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A NOTE ON DERIVATIONS OF A SULLIVAN MODEL

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Abstract. Complex Grassmann manifolds $G_{n,k}$ are a generalization of complex projective spaces and have many important features some of which are captured by the Plücker embedding $f : G_{n,k} \longrightarrow \mathbb{C}P^{N-1}$ where $N = \binom{n}{k}$. The problem of existence of cross sections of fibrations can be studied using the Gottlieb group. In a more generalized context one can use the relative evaluation subgroup of a map to describe the cohomology of smooth fiber bundles with fiber the (complex) Grassmann manifold $G_{n,k}$. Our interest lies in making use of techniques of rational homotopy theory to address problems and questions involving applications of Gottlieb groups in general.

In this paper, we construct the Sullivan minimal model of the (complex) Grassmann manifold $G_{n,k}$ for $2 \leq k < n$, and we compute the rational evaluation subgroup of the embedding $f : G_{n,k} \longrightarrow \mathbb{C}P^{N-1}$. We show that, for the Sullivan model $\phi : A \longrightarrow B$, where A and B are the Sullivan minimal models of $\mathbb{C}P^{N-1}$ and $G_{n,k}$ respectively, the evaluation subgroup $G_n(A, B; \phi)$ of ϕ is generated by a single element and the relative evaluation subgroup $G_n^{rel}(A, B; \phi)$ is zero. The triviality of the relative evaluation subgroup has its application in studying fibrations with fibre the (complex) Grassmann manifold.

1. Introduction

The discoveries by Quillen [11] and by Sullivan [12] that associate to a topological space X an explicit algebraic model, gave a computational power to rational homotopy theory. Sullivan algebras and models and Quillen models provide an effective computational approach to rational homotopy theory, where, in each case, the rational homotopy type of a given topological space is identified with the isomorphism class of its algebraic model. Similarly, the rational homotopy type of a continuous map between spaces is the same as the algebraic homotopy class of the corresponding morphism between models. Thus, the rational homotopy of a simply connected topological space is identified with the corresponding Sullivan minimal model. Let $G_{n,k}$ be the (complex) Grassmann manifold of k-dimensional vector subspaces of $\mathbb{C}^n$. The (complex)

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Grassmann manifold $G_{n,k}$ is a homogeneous space that is simply connected, so we may associate a Sullivan minimal model. Grassmann manifolds are useful in the study of problems in topology, algebraic topology and differential geometry. For instance, Grassmann manifolds provide non trivial fibrations which have applications in other areas of algebraic topology like the theory of fiber bundles, for example, in sphere bundles which are fiber bundles with fiber an n-sphere. Let $\pi : E \longrightarrow B$ be a fibration. A section (cross section) of a fiber bundle π is a continuous map $f : B \longrightarrow E$ such that $(\pi f)(x) = x$ for all $x \in B$. Since bundles do not in general have globally defined sections, one purpose of study would be to account for their existence locally. Let $\mathbb{R}^n$ be the Euclidean n-space, and let $T_x(S^{n-1}) \subset \mathbb{R}^n$ be the tangent space to the unit sphere S^{n-1} at x. A continuous tangent vector field on the sphere S^{n-1} is a function $\Upsilon : S^{n-1} \longrightarrow \mathbb{R}^n$ such that $\Upsilon(x) \in T_x(S^{n-1})$ for all $x \in S^{n-1}$. The existence of sections is closely related to the problem of existence or number of vector fields on a sphere. For the real Stiefel manifold $V_{n,k}$, it is known that S^{n-1} has $k-1$ linearly independent vector fields if and only if $V_{n,k} \longrightarrow S^{n-1}$ admits a section.

