Commun. Korean Math. Soc. **0** (0), No. 0, pp. 1–0
https://doi.org/10.4134/CKMS.c170478
pISSN: 1225-1763 / eISSN: 2234-3024

# ON EXTREMALLY DISCONNECTED SPACES VIA $m$-STRUCTURES

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Abstract. In this paper, we introduce a modification of extremally disconnected spaces which is said to be $m$-extremally disconnected. And we obtain many characterizations of $m$-extremally disconnected spaces. The concepts of $*$-extremally disconnected spaces, $*$-hyperconnected spaces, and generalized hyperconnectedness are as examples for this paper.

## 1. Introduction

In this paper, we said a topological space $(X,\tau)$ with a minimal structure $m_X$ [11] to be $m$-extremally disconnected if $mCl(U)$ is open for every open set $U$ of $(X,\tau)$. We obtain many characterizations of $m$-extremally disconnected spaces. It follows from simple examples that $m$-extremal disconnectedness and extremal disconnectedness are independent. However, if $m_X = SO(X,\tau)$ or $SPO(X,\tau)$, then the mixed space $(X,\tau,m_X)$ is $m$-extremally disconnected for every topological space $(X,\tau)$. Moreover, if $m_X = \alpha(X,\tau)$, $PO(X,\tau)$ or $BO(X,\tau)$, then $(X,\tau,m_X)$ is $m$-extremally disconnected for every extremally disconnected space $(X,\tau)$. The concepts of $*$-extremally disconnected spaces, $*$-hyperconnected spaces, and generalized hyperconnectedness are as examples for this paper. Recently papers [2–5] have introduced some new classes of sets via $m$-structures.

## 2. Minimal structures

**Definition 2.1.** Let $X$ be a nonempty set and $\mathcal{P}(X)$ the power set of $X$. A subfamily $m_X$ of $\mathcal{P}(X)$ is called a *minimal structure* (briefly *$m$-structure*) on $X$ [17] if $\emptyset \in m_X$ and $X \in m_X$. Each member of $m_X$ is said to be *$m_X$-open* and the complement of an $m_X$-open set is said to be *$m_X$-closed*.

**Definition 2.2.** Let $(X,\tau)$ be a topological space. A subset $A$ of $X$ is said to be

Received December 14, 2017; Revised March 9, 2018; Accepted March 29, 2018.
2010 *Mathematics Subject Classification.* Primary 54A05.
*Key words and phrases.* $m$-structure, mixed space, $m$-hyperconnected, $m$-extremally disconnected.

(1) $\alpha$-open [16] if $A \subset Int(Cl(Int(A)))$,
(2) semi-open [13] if $A \subset Cl(Int(A))$,
(3) preopen [15] if $A \subset Int(Cl(A))$,
(4) $b$-open [7] if $A \subset Int(Cl(A)) \cup Cl(Int(A))$,
(5) $\beta$-open [1] or semi-preopen [6] if $A \subset Cl(Int(Cl(A)))$.

The family of all $\alpha$-open (resp. semi-open, preopen, $b$-open, semi-preopen) sets in $(X, \tau)$ is denoted by $\alpha(X)$ (resp. SO$(X)$, PO$(X)$, BO$(X)$, SPO$(X)$).

**Definition 2.3.** Let $X$ be a nonempty set and $m_X$ an $m$-structure on $X$. For a subset $A$ of $X$, the *$m_X$-closure* of $A$ and the *$m_X$-interior* of $A$ are defined in [14] as follows:
(1) $\mathrm{mCl}(A) = \cap\{F : A \subset F, X \setminus F \in m_X\}$,
(2) $\mathrm{mInt}(A) = \cup\{U : U \subset A, U \in m_X\}$.

