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REPRESENTATIONS BY QUATERNARY QUADRATIC FORMS WITH COEFFICIENTS 1, 2, 5 OR 10

Ayşe Alaca and Mada Altiary

Abstract. We determine explicit formulas for the number of representations of a positive integer n by quaternary quadratic forms with coefficients 1, 2, 5 or 10. We use a modular forms approach.

1. Introduction

Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$ and $\mathbb{C}$ denote the sets of positive integers, nonnegative integers, integers and complex numbers, respectively. For $n \in \mathbb{N}$ we set $\sigma(n) = \sum_{d \mid n} d$, where d runs through the positive divisors of n. If $n \notin \mathbb{N}$ we set $\sigma(n) = 0$. For $a_1, a_2, a_3, a_4 \in \mathbb{N}$, and $n \in \mathbb{N}_0$ we define

$$\begin{aligned} &N(a_1, a_2, a_3, a_4; n) \\ &:= \operatorname{card}\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid n = a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2\}. \end{aligned}$$

It is a classical result of Jacobi [7, 21] that

$$N(1, 1, 1, 1; n) = 8\sigma(n) - 32\sigma(n/4).$$

Formulas for $N(a_1, a_2, a_3, a_4; n)$ for the quaternary quadratic forms

$$(a_1, a_2, a_3, a_4) = (1, 1, 1, 2), (1, 1, 2, 2), (1, 2, 2, 2), (1, 1, 1, 5), (1, 1, 5, 5), (1, 5, 5, 5)$$

are in the literature, see for example [1–3, 9, 12–17, 20].

There are twenty-six quaternary quadratic forms $a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2$, where $a_1, a_2, a_3, a_4 \in \{1, 2, 5, 10\}$, $a_1 \le a_2 \le a_3 \le a_4$ and $\gcd(a_1, a_2, a_3, a_4) = 1$ (see Table 2.1). In this paper, we determine an explicit formula for $N(a_1, a_2, a_3, a_4; n)$ for each of these quaternary forms in a uniform manner. We use a modular forms approach.

For $q \in \mathbb{C}$ with $|q| < 1$, Ramanujan's theta function $\varphi(q)$ is defined by

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}.$$

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For $a_1, a_2, a_3, a_4 \in \mathbb{N}$ we have

$$\sum_{n=1}^{\infty} N(a_1, a_2, a_3, a_4; n) q^n = \varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}). \tag{1.1}$$

The Dedekind eta function $\eta(z)$ is the holomorphic function defined on the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ by

$$\eta(z) = e^{\pi i z/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z}).$$

Throughout the remainder of the paper we take $q = q(z) := e^{2\pi i z}$ with $z \in \mathbb{H}$. Thus we can express $\eta(z)$ as

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n). \tag{1.2}$$

An eta quotient is defined to be a finite product of the form

$$f(z) = \prod_{\delta} \eta^{r_\delta}(\delta z),$$

where δ runs through a finite set of positive integers and the exponents r_δ are non-zero integers. It is known (see for example [5, Corollary 1.3.4]) that

$$\varphi(q) = \frac{\eta^5(2z)}{\eta^2(z)\eta^2(4z)}. \tag{1.3}$$

2. Modular spaces $M_2(\Gamma_0(40), \chi_i)$ with $i \in \{0, 1, 2, 3\}$

For $n \in \mathbb{N}$ and Dirichlet characters χ and ψ we define $\sigma_{\chi,\psi}(n)$ by

$$\sigma_{\chi,\psi}(n) := \sum_{1 \le m \mid n} \psi(m)\chi(n/m)m. \tag{2.1}$$

If $n \notin \mathbb{N}$ we set $\sigma_{\chi,\psi}(n) = 0$. Let χ_0 denote the trivial character, that is $\chi_0(n) = 1$ for all $n \in \mathbb{Z}$. Hence $\sigma_{\chi_0,\chi_0}(n)$ coincides with the sum of divisors function $\sigma(n)$. Let $N \in \mathbb{N}$. The modular subgroup $\Gamma_0(N)$ is defined by

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \; a, b, c, d \in \mathbb{Z},\ ad - bc = 1,\ c \equiv 0 \pmod{N} \right\}.$$

Let χ be a Dirichlet character of modulus dividing N and let $k \in \mathbb{Z}$. We write $M_k(\Gamma_0(N), \chi)$ to denote the space of modular forms of weight k with multiplier system χ for $\Gamma_0(N)$, and $E_k(\Gamma_0(N), \chi)$ and $S_k(\Gamma_0(N), \chi)$ to denote the subspaces of Eisenstein forms and cusp forms of $M_k(\Gamma_0(N), \chi)$, respectively. If $\chi = \chi_0$, then we write $M_k(\Gamma_0(N))$ for $M_k(\Gamma_0(N), \chi_0)$, and $S_k(\Gamma_0(N))$ for $S_k(\Gamma_0(N), \chi_0)$. It is known (see for example [19, p. 83]) that

$$M_k(\Gamma_0(N), \chi) = E_k(\Gamma_0(N), \chi) \oplus S_k(\Gamma_0(N), \chi). \tag{2.2}$$

For $n \in \mathbb{Z}$ we define three Dirichlet characters by

$$\chi_1(n) = \left(\frac{5}{n}\right),\ \chi_2(n) = \left(\frac{8}{n}\right),\ \chi_3(n) = \left(\frac{40}{n}\right). \tag{2.3}$$

We define the Eisenstein series

$$L(q) := E_{\chi_0,\chi_0}(q) = -\frac{1}{24} + \sum_{n=1}^{\infty} \sigma(n) q^n, \tag{2.4}$$

$$E_{\chi_0,\chi_1}(q) = -\frac{1}{5} + \sum_{n=1}^{\infty} \sigma_{\chi_0,\chi_1}(n) q^n,\ E_{\chi_1,\chi_0}(q) = \sum_{n=1}^{\infty} \sigma_{\chi_1,\chi_0}(n) q^n, \tag{2.5}$$

$$E_{\chi_0,\chi_2}(q) = -\frac{1}{2} + \sum_{n=1}^{\infty} \sigma_{\chi_0,\chi_2}(n) q^n,\ E_{\chi_2,\chi_0}(q) = \sum_{n=1}^{\infty} \sigma_{\chi_2,\chi_0}(n) q^n, \tag{2.6}$$

$$E_{\chi_0,\chi_3}(q) = -7 + \sum_{n=1}^{\infty} \sigma_{\chi_0,\chi_3}(n) q^n,\ E_{\chi_3,\chi_0}(q) = \sum_{n=1}^{\infty} \sigma_{\chi_3,\chi_0}(n) q^n, \tag{2.7}$$

$$E_{\chi_1,\chi_2}(q) = \sum_{n=1}^{\infty} \sigma_{\chi_1,\chi_2}(n) q^n, \qquad E_{\chi_2,\chi_1}(q) = \sum_{n=1}^{\infty} \sigma_{\chi_2,\chi_1}(n) q^n. \tag{2.8}$$

We use the following lemma to determine if certain eta quotients are modular forms. See [6, p. 174], [10, Corollary 2.3, p. 37], [8, Theorem 5.7, p. 101], [11] and [18, Theorem 1.64].