Let X be a based CW-complex. An element $[g] \in \pi_n(X, x_0)$ is said to be Gottlieb if there exist a continuous map $\phi_g : X \times S^n \longrightarrow X$ making the following diagram

$$\begin{array}{ccc} X \vee S^n & \xrightarrow{\nabla \circ (id \vee g)} & X \\ \downarrow & \nearrow_{\phi_g} & \\ X \times S^n & & \end{array}$$

commutative, where $\nabla : X \vee X \longrightarrow X$ is the folding map. The set of all Gottlieb elements $[g] \in \pi_n(X, x_0)$ is denoted by $G_n(X, x_0)$ and is called the Gottlieb group or the n-th evaluation subgroup of $\pi_n(X, x_0)$ [4]. Gottlieb groups have many applications in topology, covering spaces, fixed point theory, homotopy theory of fibrations. The existence of cross-sections can also be studied using the Gottlieb group. For instance $G_n(X) = 0$ implies that every fibration $X \longrightarrow E \longrightarrow S^{n+1}$ has a section. Thus, triviality of Gottlieb groups is related to the cross section problem of fibrations. Various authors have given results on Gottlieb groups of Stiefel manifolds, homogeneous spaces, lens spaces making use of fibrations (see [4, 6–8]).

One can use a more generalized and adapted concept of the Gottlieb group to study the cross section problem for fibrations with fiber the (complex) Grassmann manifold $G_{n,k}$ and questions involving applications of Gottlieb groups in general.

Let $f : X \longrightarrow Y$ be a based map of simply connected CW complexes, let $\operatorname{map}(X, Y; f)$ be the path component of the space of (unbased) maps from X to Y homotopic to f and let $\omega : \operatorname{map}(X, Y; f) \longrightarrow Y$ be the evaluation map. The map induced in homotopy groups $\omega_\sharp : \pi_n(\operatorname{map}(X, Y; f)) \longrightarrow \pi(Y)$ by the evaluation map satisfies that it has as image the evaluation subgroup

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$G_n(Y, X; f)$. The Gottlieb group $G_n(X)$ for a space X is special case $X = Y$ and $f = 1$. The map $\omega : \mathrm{map}(X, X; 1) \longrightarrow X$ can be identified with the connecting map in universal fibrations for fibrations with fiber X. Gottlieb groups are important objects for the study of fibrations with fiber X. There are not many explicit computations of $G_*(X)$ in literature. A major difficulty being the fact that the map $f : X \longrightarrow Y$ is not necessarily a Gottlieb map [13], since $f_\sharp(G_n(X))$ is not a subset of $G_n(Y)$ in general, but it induces a map $f_\sharp : G_n(X) \longrightarrow G_n(Y, X; f)$ which fits into a general framework of a G-sequence of f [14] given by the image of the long exact homotopy sequence of $f_* : \mathrm{map}(X, X; 1) \longrightarrow \mathrm{map}(X, Y; f)$ in the long exact homotopy sequence of f. Now, the induced homomorphism on rational homotopy groups $\omega_\sharp : \pi_n(\mathrm{map}(X, Y; f)) \longrightarrow \pi(Y)$ can be identified with the map of complexes of derivations constructed from the Sullivan minimal model of f [3]. Thus, the characterization of the evaluation subgroup $G_n(A, B; \phi)$ in terms of derivations is a consequence of the particular Sullivan model for the map (also see [1, 2]). In their paper [9] Smith and Lupton characterized the Gottlieb groups and the rational evaluation subgroups through derivations of Sullivan minimal model of the space. The construction of minimal models of the Grassmann manifolds $G_{n,k}$ can be found from the work of Sullivan [12] and others. However we have not found references on explicit descriptions depending on parameters n and k. In Section 3 we construct the Sullivan minimal model of the (complex) Grassmann manifold $G_{n,k}$. In Section 4 we discuss the evaluation subgroup of the map $f : G_{n,k} \longrightarrow \mathbb{C}P^{N-1}$ where $N = \binom{n}{k}$ and compute the relative evaluation subgroup.

2. Preliminaries

Throughout this paper, spaces are assumed to be 1-connected finite CW-complexes. Let $\mathbf{k}$ be a commutative ring with 1 and (A, d) a graded cochain algebra over $\mathbf{k}$. A Sullivan algebra is a cochain commutative graded algebra of the form $(\wedge V, d)$ where d is a differential, that is, d is a linear map $d : V \longrightarrow V$ such that $d(xy) = (d\,x)y + (-1)^{\deg x} x(d\,y)$; $x, y \in V$. The Sullivan algebra $(\wedge V, d)$ is called minimal if $dV \subset \wedge^{\geq 2} V$. Sullivan minimal models have been used to describe the rational homotopy of a space. For $\mathbf{k} = \mathbb{Q}$ any simply connected space, X has a model $(\wedge V, d)$ such that

$$H(\wedge V, d) = H^*(X, \mathbb{Q}), \quad V^n \simeq \mathrm{Hom}_{\mathbb{Z}}(\pi_n(X), \mathbb{Q}).$$