*Remark* 2.4. Let $(X, \tau)$ be a topological space and $A$ a subset of $X$. If $m_X = \tau$ (resp. SO$(X)$, PO$(X)$, BO$(X)$, SPO$(X)$), then we have
(1) $\mathrm{mCl}(A) = \mathrm{Cl}(A)$ (resp. $\mathrm{sCl}(A)$, $\mathrm{pCl}(A)$, $\mathrm{bCl}(A)$, $\mathrm{spCl}(A)$),
(2) $\mathrm{mInt}(A) = \mathrm{Int}(A)$ (resp. $\mathrm{sInt}(A)$, $\mathrm{pInt}(A)$, $\mathrm{bInt}(A)$, $\mathrm{spInt}(A)$).

**Lemma 2.5** (Maki et al. [14])**.** *Let $X$ be a nonempty set and $m_X$ a minimal structure on $X$. For subsets $A$ and $B$ of $X$, the following properties hold*:
(1) $\mathrm{mCl}(X \setminus A) = X \setminus \mathrm{mInt}(A)$ *and* $\mathrm{mInt}(X \setminus A) = X \setminus \mathrm{mCl}(A)$,
(2) *If* $(X \setminus A) \in m_X$, *then* $\mathrm{mCl}(A) = A$ *and if* $A \in m_X$, *then* $\mathrm{mInt}(A) = A$,
(3) $\mathrm{mCl}(\emptyset) = \emptyset$, $\mathrm{mCl}(X) = X$, $\mathrm{mInt}(\emptyset) = \emptyset$ *and* $\mathrm{mInt}(X) = X$,
(4) *If* $A \subset B$, *then* $\mathrm{mCl}(A) \subset \mathrm{mCl}(B)$ *and* $\mathrm{mInt}(A) \subset \mathrm{mInt}(B)$,
(5) $A \subset \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(A) \subset A$,
(6) $\mathrm{mCl}(\mathrm{mCl}(A)) = \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(\mathrm{mInt}(A)) = \mathrm{mInt}(A)$.

**Lemma 2.6** (Popa and Noiri [17])**.** *Let $X$ be a nonempty set with an $m$-structure $m_X$ and $A$ a subset of $X$. Then $x \in \mathrm{mCl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in m_X$ containing $x$.*

**Definition 2.7.** An $m$-structure $m_X$ on a nonempty set $X$ is said to have *property $\mathcal{B}$* [14] if the union of any family of subsets belong to $m_X$ belongs to $m_X$.

*Remark* 2.8. Let $(X, \tau)$ be a topological space. Then the families $\alpha(X)$, SO$(X)$, PO$(X)$, BO$(X)$ and SPO$(X)$ are $m$-structures on $X$ with property $\mathcal{B}$.

**Lemma 2.9** (Popa and Noiri [17])**.** *Let $X$ be a nonempty set and $m_X$ an $m$-structure on $X$ satisfying property $\mathcal{B}$. For a subset $A$ of $X$, the following properties hold*:
(1) $A \in m_X$ *if and only if* $\mathrm{mInt}(A) = A$,
(2) $A$ *is $m_X$-closed if and only if* $\mathrm{mCl}(A) = A$,
(3) $\mathrm{mInt}(A) \in m_X$ *and* $\mathrm{mCl}(A)$ *is $m_X$-closed.*

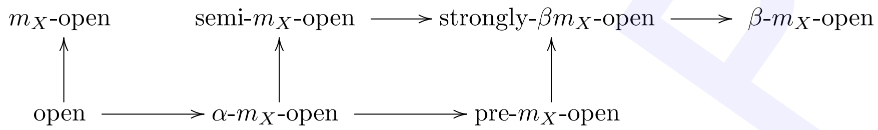
A topological space $(X, \tau)$ with an $m$-structure $m_X$ on $X$ is called a mixed space and is denoted by $(X, \tau, m_X)$.