Lemma 2.1 (Ligozat)**.** *Let $N \in \mathbb{N}$ and $f(z) = \prod_{1 \le \delta | N} \eta^{r_\delta}(\delta z)$ be an eta quotient. Let $s = \prod_{1 \le \delta | N} \delta^{|r_\delta|}$ and $k = \frac{1}{2}\sum_{1 \le \delta | N} r_\delta$. Suppose that the following conditions are satisfied*:

(L1) $\sum_{1 \le \delta | N} \delta \cdot r_\delta \equiv 0 \pmod{24}$,
(L2) $\sum_{1 \le \delta | N} \frac{N}{\delta} \cdot r_\delta \equiv 0 \pmod{24}$,
(L3) $\sum_{1 \le \delta | N} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta} \ge 0$ *for each positive divisor d of N,*
(L4) *k is an integer.*

Then $f(z) \in M_k(\Gamma_0(N), \chi)$, where the character χ is given by $\chi(m) = \left(\frac{(-1)^k s}{m}\right)$.

(L3)$'$ *In addition to the above conditions, if the inequality in* (L3) *is strict for each positive divisor d of N, then $f(z) \in S_k(\Gamma_0(N), \chi)$.*

In Table 2.1, we group our twenty-six quaternary forms (a_1, a_2, a_3, a_4) according to modular spaces $M_2(\Gamma_0(40), \chi)$ to which $\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4})$ belong.

Formulas $N(a_1, a_2, a_3, a_4; n)$ for the forms with a checkmark (✓) in Table 2.1 are known. Of the remaining nineteen forms, four are universal and identified with an asterisk $(*)$.

We deduce from [19, Sec. 6.1, p. 93] that

$$\dim(E_2(\Gamma_0(40))) = 7,\ \dim(S_2(\Gamma_0(40))) = 3. \tag{2.9}$$

Table 2.1

$M_2(\Gamma_0(40))$	$M_2(\Gamma_0(40),\chi_1)$	$M_2(\Gamma_0(40),\chi_2)$	$M_2(\Gamma_0(40),\chi_3)$
$(1,1,1,1)$✓	$(1,1,1,5)$✓	$(1,1,1,2)$✓	$(1,1,1,10)$
$(1,1,2,2)$✓	$(1,1,2,10)\ast$	$(1,1,5,10)$	$(1,1,2,5)\ast$
$(1,1,5,5)$✓	$(1,2,2,5)\ast$	$(1,2,2,2)$✓	$(1,2,2,10)$
$(1,1,10,10)$	$(1,5,5,5)$✓	$(1,2,5,5)$	$(1,5,5,10)$
$(1,2,5,10)\ast$	$(1,5,10,10)$	$(1,2,10,10)$	$(1,10,10,10)$
$(2,2,5,5)$	$(2,5,5,10)$	$(2,2,5,10)$	$(2,2,2,5)$
			$(2,5,5,5)$
			$(2,5,10,10)$

We also deduce from [19, Sec. 6.3, p. 98] that

$$\dim(E_2(\Gamma_0(40),\chi_1)) = 8,\ \dim(S_2(\Gamma_0(40),\chi_1)) = 2, \tag{2.10}$$

$$\dim(E_2(\Gamma_0(40),\chi_2)) = 4,\ \dim(S_2(\Gamma_0(40),\chi_2)) = 4, \tag{2.11}$$

$$\dim(E_2(\Gamma_0(40),\chi_3)) = 4,\ \dim(S_2(\Gamma_0(40),\chi_3)) = 4. \tag{2.12}$$

Theorem 2.1. *Let* χ_1,χ_2,χ_3 *be as in* (2.3). *If* (a_1,a_2,a_3,a_4) *is in the first, second, third or fourth column of* Table 2.1, *then*

$$\begin{aligned}
&\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}) \in M_2(\Gamma_0(40)),\\
&\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}) \in M_2(\Gamma_0(40),\chi_1),\\
&\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}) \in M_2(\Gamma_0(40),\chi_2),\\
&\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}) \in M_2(\Gamma_0(40),\chi_3),
\end{aligned}$$

respectively.

Proof. The assertion directly follows from (1.3) and Lemma 2.1. □

Let $n \in \mathbb{N}$. We define the eta quotients $A_k(q)$, $B_k(q)$, $C_k(q)$, $D_k(q)$ and integers $a_k(n)$, $b_k(n)$, $c_k(n)$, $d_k(n)$ as follows:

$$A_1(q) = \sum_{n=1}^{\infty} a_1(n)q^n = \eta^2(2z)\eta^2(10z), \tag{2.13}$$

$$A_2(q) = A_1(q^2) = \sum_{n=1}^{\infty} a_2(n)q^n = \eta^2(4z)\eta^2(20z), \tag{2.14}$$

$$A_3(q) = \sum_{n=1}^{\infty} a_3(n)q^n = \frac{\eta^5(4z)\eta(10z)\eta^2(40z)}{\eta(2z)\eta^2(8z)\eta(20z)}, \tag{2.15}$$

$$B_1(q) = \sum_{n=1}^{\infty} b_1(n)q^n = \frac{\eta(2z)\eta^4(20z)}{\eta(10z)}, \tag{2.16}$$

$$B_2(q) = \sum_{n=1}^{\infty} b_2(n)q^n = \frac{\eta^4(4z)\eta(10z)}{\eta(2z)}, \tag{2.17}$$

$$C_1(q) = \sum_{n=1}^{\infty} c_1(n) q^n = \frac{\eta^2(z)\eta(8z)\eta^2(10z)\eta(40z)}{\eta(2z)\eta(20z)}, \tag{2.18}$$

$$C_2(q) = \sum_{n=1}^{\infty} c_2(n) q^n = \frac{\eta(z)\eta(5z)\eta^2(8z)\eta^2(20z)}{\eta(4z)\eta(10z)}, \tag{2.19}$$