Let A be a cochain algebra. A derivation θ of degree p is a linear map such that $\theta(A^k) \subset A^{k-p}$ and verifies

$$\theta(xy) = \theta(x)y + (-1)^{p|x|} x\theta(y).$$

If $\theta_1, \theta_2 \in \mathrm{Der}(A)$, then $[\theta_1, \theta_2] = \theta_1\theta_2 - (-1)^{|\theta_1||\theta_2|}\theta_2\theta_1$ and $\delta\theta = [d, \theta]$. The space $(\mathrm{Der}(A), \delta)$ of derivations of A is a differential graded Lie algebra. If $(\Lambda V, d)$ is the Sullivan minimal model of a space X, an element $v \in V^n \simeq$

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$\mathrm{Hom}_{\mathbb{Z}}(\pi_n(X), \mathbb{Q})$ represents a Gottlieb element of $\pi_n(X) \otimes \mathbb{Q}$ if and only if there is a derivation θ of ΛV verifying $\theta(v) = 1$ and such that $[d, \theta] = 0$ [3].

3. Minimal model of $G_{n,k}$

In this section we will consider $G_{n,k}$ to be the complex Grassmann manifold of k-dimensional vector subspaces of $\mathbb{C}^n$. The Grassmann manifold $G_{n,k}$ is a homogeneous space that is simply connected. The following theorem gives a procedure for computing Sullivan models for homogeneous spaces.

Theorem 1 ([5, Chapter XI.4])**.** *Let H be a closed connected subgroup of a compact connected Lie group G. Denote by $i : H \to G$ the canonical inclusion and by $Bi : BH \to BG$ the induced map. Let $H^*(BG; \mathbb{Q}) = \wedge V$ and $H^*(BH; \mathbb{Q}) = \wedge W$ be the respective cohomology algebras BG and BH. Denote by sV a copy of the vector space V shifted by one degree, $|sv| = |v| - 1$, $v \in V$. Define a differential V on $\wedge W \otimes \wedge(sV)$ by $dw = 0$ and $d(sv) = H^*(Bi)(v)$ if $sv \in sV$. Then the commutative differential graded algebra $(\wedge W \otimes \wedge(sV), d)$ is a Sullivan model for the homogeneous space G/H. In particular $H^*(G/H; \mathbb{Q}) = H(\wedge W \otimes \wedge(sV), d)$.*

Note that the above model is not necessarily minimal. We compute the Sullivan minimal model of the space $G_{n,k}$ and we give the following result.

Theorem 2. (i) *The minimal Sullivan model of $G_{n,k}$ is given by*

$$(\wedge(x_2, \dots, x_{2k}, y_{2(n-k)+1}, \dots, y_{2(n-1)+1}); d),$$

where d is the differential given by $dx_i = 0$ and $dy_{2(n-t)+1} \in \wedge(x_2, \dots, x_{2k})$.

(ii) *The generators $y_{2(n-k)+1}, \dots, y_{2(n-1)+1}$ are Gottlieb elements.*

Proof. (i) The model

$$(\wedge(x_2, x_4, \dots, x_{2k}, y_{2(n-1)+1}, \dots, y_{2(n-k)+1}) \otimes \wedge(z_{2(n-s)+1}, d),$$

where $s = 1, 2, \dots, n$, is not minimal. Make change of variables and consider the acyclic ideal $I = \langle v_2, v_4, \dots, v_{2k}, z_{2(n-s)+1} \rangle$, where $dz_1 = v_2$, $dz_3 = v_4, \dots, dz_{2(n-s)+1} = v_{2k}$ for $s = k+1, \dots, n$.

Taking the quotient with the acyclic ideal we obtain the minimal model

$$(\wedge(x_2, x_4, \dots, x_{2k}) \otimes \wedge(y_{2(n-1)+1}, \dots, y_{2(n-k)+1}), d'),$$

where $d'x_i = 0$, $d'y_{2(n-t)+1} \in \wedge(x_2, x_4, \dots, x_{2k})$ for $t = 1, \dots, k$.