**Definition 2.10.** A subset $A$ of a mixed space $(X, \tau, m_X)$ is said to be:

1. $m_X$-dense if $mCl(A) = X$.
2. $m_X$-nowhere dense if $Int(mCl(A)) = \phi$.
3. $\alpha$-$m_X$-open if $A \subseteq Int(mCl(Int(A)))$.
4. semi-$m_X$-open if $A \subseteq mCl(Int(A))$.
5. pre-$m_X$-open if $A \subseteq Int(mCl(A))$.
6. $\beta$-$m_X$-open if $A \subseteq Cl(Int(mCl(A)))$.
7. semi-$m_X^*$-open if $A \subseteq Cl(mInt(A))$.
8. strongly-$\beta m_X$-open if $A \subseteq mCl(Int(mCl(A)))$.

**Lemma 2.11.** *If $\tau \subseteq m_X$, then every semi-$m_X$-open set is semi-$m_X^*$-open.*

If $\tau \subseteq m_X$, the following diagram holds:




**Lemma 2.12.** *For a subset $A$ of a mixed space $(X, \tau, m_X)$, the following properties hold*:

1. *$A$ is semi-$m_X$-open if and only if there exists $B \in \tau$ such that $B \subseteq A \subseteq mCl(B)$.*
2. *If there exists $B \in m_X$ such that $B \subseteq A \subseteq Cl(B)$, then $A$ is semi-$m_X^*$-open.*
3. *$A$ is semi-$m_X^*$-open if and only if $Cl(A) = Cl(mInt(A))$.*

## 3. Characterizations of $m$-extremally disconnected spaces

**Definition 3.1.** A mixed space $(X, \tau, m_X)$ is said to be $m$-extremally disconnected (resp. $m$-hyperconnected) if $mCl(A) \in \tau$ (resp. $mCl(A) = X$) for each $A \in \tau$.

**Example 3.2.** Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $m_X = \{X, \phi, \{a\}, \{b\}, \{c\}\}$. Then the topological space $(X, \tau)$ is not extremally disconnected and the mixed space $(X, \tau, m_X)$ is $m$-extremally disconnected.

**Example 3.3.** Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b, c\}\}$ and $m_X = \{X, \phi, \{a\}, \{b\}, \{a, c\}\}$. Then the topological space $(X, \tau)$ is extremally disconnected and the mixed space $(X, \tau, m_X)$ is not $m$-extremally disconnected.

A subfamily $\mathcal{I}$ of the power set $\mathcal{P}$(X) of a nonempty set $X$ is called an ideal if the following properties are satisfied: (1) $A \in \mathcal{I}$ and $B \subseteq A$ imply $B \in \mathcal{I}$; (2) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$. A topological space $(X, \tau)$ with an ideal $\mathcal{I}$ on $X$ is called an ideal topological space and is denoted by $(X, \tau, \mathcal{I})$. For an ideal topological space and a subset $A$ of $X$, $A^*(\mathcal{I})$ is defined as follows: $A^*(\mathcal{I}) = \{x \in X : U \cap A \notin \mathcal{I}$ for every open set $U$ containing $x\}$. In [12], $A^*(\mathcal{I})$ (briefly $A^*$) is called the local function of $A$ with respect to $\mathcal{I}$ and $\tau$

and $Cl^*(A) = A^* \cup A$ defines a Kuratowski closure operator for a topology $\tau^*$ which is finer than $\tau$. A subset $A$ is $\star$-closed if and only if $A^* \subseteq A$. Naturally, the complement of a $*$-closed set is said to be $\star$-open. $A$ is said to be $*$-dense if $Cl^*(A) = X$.

By setting $m_X = \tau^*$, as a special case of Definition 3.1 we obtain the following definitions:

**Definition 3.4** ([10])**.** An ideal space $(X, \tau, \mathcal{I})$ is said to be $*$-extremally disconnected if the $*$-closure of every open subset $A$ of $X$ is open.

**Definition 3.5** ([11])**.** An ideal space $(X, \tau, \mathcal{I})$ is said to be $*$-hyperconnected if $A$ is $*$-dense for every open subset $A \neq \phi$ of $X$.