$$C_3(q) = \sum_{n=1}^{\infty} c_3(n) q^n = \frac{\eta^6(2z)\eta(10z)\eta^2(40z)}{\eta^2(z)\eta^2(4z)\eta(20z)}, \tag{2.20}$$

$$C_4(q) = \sum_{n=1}^{\infty} c_4(n) q^n = \frac{\eta^6(4z)\eta^2(5z)\eta(20z)}{\eta^2(2z)\eta^2(8z)\eta(10z)}, \tag{2.21}$$

$$D_1(q) = \sum_{n=1}^{\infty} d_1(n) q^n = \frac{\eta^2(z)\eta^6(4z)\eta(20z)}{\eta^3(2z)\eta^2(8z)}, \tag{2.22}$$

$$D_2(q) = \sum_{n=1}^{\infty} d_2(n) q^n = \frac{\eta^2(5z)\eta(8z)\eta(10z)\eta(40z)}{\eta(20z)}, \tag{2.23}$$

$$D_3(q) = \sum_{n=1}^{\infty} d_3(n) q^n = \frac{\eta(z)\eta(5z)\eta(20z)\eta^2(40z)}{\eta(10z)}, \tag{2.24}$$

$$D_4(q) = \sum_{n=1}^{\infty} d_4(n) q^n = \frac{\eta(z)\eta(4z)\eta(5z)\eta^2(8z)}{\eta(2z)}. \tag{2.25}$$

Theorem 2.2. *Let χ_1, χ_2, χ_3 be as in* (2.3)*. Then*

$$\begin{array}{ll} \{A_1(q), A_2(q), A_3(q)\}, & \{B_1(q), B_2(q)\}, \\ \{C_1(q), C_2(q), C_3(q), C_4(q)\}, & \{D_1(q), D_2(q), D_3(q), D_4(q)\} \end{array}$$

are bases for $S_2(\Gamma_0(40))$, $S_2(\Gamma_0(40), \chi_1)$, $S_2(\Gamma_0(40), \chi_2)$ and $S_2(\Gamma_0(40), \chi_3)$, respectively.

Proof. The set $\{A_1(q), A_2(q), A_3(q)\}$ is linearly independent over $\mathbb{C}$. By Lemma 2.1, we have $A_k(q) \in S_2(\Gamma_0(40))$ for $k = 1, 2, 3$. The assertion now follows from (2.9). Similarly, the remaining three assertions follow from (2.10), (2.11), (2.12) and Lemma 2.1. $\square$

Theorem 2.3. *Let χ_0 be the trivial character and χ_1, χ_2, χ_3 be as in* (2.3)*. Then*

$$\begin{gathered} \{L(q) - tL(q^t) \mid t = 2, 4, 5, 8, 10, 20, 40\}, \\ \{E_{\chi_0,\chi_1}(q^t), E_{\chi_1,\chi_0}(q^t) \mid t = 1, 2, 4, 8\}, \\ \{E_{\chi_0,\chi_2}(q^t), E_{\chi_2,\chi_0}(q^t) \mid t = 1, 5\}, \\ \{E_{\chi_0,\chi_3}(q), E_{\chi_1,\chi_2}(q), E_{\chi_2,\chi_1}(q), E_{\chi_3,\chi_0}(q)\} \end{gathered}$$

are bases for $E_2(\Gamma_0(40))$, $E_2(\Gamma_0(40), \chi_1)$, $E_2(\Gamma_0(40), \chi_2)$ and $E_2(\Gamma_0(40), \chi_3)$, respectively.

Proof. The assertions follow from [19, Theorem 5.9] with $\chi = \psi = \chi_0$; $\epsilon = \chi_1$ and $\chi, \psi \in \{\chi_0, \chi_1\}$; $\epsilon = \chi_2$ and $\chi, \psi \in \{\chi_0, \chi_2\}$; $\epsilon = \chi_3$ and $\chi, \psi \in \{\chi_0, \chi_1, \chi_2, \chi_3\}$, respectively. ☐

Theorem 2.4. *Let χ_0 be the trivial character and χ_1, χ_2, χ_3 be as in* (2.3)*. Then*

$$\begin{aligned}
&\{L(q) - tL(q^t) \mid t = 2,4,5,8,10,20,40\} \cup \{A_1(q), A_2(q), A_3(q)\},\\
&\{E_{\chi_0,\chi_1}(q^t), E_{\chi_1,\chi_0}(q^t) \mid t = 1,2,4,8\} \cup \{B_1(q), B_2(q)\},\\
&\{E_{\chi_0,\chi_2}(q^t), E_{\chi_2,\chi_0}(q^t) \mid t = 1,5\} \cup \{C_k(q) \mid k = 1,2,3,4\},\\
&\{E_{\chi_0,\chi_3}(q), E_{\chi_1,\chi_2}(q), E_{\chi_2,\chi_1}(q), E_{\chi_3,\chi_0}(q)\} \cup \{D_k(q) \mid k = 1,2,3,4\}
\end{aligned}$$

are bases for $M_2(\Gamma_0(40))$, $M_2(\Gamma_0(40), \chi_1)$, $M_2(\Gamma_0(40), \chi_2)$, $M_2(\Gamma_0(40), \chi_3)$, respectively.

Proof. The assertions follow from (2.2), Theorems 2.2 and 2.3. ☐

We now give four theorems (Theorems 2.5–2.8) from which the theorems of Section 3 (Theorems 3.1–3.4) follow.

Theorem 2.5. *We have*

$$\begin{aligned}
\varphi^4(q) &= 8L(q) - 32L(q^4),\\
\varphi^2(q)\varphi^2(q^2) &= 4L(q) - 4L(q^2) + 8L(q^4) - 32L(q^8),\\
\varphi^2(q)\varphi^2(q^5) &= \frac{4}{3}L(q) - \frac{16}{3}L(q^4) + \frac{20}{3}L(q^5) - \frac{80}{3}L(q^{20}) + \frac{8}{3}A_1(q),\\
\varphi^2(q)\varphi^2(q^{10}) &= \frac{2}{3}L(q) - \frac{2}{3}L(q^2) + \frac{4}{3}L(q^4) + \frac{10}{3}L(q^5) - \frac{16}{3}L(q^8)\\
&\quad - \frac{10}{3}L(q^{10}) + \frac{20}{3}L(q^{20}) - \frac{80}{3}L(q^{40}) + \frac{10}{3}A_1(q)\\
&\quad + \frac{8}{3}A_2(q) + 4A_3(q),\\
\varphi(q)\varphi(q^2)\varphi(q^5)\varphi(q^{10}) &= L(q) - L(q^2) - 2L(q^4) - 5L(q^5) + 8L(q^8)\\
&\quad + 5L(q^{10}) + 10L(q^{20}) - 40L(q^{40}) + A_1(q) + 2A_3(q),\\
\varphi^2(q^2)\varphi^2(q^5) &= \frac{2}{3}L(q) - \frac{2}{3}L(q^2) + \frac{4}{3}L(q^4) + \frac{10}{3}L(q^5) - \frac{16}{3}L(q^8)\\
&\quad - \frac{10}{3}L(q^{10}) + \frac{20}{3}L(q^{20}) - \frac{80}{3}L(q^{40}) - \frac{2}{3}A_1(q)\\
&\quad + \frac{8}{3}A_2(q) - 4A_3(q).
\end{aligned}$$

