, (m_q)_l l!}$$

provided

- (1)  $\Re(\eta) > 0, \Re(\theta) > 0$ ;
- (2)  $\rho$  and  $\sigma$  are to be positive integers not both zero;
- (3) No  $m_q$  is to be zero or a negative integer;
- (4)  $p \leq q + 1$ , unless some  $l_q$  is a negative integer, in which case  $p$  may be positive integer.

*Proof.* Representing the LHS of (7.2) by  $I_{14}$

$$\begin{aligned}
 I_{14} &= \int_0^x (t)^{\eta-1} (x-t)^{\theta-1} {}_pF_q[(l_p); (m_q) : at^\rho(x-t)^\sigma]_a W_{p,b,c}^{\alpha,\mu}[zt^u(x-t)^v] dt \\
 &= \sum_{k=0}^{\infty} \frac{(-c)^k (z/2)^{2k+p+1}}{\Gamma(\alpha k + \mu) \Gamma(ak + p/\xi + (b+2)/2)} \\
 (7.4) \quad &\times \int_0^x t^{\eta+u(2k+p+1)-1} (x-t)^{\theta+v(2k+p+1)-1} {}_pF_q[(l_p); (m_q) : at^\rho(x-t)^\sigma] dt,
 \end{aligned}$$

putting  $t = sx$  and  $dt = xds$ , then we get

$$\begin{aligned}
 I_{14} &= \sum_{k=0}^{\infty} \frac{(-c)^k (z/2x^{u+v})^{2k+p+1} (x)^{\eta+\theta-1}}{\Gamma(\alpha k + \mu) \Gamma(ak + p/\xi + (b+2)/2)} \\
 &\times \int_0^1 (s)^{\eta+u(2k+p+1)-1} (1-s)^{\theta+v(2k+p+1)-1} {}_pF_q[(l_p); (m_q) : ax^{\rho+\sigma} s^\rho (1-s)^\sigma] ds.
 \end{aligned}$$

which gives the desired result.  $\square$

Similarly, we can find the following integrals.

**Fifteenth integral formula:**

$$\begin{aligned}
 &\int_0^x (t)^{\eta-1} (x-t)^{\theta-1} {}_pF_q[(l_p); (m_q) : at^\rho(x-t)^\sigma]_a W_{p,b,c}^{\alpha,\mu}[zt^{-u}(x-t)^{-v}] dt \\
 &= x^{\eta+\theta-1} B(\eta - u(2k + p + 1) + \rho l, \theta - v(2k + p + 1) + \sigma l) \\
 (7.5) \quad &\times_a W_{p,b,c,\xi}^{\alpha,\mu}(z x^{-u-v}) \sum_{l=0}^{\infty} f(l) t^{(\rho+\sigma)l},
 \end{aligned}$$

where  $f(l)$  is defined in (7.3) provided

- (1)  $\Re(\eta) > 0, \Re(\theta) > 0$ ;
- (2)  $\rho$  and  $\sigma$  are to be positive integers not both zero;

(3) No  $m_q$  is to be zero or a negative integer;

(4)  $p \leq q + 1$ , unless some  $l_q$  is a negative integer, in which case  $p$  may be positive integer.

**Sixteenth integral formula:**

$$\int_0^x (t)^{\eta-1} (x-t)^{\theta-1} {}_pF_q[(l_p); (m_q) : at^\rho(x-t)^\sigma] {}_aW_{p,b,c}^{\alpha,\mu}[zt^u(x-t)^{-v}] dt$$

$$= x^{\eta+\theta-1} B(\eta + u(2k+p+1) + \rho l, \theta - v(2k+p+1) + \sigma l)$$

(7.6)

$$\times {}_aW_{p,b,c,\xi}^{\alpha,\mu}(z x^{u-v}) \sum_{l=0}^{\infty} f(l) t^{(\rho+\sigma)l},$$

where  $f(l)$  is defined in (7.3) provided

(1)  $\Re(\eta) > 0, \Re(\theta) > 0$ ;

(2)  $\rho$  and  $\sigma$  are to be positive integers not both zero;

(3)  $m_q$  is not to be zero or a negative integer;

(4)  $p \leq q + 1$ , unless some  $l_q$  is a negative integer, in which case  $p$  may be positive integer.