Let $X$ be a nonempty set and let $\mathcal{P}(X)$ be the power set of $X$. Then $\mu \subseteq \mathcal{P}(X)$ is called a generalized topology (briefly GT) [8] on $X$ if $\emptyset \in \mu$ and $G_i \in \mu$ for $i \in I \neq \emptyset$ implies $G = \cup_{i \in I} G_i \in \mu$. We call the pair $(X, \mu)$ a generalized topological space (briefly GTS) on $X$.

For a GTS $(X, \mu)$, the elements of $\mu$ are called $\mu$-open sets and the complements of $\mu$-open sets are called $\mu$-closed sets. For $A \subseteq X$, we denote by $c_\mu(A)$ the intersection of all $\mu$-closed sets containing $A$, i.e., the smallest $\mu$-closed set containing $A$.

By setting $m_X = \mu$, where $m_X$ has property $\mathcal{B}$, as a special case of Definition 3.1 we obtain the following definition:

**Definition 3.6** ([9])**.** Let $(X, \mu)$ be a GTS and $G$ is a subset of $X$.

(1) $G$ is said to be $\mu$-dense if $c_\mu(G) = X$,
(2) $(X, \mu)$ is said to be hyperconnected if $G$ is $\mu$-dense for every $\mu$-open set $G \neq \phi$ of $(X, \mu)$.

**Lemma 3.7.** *Let $(X, \tau, m_X)$ be a mixed space. Then, the following properties hold*:

(1) *If $X$ is m-hyperconnected, then $X$ is m-extremally disconnected.*
(2) *If $m_X = SO(X, \tau)$ or $SPO(X, \tau)$, then $(X, \tau, m_X)$ is m-extremally disconnected.*
(3) *Let $(X, \tau)$ be extremally disconnected. If $m_X = \alpha(X)$, $PO(X, \tau)$ or $BO(X, \tau)$, then $(X, \tau, m_X)$ is m-extremally disconnected.*

*Proof.* (1) This is obvious.

(2) It is known in [3] that $sCl(A) = A \cup Int(Cl(A))$ and $spCl(A) = A \cup Int(Cl(Int(A)))$ for every subset $A$ of $X$. Therefore, $sCl(V)$ and $spCl(V)$ are open for every open set $V$ and hence $(X, \tau, m_X)$ is $m$-extremally disconnected for $m_X = SO(X, \tau)$ or $SPO(X, \tau)$.

(3) It is known in [3] and [5] that $\alpha(A) = A \cup Cl(Int(Cl(A)))$, $pCl(A) = A \cup Cl(Int(A))$ and $bCl(A) = sCl(A) \cap pCl(A)$ for every subset $A$ of $X$. Therefore, $\alpha(V)$, $pCl(V)$ and $bCl(V)$ are open for every open set $V$ of an extremally disconnected space $(X, \tau)$ and hence $(X, \tau, m_X)$ is $m$-extremally disconnected for $m_X = \alpha(X, \tau)$, $PO(X, \tau)$, or $BO(X, \tau)$. ∎

The following example shows that the converse of each statement of Lemma 3.7 are not true.

**Example 3.8.** Consider the mixed space $(X, \tau, m_X)$, where $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}\}$ and $m_X = \{\phi, \{a\}, \{a, c\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$. If $A = \{a\}$, then $A$ is open and $mCl(A) = \{a\} \neq X$. Hence $(X, \tau, m_X)$ is not $m$-hyperconnected. Since $m_X$-closure of every open set is open, $X$ is $m$-extremally disconnected. Moreover, since $\{b, d\}$ is not $\beta$-open, $m_X$ is not $SPO(X, \tau)$.