Proof. Let (a_1, a_2, a_3, a_4) be any of the quaternary quadratic forms listed in the first column of Table 2.1. By Theorem 2.1 we have $\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}) \in M_2(\Gamma_0(40))$. By Theorem 2.4, $\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4})$ must be a linear combination of $L(q) - tL(q^t)$ $(t = 2,4,5,8,10,20,40)$ and $A_k(q)$ $(k \in \{1,2,3\})$, namely

$$\begin{aligned} &\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}) \\ &= x_1(L(q)-2L(q^2)) + x_2(L(q)-4L(q^4)) \\ &\quad + x_3(L(q)-5L(q^5)) + x_4(L(q)-8L(q^8)) \\ &\quad + x_5(L(q)-10L(q^{10})) + x_6(L(q)-20L(q^{20})) \\ &\quad + x_7(L(q)-40L(q^{40})) + y_1A_1(q) + y_2A_2(q) + y_3A_3(q). \end{aligned} \tag{2.26}$$

We only prove the last equation in the theorem as the others can be proven similarly. Let $(a_1,a_2,a_3,a_4)=(2,2,5,5)$. Appealing to [8, Theorem 3.13], we find that the Sturm bound for the modular space $M_2(\Gamma_0(40))$ is 12. So, equating the coefficients of q^n for $0 \le n \le 12$ on both sides of (2.26), we find a system of linear equations with the unknowns x_i $(1 \le i \le 7)$, y_1, y_2 and y_3. Using MAPLE we solve the system and find that

$$x_1 = x_5 = \frac{1}{3},\ x_2 = x_6 = -\frac{1}{3},\ x_3 = y_1 = -\frac{2}{3},\ x_4 = x_7 = \frac{2}{3},\ y_2 = \frac{8}{3},\ y_3 = -4.$$

Substituting these values back in (2.26), and with the obvious simplifications, we find the asserted equation. ☐

Corollary 2.1. *Let $n \in \mathbb{N}$. We have*

$$N(1,1,10,10;n) = N(2,2,5,5;n) \ \textit{if}\ n \equiv 0 \pmod 2.$$

Proof. From Theorem 2.5, we have

$$\varphi^2(q)\varphi^2(q^{10}) - \varphi^2(q^2)\varphi^2(q^5) = 4A_1(q) + 8A_3(q). \tag{2.27}$$

It is clear from (1.2), (2.13) and (2.15) that

$$a_1(n) = a_3(n) = 0 \text{ if } n \equiv 0 \pmod 2. \tag{2.28}$$

The assertion now follows from (1.1), (2.27) and (2.28). ☐

Similarly to Theorem 2.5, Theorems 2.6–2.8 follow from Theorems 2.1 and 2.4.

Theorem 2.6. *Let χ_0 be the trivial character and χ_1 be as in* (2.3). *Then*

$$\begin{aligned} \varphi^3(q)\varphi(q^5) &= E_{\chi_0,\chi_1}(q) - 2E_{\chi_0,\chi_1}(q^2) - 4E_{\chi_0,\chi_1}(q^4) + 5E_{\chi_1,\chi_0}(q) \\ &\quad + 10E_{\chi_1,\chi_0}(q^2) - 20E_{\chi_1,\chi_0}(q^4), \\ \varphi^2(q)\varphi(q^2)\varphi(q^{10}) &= -\frac{1}{2}E_{\chi_0,\chi_1}(q) + \frac{1}{2}E_{\chi_0,\chi_1}(q^2) - E_{\chi_0,\chi_1}(q^4) \\ &\quad - 4E_{\chi_0,\chi_1}(q^8) + \frac{5}{2}E_{\chi_1,\chi_0}(q) + \frac{5}{2}E_{\chi_1,\chi_0}(q^2) \\ &\quad + 5E_{\chi_1,\chi_0}(q^4) - 20E_{\chi_1,\chi_0}(q^8) + 2B_2(q), \\ \varphi(q)\varphi^2(q^2)\varphi(q^5) &= \frac{1}{2}E_{\chi_0,\chi_1}(q) - \frac{1}{2}E_{\chi_0,\chi_1}(q^2) - E_{\chi_0,\chi_1}(q^4) \end{aligned}$$

$$\begin{aligned}
&- 4E_{\chi_0,\chi_1}(q^8) + \frac{5}{2}E_{\chi_1,\chi_0}(q) + \frac{5}{2}E_{\chi_1,\chi_0}(q^2)\\
&- 5E_{\chi_1,\chi_0}(q^4) + 20E_{\chi_1,\chi_0}(q^8) + 5B_1(q) - B_2(q),\\
\varphi(q)\varphi^3(q^5) = {}& E_{\chi_0,\chi_1}(q) - 2E_{\chi_0,\chi_1}(q^2) - 4E_{\chi_0,\chi_1}(q^4) + E_{\chi_1,\chi_0}(q)\\
&+ 2E_{\chi_1,\chi_0}(q^2) - 4E_{\chi_1,\chi_0}(q^4),\\
\varphi(q)\varphi(q^5)\varphi^2(q^{10}) = {}& \frac{1}{2}E_{\chi_0,\chi_1}(q) - \frac{1}{2}E_{\chi_0,\chi_1}(q^2) - E_{\chi_0,\chi_1}(q^4)\\
&- 4E_{\chi_0,\chi_1}(q^8) + \frac{1}{2}E_{\chi_1,\chi_0}(q) + \frac{1}{2}E_{\chi_1,\chi_0}(q^2)\\
&- E_{\chi_1,\chi_0}(q^4) + 4E_{\chi_1,\chi_0}(q^8) - B_1(q) + B_2(q),\\
\varphi(q^2)\varphi^2(q^5)\varphi(q^{10}) = {}& -\frac{1}{2}E_{\chi_0,\chi_1}(q) + \frac{1}{2}E_{\chi_0,\chi_1}(q^2) - E_{\chi_0,\chi_1}(q^4)\\
&- 4E_{\chi_0,\chi_1}(q^8) + \frac{1}{2}E_{\chi_1,\chi_0}(q) + \frac{1}{2}E_{\chi_1,\chi_0}(q^2)\\
&+ E_{\chi_1,\chi_0}(q^4) - 4E_{\chi_1,\chi_0}(q^8) - 2B_1(q).
\end{aligned}$$