(ii) Denote by $\langle y_{2(n-t)+1}, 1 \rangle$ the derivation $\theta_{2(n-t)+1}$ such that

$$\theta_{2(n-t)+1}(y_{2(n-t)+1}) = 1$$

and zero on other elements of the basis.

Therefore $[d', \theta_{2(n-t)+1}](y_{2(n-t)+1}) = 0$.

Hence the generators $y_{2(n-t)+1}$ are Gottlieb elements for $t = 1, \dots, k$. $\square$

We give the following example for $k = 2$.

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Example 3. (i) The Sullivan minimal model of $G_{n,2}$ is given by

$$(\wedge(x_2, x_4, y_{2n-3}, y_{2n-1}), d),$$

where $dx_2 = dx_4 = 0$ and $dy_i \in \wedge(x_2, x_4)$ where $i = 2n-3,\ 2n-1$.

(ii) The generators y_{2n-3} and y_{2n-1} are Gottlieb elements.

First we observe that the model

$$(\wedge(x_2, x_4, y_2, y_4, \ldots, y_{2n-4}) \otimes \wedge(z_1, z_3, \ldots, z_{2n-1}), d)$$

is not minimal. We make change of variables and the following substitutions, let

$$\begin{aligned} v_2 &= x_2 + y_2 \\ y_2 &= v_2 - x_2 \\ v_4 &= x_2 y_2 + x_4 + y_4 \\ y_4 &= v_4 - x_2 y_2 - x_4 \\ v_{j+1} &= x_2 y_{j-1} + x_4 y_{j-3} + y_{j+1} \\ y_{j+1} &= v_{j+1} - x_2 y_{j-1} - x_4 y_{j-3};\ j = 3, 5, \ldots, 2n-5. \end{aligned}$$

Therefore $dz_1 = v_2, \ldots, dz_j = v_{j+1},\ j = 3, 5, \ldots, 2n-5$. Let

$$I = \langle v_2, v_4, \ldots, v_{2n-4}, z_1, z_3, \ldots, z_{2n-5} \rangle.$$

The ideal I is acyclic and $dz_i = 0$ for $i = 1, 3, \ldots, 2n-5$.

Secondly, let $\mathrm{Der}_n(\wedge(x_2, x_4, y_{2n-3}, y_{2n-1}), d)$ be the vector space of positive derivations that reduce degree by n. Denote by $\langle y_i, 1\rangle$, $i = 2n-3, 2n-1$ the derivation θ_i such that $\theta_i(y_i) = 1$ and zero on other elements of the basis. We have $[d, \theta_i] = d\theta_i - (-1)^{|\theta_i|}\theta_i d = d\theta_i = 0$ since $dy_i \in \wedge(x_2, x_4)$. Therefore $[d, \theta_i](y_i) = 0$ for $i = 2n-3,\ 2n-1$.

4. Evaluation subgroups of a map

Consider the embedding $f : G_{n,k} \hookrightarrow \mathbb{C}P^{N-1}$, where $N = \binom{n}{k}$, $2 \le k \le \left[\frac{n}{2}\right]$, $n > 3$.

The hypothesis that $k \le \left[\frac{n}{2}\right]$ is not a restriction since $G_{n,k} \equiv G_{n,n-k}$.

The minimal Sullivan model of $\mathbb{C}P^{N-1}$ is given by $(\bigwedge(x_2, y_{2n-1}), d)$ where d is the differential given by $dx_2 = 0,\ dy_{2N-1} = x_2^{2N-1}$ and the Sullivan minimal model of f is given by

$$\phi : (\wedge(x_2, y_{2N-1}), d) \longrightarrow (\wedge(x_2, \ldots, x_{2k}, y_{2(n-k)+1}, \ldots, y_{2(n-1)+1}), d'),$$

where ϕ is defined as follows: $\phi(x_2) = x_2,\ \phi(y_{2N-1}) = \alpha$ where $d'(\alpha) = x_2^N$.

We study the evaluation subgroups of ϕ.