**Theorem 3.9.** *Let $(X, \tau, m_X)$ be a mixed space, the following properties are equivalent*:

(1) *$X$ is $m$-extremally disconnected*;
(2) *$mInt(A)$ is closed for every closed subset $A$ of $X$;*
(3) *$mCl(Int(A)) \subseteq Int(mCl(A))$ for every subset $A$ of $X$;*
(4) *Every semi-$m_X$-open set is pre-$m_X$-open*;
(5) *$mCl(A) \in \tau$ for every strongly-$\beta m_X$-open set $A$*;
(6) *Every strongly-$\beta m_X$-open set is pre-$m_X$-open*;
(7) *$A$ is $\alpha$-$m_X$-open if and only if it is semi-$m_X$-open for every $A \subseteq X$.*

*Proof.* (1) $\Rightarrow$ (2): Let $A$ be a closed set in $X$. Then $X - A$ is open. By (1) $mCl(X - A) = X - mInt(A)$ is open. Thus, $mInt(A)$ is closed.

(2) $\Rightarrow$ (3): Let $A$ be any set of $X$. Then $X - Int(A)$ is closed in $X$ and by (2) $mInt[X - Int(A)]$ is closed in $X$. Therefore, $mCl(Int(A))$ is open in $X$ and hence $mCl(Int(A)) \subseteq Int(mCl(A))$.

(3) $\Rightarrow$ (4): Let $A$ be semi-$m_X$-open. By (3), we have $A \subseteq mCl(Int(A)) \subseteq Int(mCl(A))$. Thus, $A$ is pre-$m_X$-open.

(4) $\Rightarrow$ (5): Let $A$ be a strongly-$\beta m_X$-open set. Then $mCl(A)$ is semi-$m_X$-open. By (4), $mCl(A)$ is pre-$m_X$-open. Thus, $mCl(A) \subseteq Int(mCl(A))$ and hence $mCl(A)$ is open.

(5) $\Rightarrow$ (6): Let $A$ be a strongly-$\beta m_X$-open set. By (5), $mCl(A) = Int(mCl(A))$. Thus, $A \subseteq mCl(A) = Int(mCl(A))$ and hence $A$ is pre-$m_X$-open.

(6) $\Rightarrow$ (7): Let $A$ be a semi-$m_X$-open set. Since a semi-$m_X$-open set is strongly-$\beta m_X$-open, then by (6) it is pre-$m_X$-open. Since $A$ is semi-$m_X$-open and pre-$m_X$-open, it is $\alpha$-$m_X$-open.

(7) $\Rightarrow$ (1): Let $A$ be an open set of $X$. Then $mCl(A)$ is semi-$m_X$-open and by (7) $mCl(A)$ is $\alpha$-$m_X$-open. Therefore, $mCl(A) \subseteq Int(mCl(Int(mCl(A)))) = Int(mCl(A))$ and hence $mCl(A) = Int(mCl(A))$. Hence $mCl(A)$ is open and $X$ is $m$-extremally disconnected. □

**Corollary 3.10.** *Let $(X, \tau, m_X)$ be a mixed space. Then, the following properties are equivalent*:

(1) *$X$ is $m$-extremally disconnected*;
(2) *$mCl(A) \in \tau$ for every $\alpha$-$m_X$-open set $A$ of $X$;*
(3) *$mCl(A) \in \tau$ for every semi-$m_X$-open set $A$ of $X$;*

(4) $mCl(A) \in \tau$ *for every pre-*$m_X$*-open set* $A$ *of* $X$.

*Proof.* This follows from Theorem 3.9 and Diagram. □

**Theorem 3.11.** *Let* $(X, \tau, m_X)$ *be a mixed space and* $m_X$ *have property* $\mathcal{B}$. *Then, the following properties are equivalent*:

(1) $X$ *is* $m$*-extremally disconnected*;
(2) *For any* $A \in \tau$ *and* $B \in m_X$ *such that* $A \cap B = \phi$, *there exist disjoint a* $m_X$*-closed set* $U$ *and a closed set* $V$ *such that* $A \subseteq U$ *and* $B \subseteq V$;
(3) $mCl(U) \cap Cl(V) = \phi$ *for every* $U \in \tau$ *and* $V \in m_X$ *with* $U \cap V = \phi$;
(4) $mCl[Int(mCl(U))] \cap Cl(V) = \phi$ *for every* $U \subseteq X$ *and* $V \in m_X$ *with* $U \cap V = \phi$.