Theorem 2.7. *Let* χ_0 *be the trivial character and* χ_2 *be as in* (2.3). *Then*

$$\begin{aligned}
\varphi^3(q)\varphi(q^2) = {}& -2E_{\chi_0,\chi_2}(q) + 8E_{\chi_2,\chi_0}(q),\\
\varphi^2(q)\varphi(q^5)\varphi(q^{10}) = {}& \frac{2}{13}\big(2E_{\chi_0,\chi_2}(q) - 15E_{\chi_0,\chi_2}(q^5) + 8E_{\chi_2,\chi_0}(q)\\
&+ 60E_{\chi_2,\chi_0}(q^5)\big) + \frac{8}{13}\big(6C_1(q) - 4C_2(q) - 3C_3(q)\\
&+ 4C_4(q)\big),\\
\varphi(q)\varphi^3(q^2) = {}& -2E_{\chi_0,\chi_2}(q) + 4E_{\chi_2,\chi_0}(q),\\
\varphi(q)\varphi(q^2)\varphi^2(q^5) = {}& \frac{2}{13}\big(-3E_{\chi_0,\chi_2}(q) - 10E_{\chi_0,\chi_2}(q^5) + 12E_{\chi_2,\chi_0}(q)\big)\\
&- \frac{80}{13}E_{\chi_2,\chi_0}(q^5) + \frac{8}{13}\big(-2C_2(q) - 5C_3(q) + C_4(q)\big),\\
\varphi(q)\varphi(q^2)\varphi^2(q^{10}) = {}& \frac{2}{13}\big(-3E_{\chi_0,\chi_2}(q) - 10E_{\chi_0,\chi_2}(q^5) + 6E_{\chi_2,\chi_0}(q)\big)\\
&- \frac{40}{13}E_{\chi_2,\chi_0}(q^5) + \frac{4}{13}\big(2C_1(q) - 2C_3(q) + 5C_4(q)\big),\\
\varphi^2(q^2)\varphi(q^5)\varphi(q^{10}) = {}& \frac{2}{13}\big(2E_{\chi_0,\chi_2}(q) - 15E_{\chi_0,\chi_2}(q^5) + 4E_{\chi_2,\chi_0}(q)\\
&+ 30E_{\chi_2,\chi_0}(q^5)\big) + \frac{4}{13}\big(-4C_1(q) + 12C_2(q) + 8C_3(q)\\
&- 3C_4(q)\big).
\end{aligned}$$

Theorem 2.8. *Let* χ_0 *be the trivial character and* χ_1, χ_2, χ_3 *be as in* (2.3). *Then*

$$\varphi^3(q)\varphi(q^{10}) = \frac{1}{7}\big(-E_{\chi_0,\chi_3}(q) - 5E_{\chi_1,\chi_2}(q) + 4E_{\chi_2,\chi_1}(q) + 20E_{\chi_3,\chi_0}(q)\big)$$

$$\begin{aligned}
&\quad + \frac{4}{7}\big(- 3D_1(q) + 15D_2(q) - 15D_3(q) + 9D_4(q)\big),\\
\varphi^2(q)\varphi(q^2)\varphi(q^5) &= \frac{1}{7}\big(- E_{\chi_0,\chi_3}(q) + 5E_{\chi_1,\chi_2}(q) - 4E_{\chi_2,\chi_1}(q) + 20E_{\chi_3,\chi_0}(q)\big)\\
&\quad + \frac{8}{7}\big(- D_1(q) + 2D_4(q)\big),\\
\varphi(q)\varphi^2(q^2)\varphi(q^{10}) &= \frac{1}{7}\big(- E_{\chi_0,\chi_3}(q) - 5E_{\chi_1,\chi_2}(q) + 2E_{\chi_2,\chi_1}(q) + 10E_{\chi_3,\chi_0}(q)\big)\\
&\quad + \frac{4}{7}\big(D_1(q) + 5D_2(q) + 5D_3(q) + D_4(q)\big),\\
\varphi(q)\varphi^2(q^5)\varphi(q^{10}) &= \frac{1}{7}\big(- E_{\chi_0,\chi_3}(q) - E_{\chi_1,\chi_2}(q) + 4E_{\chi_2,\chi_1}(q) + 4E_{\chi_3,\chi_0}(q)\\
&\quad + \frac{8}{7}\big(D_2(q) - D_3(q) + D_4(q)\big),\\
\varphi(q)\varphi^3(q^{10}) &= -\frac{1}{7}\big(E_{\chi_0,\chi_3}(q) + E_{\chi_1,\chi_2}(q) - 2E_{\chi_2,\chi_1}(q) - 2E_{\chi_3,\chi_0}(q)\big)\\
&\quad + \frac{12}{7}\big(D_2(q) + D_3(q) + D_4(q)\big),\\
\varphi^3(q^2)\varphi(q^5) &= \frac{1}{7}\big(- E_{\chi_0,\chi_3}(q) + 5E_{\chi_1,\chi_2}(q) - 2E_{\chi_2,\chi_1}(q) + 10E_{\chi_3,\chi_0}(q)\\
&\quad - 12D_1(q)\big),\\
\varphi(q^2)\varphi^3(q^5) &= \frac{1}{7}\big(- E_{\chi_0,\chi_3}(q) + E_{\chi_1,\chi_2}(q) - 4E_{\chi_2,\chi_1}(q) + 4E_{\chi_3,\chi_0}(q)\big)\\
&\quad - \frac{12}{7}\big(D_1(q) + D_2(q) + 3D_3(q) - D_4(q)\big),\\
\varphi(q^2)\varphi(q^5)\varphi^2(q^{10}) &= \frac{1}{7}\big(- E_{\chi_0,\chi_3}(q) + E_{\chi_1,\chi_2}(q) - 2E_{\chi_2,\chi_1}(q) + 2E_{\chi_3,\chi_0}(q)\big)\\
&\quad + \frac{4}{7}\big(- D_1(q) + D_2(q) - 3D_3(q) + D_4(q)\big).
\end{aligned}$$