Following [9], let $\phi : A \longrightarrow B$ be a map of differential graded algebras, define a ϕ-derivation of degree n to be a linear map $\theta : A \longrightarrow B$ that reduces degree by n and satisfy $\theta(xy) = \theta(x)\phi(y) + (-1)^{n|x|}\phi(x)\theta(y)$. Consider the vector space of ϕ-derivations $\mathrm{Der}_*(A, B; \phi)$ which are derivations that decrease degree by some positive number n. When $n = 1$, we restrict to derivations θ such that

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$d_B \circ \theta + \theta \circ d_A = 0$. The differential D is defined $D(\theta) = d_B \circ \theta - (-1)^{|\theta|}\theta \circ d_A$ where d_A and d_B are the differentials for A and B respectively. Pre-composition with ϕ gives a chain complex map $\phi^* : \mathrm{Der}(B, B; 1) \longrightarrow \mathrm{Der}(A, B; \phi)$ given by $\phi^*(\theta)(a) = \theta(\phi(a))$ and post-composition by the augmentation $\varepsilon : B \to \mathbb{Q}$ gives $\varepsilon_* : \mathrm{Der}_*(A, B; \phi) \longrightarrow \mathrm{Der}_*(A, \mathbb{Q}; \varepsilon)$ given by $\varepsilon_*(\varphi)(a) = \varepsilon(\varphi(a))$ for $\varphi \in \mathrm{Der}_*(A, B; \phi)$. Note that the augmentation ε will be either of A or B.

The evaluation subgroup of ϕ is defined by

$$G_n(A, B; \phi) = \mathrm{im}\{H(\varepsilon_*) : H_n(\mathrm{Der}(A, B; \phi) \longrightarrow H_n(\mathrm{Der}(A, \mathbb{Q}; \varepsilon))\}.$$

The Gottlieb group of the differential graded algebra (A, d_A) is obtained as a special case

$$G_n(A, A; \mathrm{id}) = \mathrm{im}\{H(\varepsilon_*) : H_n(\mathrm{Der}(A, A; \mathrm{id}) \longrightarrow H_n(\mathrm{Der}(A, \mathbb{Q}; \varepsilon))\}.$$

In particular $G_n(A, A; \mathrm{id}) \cong G_n(X)$ where A is the Sullivan minimal model of X.

Proposition 4. *Let A and B denote the minimal models of $\mathbb{C}P^{N-1}$ and $G_{n,k}$ respectively where $N = \binom{n}{k}$ and let $f : G_{n,k} \longrightarrow \mathbb{C}P^{N-1}$ be the embedding. Define $\phi : A \longrightarrow B$ as follows: $\phi(x_2) = x_2$ and $\phi(y_{2N-1}) = \alpha, \mid d'\alpha = x_2^N$. Then $G_n(A, B; \phi) = \langle \varphi_{2N-1} \rangle$ where $\varphi_{2N-1} \in \mathrm{Der}(A, \mathbb{Q}; \epsilon)$.*

Proof. We have $H_*(\mathrm{Der}\,(A, \mathbb{Q}; \varepsilon)) = \langle \varphi_{2N-1} \rangle$. Let $\lambda \in \mathrm{Der}(A, B; \phi)$ such that $\lambda(y_{2N-1}) = 1$ and λ is zero on x_2. Clearly $\lambda \in H_*(\mathrm{Der}(A, B; \phi))$ and $H(\varepsilon_*)(\lambda) = \varphi_{2N-1}$. □

Following [10], let $\phi : A \longrightarrow B$ be a map of differential graded vector spaces, the mapping cone of ϕ denoted by $Rel_*(\phi)$ is defined as follows: $Rel_n(\phi) = A_{n-1} \oplus B_n$ with the differential $\delta(a, b) = (-d_A(a), \phi(a) + d_B(b))$. Define inclusion and projection chain maps $J : B_n \longrightarrow Rel_n(\phi)$ and $P : Rel_n(\phi) \longrightarrow A_{n-1}$ by $J(b) = (0, b),\ P(a, b) = a$.