**Seventeenth integral formula:**

$$\int_0^x (t)^{\eta-1} (x-t)^{\theta-1} {}_pF_q[(l_p); (m_q) : at^\rho(x-t)^\sigma] {}_aW_{p,b,c}^{\alpha,\mu}[zt^{-u}(x-t)^v] dt$$

$$= x^{\eta+\theta-1} B(\eta - u(2k+p+1) + \rho l, \theta + v(2k+p+1) + \sigma l)$$

(7.7)  $\times {}_aW_{p,b,c,\xi}^{\alpha,\mu}(z x^{-u+v}) \sum_{l=0}^{\infty} f(l) t^{(\rho+\sigma)l},$

provided

(1)  $\Re(\eta) > 0, \Re(\theta) > 0$ ;

(2)  $\rho$  and  $\sigma$  are to be positive integers not both zero;

(3) No  $m_q$  is to be zero or a negative integer;

(4)  $p \leq q + 1$ , unless some  $l_q$  is a negative integer, in which case  $p$  may be positive integer.

## References

- [1] A. Baricz, *Generalized Bessel Functions of the First Kind*, Lecture Notes in Mathematics, **1994**, Springer-Verlag, Berlin, 2010.
- [2] K. N. Bhowmick, *Some relations between a generalized Struve's function and hypergeometric functions*, Vijnana Parishad Anusandhan Patrika **5** (1962), 93–99.
- [3] ———, *A generalized Struve's function and its recurrence formula*, Vijnana Parishad Anusandhan Patrika **6** (1963), 1–11.
- [4] A. Erdelyi and H. Bateman, *Higher Transcendental Functions. Vol. 1*, McGraw Hill, 1953.
- [5] A. Erdelyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions. Vol. 3*, McGraw-Hill, New York, NY USA, 1955.
- [6] L. Galuè, *A generalized Bessel function*, Integral Transforms Spec. Funct. **14** (2003), no. 5, 395–401.

- [7] B. N. Kanth, *Integrals involving generalised Struve's function*, Nepali Math. Sci. Rep. **6** (1981), no. 1-2, 61–64.
- [8] V. Kiryakova, *Generalized Fractional Calculus and Applications*, Pitman Research Notes in Mathematics Series, **301**, Longman Scientific & Technical, Harlow, 1994.
- [9] K. S. Nisar, D. Baleanu, and M. M. Al-Qurashi, *Fractional calculus and application of generalized Struve function*, SpringerPlus, **2016** (2016), DOI 10.1186/s40064-016-2560-3.
- [10] H. Orhan and N. Yagmur, *Geometric properties of generalized Struve functions*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.) **63** (2017), no. 2, 229–244.
- [11] E. D. Rainville, *Special Functions*, reprint of 1960 first edition, Chelsea Publishing Co., Bronx, NY, 1971.
- [12] V. P. Saxena, *The I-Function*, Anamaya publisher, New Delhi, 2008.
- [13] R. P. Singh, *Generalized Struve's function and its recurrence relations*, Ranchi Univ. Math. J. **5** (1974), 67–75.
- [14] ———, *Generalized Struve's function and its recurrence equation*, Vijnana Parishad Anusandhan Patrika **28** (1985), no. 3, 287–292.
- [15] ———, *Some integral representation of generalized Struve's function*, Math. Ed. (Siwan) **22** (1988), no. 3, 91–94.
- [16] ———, *Infinite integrals involving generalized Struve function*, Math. Ed. (Siwan) **23** (1989), no. 1, 30–36.
- [17] H. M. Srivastava and J. Choi, *Zeta and q-Zeta functions and associated series and integrals*, Elsevier, Inc., Amsterdam, 2012.
- [18] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Ellis Horwood Series: Mathematics and its Applications, Ellis Horwood Ltd., Chichester, 1985.
- [19] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Ellis Horwood Series: Mathematics and its Applications, Ellis Horwood Ltd., Chichester, 1984.
- [20] E. M. Wright, *The asymptotic expansion of the generalized Bessel function*, Proc. London Math. Soc. (2) **38** (1935), 257–270.
- [21] ———, *The asymptotic expansion of the generalized hypergeometric function*, J. London Math. Soc. **10** (1935), 286–293.
- [22] N. Yagmur and H. Orhan, *Starlikeness and convexity of generalized Struve functions*, Abstr. Appl. Anal. **2013** (2013), Art. ID 954513, 6 pp.

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