*Proof.* (1) ⇒ (2): Let $X$ be $m$-extremally disconnected. Let $A$ and $B$ be two disjoint open and $m_X$-open sets, respectively. Then $mCl(A)$ and $X - mCl(A)$ are disjoint $m_X$-closed and closed sets containing $A$ and $B$, respectively.

(2) ⇒ (3): Let $U \in \tau$ and $V \in m_X$ with $U \cap V = \phi$. By (2), there exist disjoint an $m_X$-closed set $F$ and a closed set $G$ such that $U \subseteq F$ and $V \subseteq G$. Therefore, $mCl(U) \cap Cl(V) \subseteq F \cap G = \phi$. Thus, $mCl(U) \cap Cl(V) = \phi$.

(3) ⇒ (4): Let $U \subseteq X$ and $V \in m_X$ with $U \cap V = \phi$. Since $Int(mCl(U)) \in \tau$ and $Int(mCl(U)) \cap V = \phi$, by (3) $mCl[Int(mCl(U))] \cap Cl(V) = \phi$.

(4) ⇒ (1): Let $U$ be any open set. Then $[X - mCl(U)] \cap U = \phi$. Since $m_X$ has property $\mathcal{B}$, $X - mCl(U) \in m_X$ and by (4) $mCl(Int(mCl(U))) \cap Cl(X - mCl(U)) = \phi$. Since $U \in \tau$, we have $mCl(U) \cap [X - Int(mCl(U))] = \phi$. Therefore, $mCl(U) \subseteq Int(mCl(U))$ and $mCl(U)$ is open. This shows that $X$ is $m$-extremally disconnected. □

**Definition 3.12.** A subset $A$ of a mixed space $(X, \tau, m_X)$ is called an $R_m$-open set if $A = Int(mCl(A))$. The complement of an $R_m$-open set is said to be $R_m$-closed.

**Theorem 3.13.** *Let* $(X, \tau, m_X)$ *be a mixed space and* $m_X$ *have property* $\mathcal{B}$. *Then, the following properties are equivalent*:

(1) $X$ *is* $m$*-extremally disconnected*;
(2) *Every* $R_m$*-open set of* $X$ *is* $m_X$*-closed in* $X$;
(3) *Every* $R_m$*-closed set of* $X$ *is* $m_X$*-open in* $X$.

*Proof.* (1) ⇒ (2): Let $X$ be $m$-extremally disconnected. Let $A$ be an $R_m$-open set of $X$. Then $A = Int(mCl(A))$. Since $A$ is an open set, then $mCl(A)$ is open. Thus, $A = Int(mCl(A)) = mCl(A)$ and hence $A$ is $m_X$-closed.

(2) ⇒ (1): Suppose that every $R_m$-open subset of $X$ is $m_X$-closed in $X$. Let $A$ be an open subset of $X$. Since $Int(mCl(A))$ is $R_m$-open, then it is $m_X$-closed. This implies that $mCl(A) \subseteq mCl(Int(mCl(A))) = Int(mCl(A))$ since $A \subseteq Int(mCl(A))$. Thus, $mCl(A)$ is open and hence $X$ is $m$-extremally disconnected.

(2) ⇔ (3): It is obvious. □

**Theorem 3.14.** *Let $(X, \tau, m_X)$ be a mixed space. Then the following properties are equivalent*:

(1) *$X$ is $m$-extremally disconnected*;
(2) *$mCl(A) \in \tau$ for every $R_m$-open set $A$ of $X$.*

*Proof.* (1) $\Rightarrow$ (2): Let $A$ be an $R_m$-open set of $X$. Then $A$ is open and $mCl(A) \in \tau$.