We have short exact sequences of chain complexes

$$0 \to B_* \xrightarrow{J} Rel_*(\phi) \xrightarrow{P} A_{*-1} \longrightarrow 0$$

and a long exact sequence in homology

$$\cdots \to H_{n+1}(Rel_*(\phi)) \to H_n(A) \xrightarrow{H(\phi)} H_n(B) \to H_n(Rel(\phi)) \to \cdots .$$

We consider the commutative diagram:

$$\begin{array}{ccc} \mathrm{Der}_*(B, B; 1) & \xrightarrow{\phi^*} & \mathrm{Der}_*(A, B; \phi) \\ \downarrow & & \downarrow \\ \mathrm{Der}_*(B, \mathbb{Q}; \varepsilon) & \xrightarrow{\hat{\phi}^*} & Der_*(A, \mathbb{Q}; \varepsilon) \end{array}$$

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and we have the following homology ladder,

$$\begin{array}{ccccccccc}
\cdots \xrightarrow{H(J)} & H_{n+1}(Rel(\phi^*)) & \xrightarrow{H(P)} & H_n(\mathrm{Der}(B,B;1)) & \xrightarrow{H(\phi^*)} & H_n(\mathrm{Der}(A,B;\phi)) & \xrightarrow{H(J)} \cdots \\
& \downarrow{\scriptstyle H(\varepsilon_*,\varepsilon_*)} & & \downarrow{\scriptstyle H(\varepsilon_*)} & & \downarrow{\scriptstyle H(\varepsilon_*)} & \\
\cdots \xrightarrow{H(J)} & H_{n+1}(Rel(\hat{\phi}^*)) & \xrightarrow{H(\hat{P})} & H_n(\mathrm{Der}(B,\mathbb{Q};\varepsilon)) & \xrightarrow{H(\hat{\phi}^*)} & H_n(\mathrm{Der}(A,\mathbb{Q};\varepsilon)) & \xrightarrow{H(J)} \cdots
\end{array}$$

for $n \geq 2$.

The n^{th} relative evaluation subgroup of ϕ is defined by

$$G_n^{rel}(A,B;\phi) = \mathrm{im}\{H(\varepsilon_*,\varepsilon_*) : H_n(Rel(\phi^*)) \longrightarrow H_n(Rel(\hat{\phi}^*))\}.$$

Theorem 5. $G_*^{rel}(A,B;\phi) = 0$.

Proof. $H_n(Rel(\hat{\phi}^*)) = 0$. We have $Rel_*(\hat{\phi}^*) = \mathrm{Der}_{*-1}(A,\mathbb{Q};\varepsilon) \oplus \mathrm{Der}_*(B,\mathbb{Q};\varepsilon)$ and $H_*(\mathrm{Der}_*(A,\mathbb{Q};\varepsilon)) = \langle \varphi_{2N-1} \rangle$, $H_*(\mathrm{Der}_*(B,\mathbb{Q};\varepsilon)) = \langle \theta_{2(n-t)+1} \rangle$, where $\theta_{2(n-t)+1} \in \mathrm{Der}(B,\mathbb{Q};\varepsilon)$, $t = 1,\dots,k$.

We have $G_*(B) = \langle \theta_{2(n-k)+1}, \dots, \theta_{2(n-1)+1} \rangle$ and $G_*(A,B;\phi) = \langle \varphi_{2N-1} \rangle$.

We claim that $2n-1 \leq 2N-1$, that is, $n \leq N$ where $N = \binom{n}{k}$, $2 \leq k \leq \left[\frac{n}{2}\right]$, $n > 3$.

For $k = 2$, we have $N = \binom{n}{k} = \frac{n!}{(n-2)!2!} = \frac{n(n-1)}{2} \geq n$. Suppose the inequality holds for k, that is, $\frac{n!}{(n-k)!k!} \geq n$ and we prove for $k+1$. We show that $N \geq n$ for $2 \leq (k+1) \leq \left[\frac{n}{2}\right]$. Now $\binom{n}{k+1} = \frac{n!}{(k+1)!(n-k-1)!} = \frac{n!}{k!(n-k)!}\frac{n-k}{k+1} \geq n$ since $\frac{n-k}{k+1} \geq 1$.

Since $N \geq n$ therefore $Rel_*(\hat{\phi}^*) = 0$. From

$$G_n^{rel}(A,B;\phi) = \mathrm{im}\{H(\varepsilon_*,\varepsilon_*) : H_n(Rel(\phi^*)) \longrightarrow H_n(Rel(\hat{\phi}^*))\}$$

we have $G_n^{rel}(A,B;\phi) = 0$ □

Question 6. Does triviality of the relative evaluation subgroup provide us with information on the cohomology of smooth fiber bundles with fiber complex Grassmann manifold $G_{n,k}$?

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