3. Main results

Theorem 3.1. *Let* $n \in \mathbb{N}$. *We have*

$$\begin{aligned}
N(1,1,5,5;n) &= \frac{4}{3}\sigma(n) - \frac{16}{3}\sigma(n/4) + \frac{20}{3}\sigma(n/5) - \frac{80}{3}\sigma(n/20) + \frac{8}{3}a_1(n),\\
N(1,1,10,10;n) &= \frac{2}{3}\sigma(n) - \frac{2}{3}\sigma(n/2) + \frac{4}{3}\sigma(n/4) + \frac{10}{3}\sigma(n/5) - \frac{16}{3}\sigma(n/8)\\
&\quad - \frac{10}{3}\sigma(n/10) + \frac{20}{3}\sigma(n/20) - \frac{80}{3}\sigma(n/40) + \frac{10}{3}a_1(n)\\
&\quad + \frac{8}{3}a_2(n) + 4a_3(n),\\
N(1,2,5,10;n) &= \sigma(n) - \sigma(n/2) - 2\sigma(n/4) - 5\sigma(n/5) + 8\sigma(n/8)\\
&\quad + 5\sigma(n/10) + 10\sigma(n/20) - 40\sigma(n/40) + a_1(n) + 2a_3(n),
\end{aligned}$$

$$\begin{aligned} N(2,2,5,5;n) = {} & \frac{2}{3}\sigma(n) - \frac{2}{3}\sigma(n/2) + \frac{4}{3}\sigma(n/4) + \frac{10}{3}\sigma(n/5) - \frac{16}{3}\sigma(n/8) \\ & - \frac{10}{3}\sigma(n/10) + \frac{20}{3}\sigma(n/20) - \frac{80}{3}\sigma(n/40) - \frac{2}{3}a_1(n) \\ & + \frac{8}{3}a_2(n) - 4a_3(n). \end{aligned}$$

Proof. The assertions follow from (1.1), (2.4) and Theorem 2.5. □

Theorem 3.2. *Let $n \in \mathbb{N}$. Let $\sigma_{\chi_i,\chi_j}(n)$ be as in* (2.1) *for $i,j \in \{0,1\}$. We have*

$$\begin{aligned} N(1,1,1,5;n) = {} & \sigma_{\chi_0,\chi_1}(n) - 2\sigma_{\chi_0,\chi_1}(n/2) - 4\sigma_{\chi_0,\chi_1}(n/4) \\ & + 5\sigma_{\chi_1,\chi_0}(n) + 10\sigma_{\chi_1,\chi_0}(n/2) - 20\sigma_{\chi_1,\chi_0}(n/4), \\ N(1,1,2,10;n) = {} & -\frac{1}{2}\sigma_{\chi_0,\chi_1}(n) + \frac{1}{2}\sigma_{\chi_0,\chi_1}(n/2) - \sigma_{\chi_0,\chi_1}(n/4) \\ & - 4\sigma_{\chi_0,\chi_1}(n/8) + \frac{5}{2}\sigma_{\chi_1,\chi_0}(n) + \frac{5}{2}\sigma_{\chi_1,\chi_0}(n/2) \\ & + 5\sigma_{\chi_1,\chi_0}(n/4) - 20\sigma_{\chi_1,\chi_0}(n/8) + 2b_2(n), \\ N(1,2,2,5;n) = {} & \frac{1}{2}\sigma_{\chi_0,\chi_1}(n) - \frac{1}{2}\sigma_{\chi_0,\chi_1}(n/2) - \sigma_{\chi_0,\chi_1}(n/4) - 4\sigma_{\chi_0,\chi_1}(n/8) \\ & + \frac{5}{2}\sigma_{\chi_1,\chi_0}(n) + \frac{5}{2}\sigma_{\chi_1,\chi_0}(n/2) - 5\sigma_{\chi_1,\chi_0}(n/4) \\ & + 20\sigma_{\chi_1,\chi_0}(n/8) + 5b_1(n) - b_2(n), \\ N(1,5,5,5;n) = {} & \sigma_{\chi_0,\chi_1}(n) - 2\sigma_{\chi_0,\chi_1}(n/2) - 4\sigma_{\chi_0,\chi_1}(n/4) \\ & + \sigma_{\chi_1,\chi_0}(n) + 2\sigma_{\chi_1,\chi_0}(n/2) - 4\sigma_{\chi_1,\chi_0}(n/4), \\ N(1,5,10,10;n) = {} & \frac{1}{2}\sigma_{\chi_0,\chi_1}(n) - \frac{1}{2}\sigma_{\chi_0,\chi_1}(n/2) - \sigma_{\chi_0,\chi_1}(n/4) - 4\sigma_{\chi_0,\chi_1}(n/8) \\ & + \frac{1}{2}\sigma_{\chi_1,\chi_0}(n) + \frac{1}{2}\sigma_{\chi_1,\chi_0}(n/2) - \sigma_{\chi_1,\chi_0}(n/4) \\ & + 4\sigma_{\chi_1,\chi_0}(n/8) - b_1(n) + b_2(n), \\ N(2,5,5,10;n) = {} & -\frac{1}{2}\sigma_{\chi_0,\chi_1}(n) + \frac{1}{2}\sigma_{\chi_0,\chi_1}(n/2) - \sigma_{\chi_0,\chi_1}(n/4) \\ & - 4\sigma_{\chi_0,\chi_1}(n/8) + \frac{1}{2}\sigma_{\chi_1,\chi_0}(n) + \frac{1}{2}\sigma_{\chi_1,\chi_0}(n/2) \\ & + \sigma_{\chi_1,\chi_0}(n/4) - 4\sigma_{\chi_1,\chi_0}(n/8) - 2b_1(n). \end{aligned}$$