(2) $\Rightarrow$ (1): Suppose that $mCl(A) \in \tau$ for every $R_m$-open set $A$ of $X$. Let $V$ be any open set of $X$. Then $Int(mCl(V))$ is an $R_m$-open set and $mCl(V) = mCl(Int(mCl(V))) \in \tau$. Thus $mCl(V) \in \tau$ and hence $X$ is $m$-extremally disconnected. □

**Theorem 3.15.** *Let $(X, \tau, m_X)$ be a mixed space and $m_X$ have property $\mathcal{B}$. Then, the following properties are equivalent*:

(1) *$X$ is $m$-extremally disconnected*;
(2) *If $A$ is semi-$m_X$-open, $B$ is semi-$m_X^*$-open and $A \cap B = \phi$, then $mCl(A) \cap Cl(B) = \phi$.*

*Proof.* (1) $\Rightarrow$ (2): Let $A$ be semi-$m_X$-open, $B$ semi-$m_X^*$-open and $A \cap B = \phi$. Since $m_X$ is property $\mathcal{B}$, $mInt(B)$ is $m_X$-open and $mCl(A) \cap mInt(B) = \phi$. By Corollary 3.10, $mCl(A)$ is open and $mCl(A) \cap Cl(mInt(B)) = \phi$. Since $B$ is semi-$m_X^*$-open, $Cl(B) = Cl(mInt(B))$ and hence $mCl(A) \cap Cl(B) = \phi$.

(2) $\Rightarrow$ (1): Let $A$ be a semi-$m_X$-open set. Since $A$ and $X - mCl(A)$ are disjoint semi-$m_X$-open and semi-$m_X^*$-open, respectively, by (2) we have $mCl(A) \cap Cl[X - mCl(A)] = \phi$. This implies that $mCl(A) \subseteq Int(mCl(A))$. Thus $mCl(A)$ is open. Hence, by Corollary 3.10, $X$ is $m$-extremally disconnected. □

**Theorem 3.16.** *Let $(X, \tau, m_X)$ be a mixed space and $m_X$ have property $\mathcal{B}$. Then $X$ is $m$-extremally disconnected if and only if for every open set $G$ and every $m_X$-closed set $F$ with $G \subseteq F$, there exist an open set $G_1$ and an $m_X$-closed set $F_1$ such that $G \subseteq F_1 \subseteq G_1 \subseteq F$.*

*Proof.* Suppose $X$ is $m$-extremally disconnected. Let $G$ be an open set and $F$ an $m_X$-closed set in $X$ such that $G \subseteq F$. Then $G \cap (X - F) = \phi$. Then by Theorem 3.11 $mCl(G) \cap Cl(X - F) = \phi$, that is, $mCl(G) \subseteq X - Cl(X - F)$. Using the fact that $X - Cl(X - F) \subseteq F$ and writing $mCl(G) = F_1$, $X - Cl(X - F) = G_1$, we get $G \subseteq F_1 \subseteq G_1 \subseteq F$.

Conversely, let the condition hold. Let $U$ be an open set and V be an $m_X$-open set in $X$ such that $U \cap V = \phi$. Then, $U \subseteq X - V$ and $X - V$ is $m_X$-closed. Accordingly, there exist an open set $G$ and an $m_X$-closed set $F$ such that $U \subseteq F \subseteq G \subseteq X - V$. This implies that $mCl(U) \cap X - [Int(X - V)] = \phi$. But $X - [Int(X - V)] = Cl(V)$. That is, $mCl(U) \cap Cl(V) = \phi$ and by Theorem 3.11 $X$ is $m$-extremally disconnected. □

**Definition 3.17.** Let $(X, \tau, m_X)$ be a mixed space and $R$ be the real line with the usual topology. A mapping $f : X \to R$ is said to be upper semicontinuous or *u.s.c.* in brief (resp. lower minimal semi-continuous or *l.m.s.c.* in brief) if for each $a \in R$, the set $\{x : x \in X, f(x) < a\}$ is open (resp. $\{x : x \in X, f(x) > a\}$ is $m_X$-open).