Proof. The assertions follow from (1.1), (2.5) and Theorem 2.6. □

Theorem 3.3. *Let $n \in \mathbb{N}$. Let $\sigma_{\chi_i,\chi_j}(n)$ be as in* (2.1) *for $i,j \in \{0,2\}$. Then*

$$\begin{aligned} N(1,1,1,2;n) = {} & -2\sigma_{\chi_0,\chi_2}(n) + 8\sigma_{\chi_2,\chi_0}(n), \\ N(1,1,5,10;n) = {} & \frac{2}{13}\big(2\sigma_{\chi_0,\chi_2}(n) - 15\sigma_{\chi_0,\chi_2}(n/5) + 8\sigma_{\chi_2,\chi_0}(n) \end{aligned}$$

$$\begin{aligned}
&+60\sigma_{\chi_2,\chi_0}(n/5)\big)+\frac{8}{13}\big(6c_1(n)-4c_2(n)-3c_3(n)\\
&+4c_4(n)\big),\\
N(1,2,2,2;n)=\ &-2\sigma_{\chi_0,\chi_2}(n)+4\sigma_{\chi_2,\chi_0}(n),\\
N(1,2,5,5;n)=\ &-\frac{2}{13}\big(3\sigma_{\chi_0,\chi_2}(n)+10\sigma_{\chi_0,\chi_2}(n/5)-12\sigma_{\chi_2,\chi_0}(n)\\
&+40\sigma_{\chi_2,\chi_0}(n/5)\big)+\frac{8}{13}\big(-2c_2(n)-5c_3(n)+c_4(n)\big),\\
N(1,2,10,10;n)=\ &\frac{2}{13}\big(-3\sigma_{\chi_0,\chi_2}(n)-10\sigma_{\chi_0,\chi_2}(n/5)+6\sigma_{\chi_2,\chi_0}(n)\\
&-20\sigma_{\chi_2,\chi_0}(n/5)\big)+\frac{4}{13}\big(2c_1(n)-2c_3(n)+5c_4(n)\big),\\
N(2,2,5,10;n)=\ &\frac{2}{13}\big(2\sigma_{\chi_0,\chi_2}(n)-15\sigma_{\chi_0,\chi_2}(n/5)+4\sigma_{\chi_2,\chi_0}(n)\\
&+30\sigma_{\chi_2,\chi_0}(n/5)\big)+\frac{4}{13}\big(-4c_1(n)+12c_2(n)+8c_3(n)\\
&-3c_4(n)\big).
\end{aligned}$$

Proof. The assertions follow from (1.1), (2.6) and Theorem 2.7. ☐

Theorem 3.4. *Let $n\in\mathbb{N}$. Let $\sigma_{\chi_i,\chi_j}(n)$ be as in (2.1) for $i,j\in\{0,1,2,3\}$. Then*

$$\begin{aligned}
N(1,1,1,10;n)=\ &\frac{1}{7}\big(-\sigma_{\chi_0,\chi_3}(n)-5\sigma_{\chi_1,\chi_2}(n)+4\sigma_{\chi_2,\chi_1}(n)+20\sigma_{\chi_3,\chi_0}(n)\big)\\
&+\frac{4}{7}\big(-3d_1(n)+15d_2(n)-15d_3(n)+9d_4(n)\big),\\
N(1,1,2,5;n)=\ &\frac{1}{7}\big(-\sigma_{\chi_0,\chi_3}(n)+5\sigma_{\chi_1,\chi_2}(n)-4\sigma_{\chi_2,\chi_1}(n)+20\sigma_{\chi_3,\chi_0}(n)\big)\\
&+\frac{8}{7}\big(-d_1(n)+2d_4(n)\big),\\
N(1,2,2,10;n)=\ &\frac{1}{7}\big(-\sigma_{\chi_0,\chi_3}(n)-5\sigma_{\chi_1,\chi_2}(n)+2\sigma_{\chi_2,\chi_1}(n)+10\sigma_{\chi_3,\chi_0}(n)\big)\\
&+\frac{4}{7}\big(d_1(n)+5d_2(n)+5d_3(n)+d_4(n)\big),\\
N(1,5,5,10;n)=\ &\frac{1}{7}\big(-\sigma_{\chi_0,\chi_3}(n)-\sigma_{\chi_1,\chi_2}(n)+4\sigma_{\chi_2,\chi_1}(n)+4\sigma_{\chi_3,\chi_0}(n)\big)\\
&+\frac{8}{7}\big(d_2(n)-d_3(n)+d_4(n)\big),\\
N(1,10,10,10;n)=\ &\frac{1}{7}\big(-\sigma_{\chi_0,\chi_3}(n)-\sigma_{\chi_1,\chi_2}(n)+2\sigma_{\chi_2,\chi_1}(n)+2\sigma_{\chi_3,\chi_0}(n)\big)\\
&+\frac{12}{7}\big(d_2(n)+d_3(n)+d_4(n)\big),\\
N(2,2,2,5;n)=\ &\frac{1}{7}\big(-\sigma_{\chi_0,\chi_3}(n)+5\sigma_{\chi_1,\chi_2}(n)-2\sigma_{\chi_2,\chi_1}(n)+10\sigma_{\chi_3,\chi_0}(n)\big)
\end{aligned}$$

$$
\begin{aligned}
&\quad -\frac{12}{7}d_1(n),\\
N(2,5,5,5;n) &= \frac{1}{7}\big(-\sigma_{\chi_0,\chi_3}(n)+\sigma_{\chi_1,\chi_2}(n)-4\sigma_{\chi_2,\chi_1}(n)+4\sigma_{\chi_3,\chi_0}(n)\big)\\
&\quad +\frac{12}{7}\big(-d_1(n)-d_2(n)-3d_3(n)+d_4(n)\big),\\
N(2,5,10,10;n) &= \frac{1}{7}\big(-\sigma_{\chi_0,\chi_3}(n)+\sigma_{\chi_1,\chi_2}(n)-2\sigma_{\chi_2,\chi_1}(n)+2\sigma_{\chi_3,\chi_0}(n)\big)\\
&\quad +\frac{4}{7}\big(-d_1(n)+d_2(n)-3d_3(n)+d_4(n)\big).
\end{aligned}
$$

Proof. The assertions follow from (1.1), (2.7), (2.8) and Theorem 2.8. □

4. Remarks

Remark 4.1. Replacing q by $-q$ in $\varphi^3(q)\varphi(q^5)$ in Theorem 2.6, we have

$$
\begin{aligned}
\varphi^3(-q)\varphi(-q^5) &= E_{\chi_0,\chi_1}(-q)-2E_{\chi_0,\chi_1}(q^2)-4E_{\chi_0,\chi_1}(q^4)\\
&\quad +5E_{\chi_1,\chi_0}(-q)+10E_{\chi_1,\chi_0}(q^2)-20E_{\chi_1,\chi_0}(q^4).
\end{aligned} \tag{4.1}
$$

Appealing to Theorem 2.3, we obtain

$$E_{\chi_0,\chi_1}(-q) = -E_{\chi_0,\chi_1}(q)-2E_{\chi_0,\chi_1}(q^2)+4E_{\chi_0,\chi_1}(q^4), \tag{4.2}$$

$$E_{\chi_1,\chi_0}(-q) = -E_{\chi_1,\chi_0}(q)+2E_{\chi_1,\chi_0}(q^2)+4E_{\chi_1,\chi_0}(q^4). \tag{4.3}$$

Substituting (4.2) and (4.3) in (4.1), we obtain

$$\varphi^3(-q)\varphi(-q^5) = -E_{\chi_0,\chi_1}(q)-4E_{\chi_0,\chi_1}(q^2)-5E_{\chi_1,\chi_0}(q)+20E_{\chi_1,\chi_0}(q^2). \tag{4.4}$$

It can easily be seen that

$$-E_{\chi_0,\chi_1}(q)-4E_{\chi_0,\chi_1}(q^2) = 1+\sum_{n=1}^{\infty}\Big(\sum_{d\mid n}(-1)^d\Big(\frac{5}{d}\Big)d\Big)q^n, \tag{4.5}$$

$$-E_{\chi_1,\chi_0}(q)+4E_{\chi_1,\chi_0}(q^2) = \sum_{n=1}^{\infty}\Big(\sum_{d\mid n}(-1)^d\Big(\frac{5}{n/d}\Big)d\Big)q^n. \tag{4.6}$$

Now, appealing to (1.1) and (4.4)–(4.6), we obtain

$$
\begin{aligned}
&\sum_{n=0}^{\infty}N(1,1,1,5;n)(-q)^n\\
&= \varphi^3(-q)\varphi(-q^5)\\
&= 1+\sum_{n=1}^{\infty}\Big(\sum_{d\mid n}(-1)^d\Big(\frac{5}{d}\Big)d\Big)q^n+5\sum_{n=1}^{\infty}\Big(\sum_{d\mid n}(-1)^d\Big(\frac{5}{n/d}\Big)d\Big)q^n,
\end{aligned}
$$

from which we deduce

$$N(1,1,1,5;n) = \sum_{d\mid n}(-1)^{n+d}\Big(\frac{5}{d}\Big)d+5\sum_{d\mid n}(-1)^{n+d}\Big(\frac{5}{n/d}\Big)d,$$

which agrees with known results, see for example [3, Theorem 5.1]. Similarly, one can show that our formula for $N(1,5,5,5;n)$ given in Theorem 3.2 agrees with the result in [3, Theorem 6.1].

Remark 4.2. Appealing to Lemma 2.1 and Theorem 2.3, we obtain the following identities:

$$\begin{aligned}
&L(q) - 4L(q^4) = \frac{1}{8}\,\frac{\eta^{20}(2z)}{\eta^8(z)\eta^8(4z)},\\
&E_{\chi_0,\chi_1}(q) = -\frac{1}{5}\,\frac{\eta^5(z)}{\eta(5z)},\\
&E_{\chi_1,\chi_0}(q) = \frac{\eta^5(5z)}{\eta(z)},\\
&E_{\chi_0,\chi_2}(q) = -\frac{1}{2}\,\frac{\eta^2(z)\eta(2z)\eta^3(4z)}{\eta^2(8z)},\\
&E_{\chi_2,\chi_0}(q) = \frac{\eta^3(2z)\eta(4z)\eta^2(8z)}{\eta^2(z)},\\
&E_{\chi_0,\chi_1}(q) + 4E_{\chi_0,\chi_1}(q^2) = -\frac{\eta(z)\eta^2(2z)\eta^3(5z)}{\eta^2(10z)},\\
&E_{\chi_1,\chi_0}(q) + E_{\chi_1,\chi_0}(q^2) = \frac{\eta^3(2z)\eta^2(5z)\eta(10z)}{\eta^2(z)},\\
&E_{\chi_1,\chi_0}(q) - 4E_{\chi_1,\chi_0}(q^2) = \frac{\eta^3(z)\eta(5z)\eta^2(10z)}{\eta^2(2z)},\\
&E_{\chi_0,\chi_2}(q) - 2E_{\chi_2,\chi_0}(q) = -\frac{1}{2}\,\frac{\eta^{13}(4z)}{\eta^2(z)\eta(2z)\eta^6(8z)},\\
&E_{\chi_0,\chi_2}(q) - 4E_{\chi_2,\chi_0}(q) = -\frac{1}{2}\,\frac{\eta^{13}(2z)}{\eta^6(z)\eta(4z)\eta^2(8z)},\\
&E_{\chi_0,\chi_1}(q) - 2E_{\chi_0,\chi_1}(q^2) - 4E_{\chi_0,\chi_1}(q^4) = \frac{\eta^5(2z)\eta^7(10z)}{\eta(z)\eta(4z)\eta^3(5z)\eta^3(20z)},\\
&E_{\chi_1,\chi_0}(q) + 2E_{\chi_1,\chi_0}(q^2) - 4E_{\chi_1,\chi_0}(q^4) = \frac{\eta^7(2z)\eta^5(10z)}{\eta^3(z)\eta^3(4z)\eta(5z)\eta(20z)}.
\end{aligned}$$

Remark 4.3. Set $a := \varphi(q)$, $b := \varphi(q^2)$, $c := \varphi(q^5)$ and $d := \varphi(q^{10})$. We obtain the following identities from Theorem 2.8:

$$\begin{aligned}
&ad(-a^2 - b^2 + 5c^2 - 5d^2) + bc(5a^2 - 8b^2 - 5c^2 + 10d^2) = 12D_1(q),\\
&ad(2a^2 - b^2 - 4c^2 + d^2) + bc(-a^2 + b^2 - 5c^2 + 7d^2) = 24D_2(q),\\
&ad(-a^2 + 2b^2 - 7c^2 + 10d^2) + bc(-a^2 + 4b^2 + c^2 - 8d^2) = 48D_3(q),\\
&ad(a^2 - 8b^2 - 5c^2 + 20d^2) + bc(7a^2 - 10b^2 + 5c^2 - 10d^2) = 48D_4(q).
\end{aligned}$$

Remark 4.4. It would be interesting to determine general formulas for the number of representations of a positive integer n by the quaternary quadratic forms with coefficients in $\{1, p, q, pq\}$, where p and q are distinct prime numbers. The case when $p = 2$ and $q = 7$ is treated in [4].

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Ayşe Alaca
School of Mathematics and Statistics
Carleton University
Ottawa, ON K1S 5B6, Canada
Email address: `AyseAlaca@cunet.carleton.ca`

Mada Altiary
School of Mathematics and Statistics
Carleton University
Ottawa, ON K1S 5B6, Canada
Email address: `MadaAltiary@gmail.ca`