**Theorem 3.18.** *Let $(X, \tau, m_X)$ be an m-extremally disconnected mixed space and $m_X$ have property $\mathcal{B}$. Let $U$ be an open set, $V$ be an $m_X$-open set in $X$ such that $U \cap V = \phi$. Then there exists a real-valued u.s.c. and l.m.s.c. mapping $f : X \to [0, 1]$ such that $f(U) = \{0\}$ and $f(V) = \{1\}$.*

*Proof.* Since $U \subseteq X - V$, by Theorem 3.16, there exist an open set $G_{1/2}$ and an $m_X$-closed set $F_{1/2}$ such that $U \subseteq F_{1/2} \subseteq G_{1/2} \subseteq X - V$. Again, since $U \subseteq F_{1/2}$ and $G_{1/2} \subseteq X - V$, by the same reasoning, there exist open sets $G_{1/4}$, $G_{3/4}$ and $m_X$-closed sets $F_{1/4}$, $F_{3/4}$ such that

$$U \subseteq F_{1/4} \subseteq G_{1/4} \subseteq F_{1/2} \subseteq G_{1/2} \subseteq F_{3/4} \subseteq G_{3/4} \subseteq X - V.$$

We continue this process for each dyadic rational number of the form $t = \frac{m}{2^n}$ (where $n = 1, 2, 3, \ldots$ and $m = 1, 2, \ldots, 2n - 1$). We find that for $t_1 < t_2$, there exist open sets $G_{t_1}$ and $G_{t_2}$ and $m_X$-closed sets $F_{t_1}$ and $F_{t_2}$ such that

$$U \subseteq F_{t_1} \subseteq G_{t_1} \subseteq F_{t_2} \subseteq G_{t_2} \subseteq X - V.$$

Now, we define a mapping $f$ on $X$ as follows:

$$f(x) = \begin{cases} 0, & x \in G_t \text{ for some } t; \\ \sup\{t : t \notin G_t\}, & \text{otherwise.} \end{cases}$$

Clearly, the values of $f$ lies in $[0, 1]$. Also, $f(U) = \{0\}$ and $f(V) = \{1\}$.

(1) $f$ is *u.s.c.*: We show that $f^{-1}([0, a))$ is open for each $a$, where $0 < a < 1$. Let $x \in f^{-1}([0, a))$. Then $f(x) < a$, hence there must be some dyadic rational $t < a$ such that $x \in G_t$. Thus $f^{-1}([0, a)) \subseteq \cup_{t<a} G_t$. Again, if $x \in \cup_{t<a} G_t$, then $x \in G_{t_0}$ for some $t_0 < a$, so that $x \in f^{-1}([0, a))$ . Thus, $f^{-1}([0, a)) = \cup_{t<a} G_t$, which is open. Hence, $f$ is *u.s.c.*

(2) $f$ is *l.m.s.c.*: We show that $f^{-1}((a, 1])$ is $m_X$-open for each $a$, where $0 < a < 1$. Let $y \in f^{-1}((a, 1])$. Then there exists a dyadic rational $t > a$ such that $y \notin G_t$. Then $y \notin F_t$. Thus $f^{-1}((a, 1]) \subseteq \cup_{t>a}(X - F_t)$. Also $y \in X - F_t$ for some $t > a$ implies that $y \notin G_{t_0}$ for some $t_0$ with $t > t_0 > a$. Hence $f(y) > a$, that is, $y \in f^{-1}((a, 1])$. Thus we get, $f^{-1}((a, 1]) = \cup_{t>a}(X - F_t)$, which is $m_X$-open. Hence $f$ is *l.m.s.c.* □

**Acknowledgement.** The authors wish to thank the referees for useful comments and suggestions.

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